

Starlikeness of Libera transformation II

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1. Introduction.

Let A denote the class of function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in $E = \{z : |z| < 1\}$.

A function $f(z) \in A$ is called to be starlike if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.$$

Similarly, $f(z) \in A$ is called to be convex if and only if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } E.$$

The following interesting results are due to Libera [2].

Theorem A. If $f(z)$ is starlike in E , then so does the function $F(z)$, defined by

$$F(z) = \frac{2}{z} \int_0^z f(t) dt.$$

Theorem B. If $f(z)$ is convex in E , then so does the function $F(z)$, defined by

$$F(z) = \frac{2}{z} \int_0^z f(t) dt.$$

On the other hand, S.Singh and R.Singh [5, Theorem 1] and [4, Theorem 1] proved the following Theorem C and D and Nunokawa [3] proved Theorem E.

Theorem C. If $f(z) \in A$ and $\operatorname{Re} f'(z) > 0$ in E , then the function

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt,$$

is starlike in E for all c , $-1 < c \leq 0$.

Theorem D. If $f(z) \in A$ and $|zf''(z)/f'(z)| < 3/2$ in E , then the function

$$F(z) = \frac{2}{z} \int_0^z f(t) dt.$$

is convex.

Theorem E. Let $f(z) \in A$ and suppose that

$$\operatorname{Re} f'(z) > \frac{1}{12} \left(\log \frac{4}{e} \right) (\tan^2 \alpha^* \frac{\pi}{2} - 3) \quad \text{in } E.$$

where

$$-0.01759 < \frac{1}{12} \left(\log \frac{4}{e} \right) (\tan^2 \alpha^* \frac{\pi}{2} - 3) < -0.01751.$$

Then $F(z)$ is starlike in E , where

$$F(z) = \frac{2}{z} \int_0^z f(t) dt.$$

2. Preliminary.

Lemma. Let $f(z) \in A$ and $f(z) \neq 0$ in $0 < |z| < 1$ and suppose that

$$|\arg f'(z)| < \frac{\pi}{2} \alpha \quad \text{in } E$$

where

$$\alpha = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2} \quad \text{and} \quad \beta > 0.$$

Then we have

$$|\arg g'(z)| < \frac{\pi}{2} \alpha \quad \text{in } E$$

and

$$|\arg \frac{g(z)}{z}| < \frac{\pi}{2} \gamma \quad \text{in } E,$$

where

$$0 < \gamma, \quad \beta = \gamma + \frac{2}{\pi} \tan^{-1} \gamma$$

and

$$g(z) = \frac{2}{z} \int_0^z f(t) dt.$$

Proof. From the hypothesis, it follows that

$$f(z) = g(z) + zg'(z),$$

$$f'(z) = 2g'(z) + zg''(z)$$

and so

$$\arg f'(z) = \arg g'(z) + \arg\left(2 + \frac{zg''(z)}{g'(z)}\right).$$

Now, if there exists a point $z_0 \in E$ such that

$$|\arg g'(z)| < \frac{\pi}{2}\beta \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg g'(z_0)| = \frac{\pi}{2}\beta$$

Then, from [3, Lemma], we have

$$\frac{z_0 g''(z_0)}{g'(z_0)} = i\beta k$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg g'(z_0) = \frac{\pi}{2}\beta$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg g'(z_0) = -\frac{\pi}{2}\beta$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia, \quad a > 0.$$

If $\arg g'(z_0) = \pi\beta/2$, then we have

$$\arg f'(z_0) = \arg g'(z_0) + \arg\left(2 + \frac{zg''(z_0)}{g'(z_0)}\right)$$

$$= \frac{\pi}{2}\beta + \arg(2 + i\beta k)$$

$$\geq \frac{\pi}{2}\left(\beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2}\right).$$

This contradicts the hypothesis. On the other hand, if $\arg g'(z_0) = -\pi\beta/2$, then we have

$$\begin{aligned}\arg f'(z_0) &= -\frac{\pi}{2}\beta + \arg(2 + i\beta k) \\ &\leq -\frac{\pi}{2} - \tan^{-1} \frac{\beta}{2} \\ &= -\frac{\pi}{2}\left(\beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2}\right).\end{aligned}$$

This also contradicts the hypothesis. Therefore we must have

$$|\arg g(z)| < \frac{\pi}{2}\beta \quad \text{in } E.$$

Putting

$$p(z) = \frac{g(z)}{z}, \quad p(0) = 1,$$

then we have

$$g'(z) = p(z) + zp'(z)$$

and

$$\arg g'(z) = \arg p(z) + \arg\left(1 + \frac{zp'(z)}{p(z)}\right).$$

If there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\gamma \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\gamma,$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\gamma k$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2}\gamma$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2}\gamma$$

where

$$p(z_0)^{\frac{1}{\gamma}} = \pm ia, \quad a > 0.$$

Then, applying the same method as the above, we have

$$|\arg g'(z_0)| < \frac{\pi}{2}(\gamma + \frac{2}{\pi} \tan^{-1} \gamma).$$

This contradicts the hypothesis. Therefore we must have

$$|\arg \frac{g(z)}{z}| < \frac{\pi}{2} \gamma \quad \text{in } E.$$

3. Main result.

Theorem. Let $f(z) \in A$, $f(z) \neq 0$ in $0 < |z| < 1$,

$$(1) \quad |\arg f'(z)| < \frac{\pi}{2}(\frac{5}{3} - \gamma_0) \quad \text{in } E$$

where γ_0 is the smallest positive root of the equation

$$\frac{5}{6} = 2\gamma + \frac{2}{\pi} \tan^{-1} \gamma + \frac{2}{\pi} \tan^{-1} \frac{1}{2}(\gamma + \frac{2}{\pi} \tan^{-1} \gamma)$$

and $0.266 < \gamma_0 < 0.267$.

Let us put

$$(2) \quad g(z) = \frac{2}{z} \int_0^z f(t) dt.$$

Then $g(z)$ is starlike in E .

Proof. From (1) and Lemma, we easily have

$$|\arg \frac{g(z)}{z}| < \frac{\pi}{2} \gamma_0 \quad \text{in } E.$$

From (2), we have

$$2f(z) = g(z) + zg'(z)$$

and

$$2f'(z) = 2g'(z) + zg''(z)$$

Let us put

$$\frac{zg'(z)}{g(z)} = \frac{1+w(z)}{1-w(z)}, \quad w(0) = 0$$

then $w(z)$ is analytic in E and $w(z) \neq 1$ in E .

Then it follows that

$$\begin{aligned} 2f'(z) &= 2g'(z) + zg''(z) \\ &= \frac{zg'(z)}{g(z)} \left[\left(\frac{1+w(z)}{1-w(z)} \right)^2 + \frac{2zw'(z)}{(1-w(z))^2} + \frac{1+w(z)}{1-w(z)} \right]. \end{aligned}$$

If there exists a point $z_0 \in E$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then from Jack's lemma [1, Lemma], we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1$$

Putting $w(z_0) = e^{i\theta}$, $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} 2f'(z_0) &= 2g'(z_0) + z_0 g''(z_0) \\ &= \frac{g(z_0)}{z_0} \left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}} \right)^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2} + \frac{1+e^{i\theta}}{1-e^{i\theta}} \right] \\ &= \frac{g(z_0)}{z_0} \left[-\frac{\sin^2 \theta}{(1-\cos \theta)^2} - \frac{k}{1-\cos \theta} + \frac{i \sin \theta}{1-\cos \theta} \right]. \end{aligned}$$

Then we have

$$\begin{aligned} \arg f'(z_0) &\geq \arg \left(-\frac{\sin^2 \theta + k(1-\cos \theta)}{(1-\cos \theta)^2} - \frac{i \sin(1-\cos \theta)}{(1-\cos \theta)^2} \right) - \left| \arg \left(\frac{g(z_0)}{z_0} \right) \right| \\ &\geq \arg \left(-\frac{\sin^2 \theta + 1 - \cos \theta}{(1-\cos \theta)^2} - \frac{i \sin \theta(1-\cos \theta)}{(1-\cos \theta)^2} \right) - \frac{\pi}{2} \gamma_0 \\ &= \pi - \tan^{-1} \frac{|\sin \theta|(1-\cos \theta)}{\sin^2 \theta + 1 - \cos \theta} - \frac{\pi}{2} \gamma_0 \\ &= \pi - \tan^{-1} \frac{|\sin \theta|}{2 + \cos \theta} - \frac{\pi}{2} \gamma_0 \\ &\geq \pi - \tan^{-1} \frac{1}{\sqrt{3}} - \frac{\pi}{2} \gamma_0 \\ &= \frac{5}{6} \pi - \frac{\pi}{2} \gamma_0 \\ &= \frac{\pi}{2} \left(\frac{5}{3} - \gamma_0 \right). \end{aligned}$$

This contradicts (1). Therefore we must have

$$|w(z)| < 1 \quad \text{in } E.$$

This show that

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > 0 \quad \text{in } E.$$

or $g(z)$ is starlike in E .

References

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