

On integral bases of real octic 2-elementary abelian extensions (実 8 次 2-基本アーベル拡大体の整数基について)

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Abstract. Let K be an abelian field whose Galois group is 2-elementary abelian over the rationals $\mathbb{Q}$. If an octic field K is monogenic and a quadratic subfield with odd discriminant and a quartic subfield of K are linearly disjoint, then K coincides with the field $\mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{-3})$, namely K is equal to the cyclotomic field $\mathbb{Q}(\zeta_{24})$ [MN]. In this article, we explain how to prove that all the real octic fields K are non-monogenic, that is, the rings Z_K of integers in K do not have any power integral basis. Finally, we propose a few problems on the evaluation on the field index of K and the non-essential factor (außerwesentliche Diskriminantenteiler) of K .

§1. Introduction

Let K be an algebraic number field over the rationals $\mathbb{Q}$. We denote the ring of integers in K by Z_K . When $Z_K = \mathbb{Z}[\alpha]$ for some element α of Z_K , it is said that α generates a power integral basis of the ring Z_K or simply Z_K has a power integral basis. The field K is called monogenic if Z_K has a power integral basis. It is known as a problem of Hasse to characterize whether a field K is monogenic or not [Gy]. In this article, we consider the fields K whose Galois groups are 2-elementary abelian. Since the field K for $[K : \mathbb{Q}] \geq 16$

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is non-monogenic, i.e., the ring Z_K of integers in K has no power integral basis by virtue of the decomposition theory of a prime number ([Lemma 1, SN], [MNS], [Wa]) and by the works of K. S. Williams, M.-N. Gras and F. Tanoé for Dirichlet fields K , ([Wi], [GT]) it is enough for us to investigate the octic 2-elementary abelian fields. Let k and L be a quadratic subfield of odd discriminant and a quartic subfield of K , respectively. If k and L are linearly disjoint, then such an octic field $K = kL$ is non-monogenic except for the cyclotomic field $\mathbf{Q}(\zeta_{24})$ of conductor 24 [MN]. In this paper, we will show an integral basis of the ring Z_K over the ring $\mathbf{Z}$ of rational integers in an octic field K [Theorem 1]. Next, being based on the linear equations

$$a_{i1}E_{i1} + a_{i2}E_{i2} + a_{i3}E_{i3} = 0 \quad (1 \leq i \leq 7)$$

with suitable factors a_{ij} of the field discriminant D_K , where $(a_{ij}, D_i) = 1$ and units E_{ij} as coefficients of *valuables* a_{ij} in each quadratic subfield $k_j = \mathbf{Q}(\sqrt{D_j})$ [Proposition 2], we can prove that all the real 2-elementary abelian fields K of degree 8 have no power integral basis [Theorem 2].

§2. Integral bases

We determine explicit integral bases of some octic fields K whose Galois groups are 2-elementary abelian. We denote the Galois group

$$\langle \tau, \sigma, \rho \mid \tau : \sqrt{mn} \mapsto -\sqrt{mn}, \sigma : \sqrt{dn} \mapsto -\sqrt{dn}, \rho : \sqrt{d_1 m_1 n_1 \ell} \mapsto -\sqrt{d_1 m_1 n_1 \ell} \rangle$$

of $K/\mathbf{Q}$ by G .

The following lemma and proposition are available to deduce the type of 2-elementary abelian extension fields K which would have power integral bases.

Lemma 1 ([SN]). *Let ℓ be a prime number and let $F/\mathbf{Q}$ be a Galois extension of degree $n = efg$ with ramification index e and the relative degree f with respect to ℓ . If one of the following conditions is satisfied, then Z_F has no power integral basis, i.e., F is non-monogenic;*

$$(1) \ell^f < n \text{ if } f = 1;$$

or

$$(2) \ell^f \leq n + e - 1 \text{ if } f \geq 2.$$

Proposition 1 ([MN]). *Let $a_1, a_2, \dots, a_r$ be square free rational integers and F be the field $\mathbf{Q}(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_r})$ of degree $2^r, r \geq 4$. Then F is non-monogenic.*

Proof. Without loss of generality, we may assume that there exists at most two generators $\sqrt{a_1}, \sqrt{a_2}$ of F with $a_j \not\equiv 1 \pmod{4} (1 \leq j \leq 2)$. Then the ramification index e of the prime

is at most 2^2 . Since the Galois group $G = \text{Gal}(F/\mathbb{Q})$ is 2-elementary, the relative degree f of the prime 2 is at most 2, because the inertia subgroup of G is cyclic. In Lemma 1 let ℓ be equal to 2. Then we can deduce $e\ell^f \leq 2^2 \cdot 2^1 < 2^r$ if $f = 1$ and $e\ell^f \leq 2^2 \cdot 2^2 \leq 2^r + e - 1$ if $f = 2$. Thus F is non-monogenic. $\square$

By the proof of Proposition 1, if an octic field K is monogenic, it is sufficient to consider that K contains *two* quadratic subfields of even discriminant and *one* of odd discriminant.

The main theorem is based on the following theorem, which is an extension of a result of the case of quartic fields [M₁, M₂, Wi].

Theorem 1 ([PMN]). *Let K be an octic field $\mathbb{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1\ell})$ with $d = d_1d_2$, $m = m_1m_2$, $n = n_1n_2$, $mn \equiv 3$, $dn \equiv 2$, $d_1m_1n_1\ell \equiv 1$, $d_2 \equiv 2 \pmod{4}$, $d_1, m_1, n_1 \geq 1$ and $dmn\ell$ is square free. Let D_K be the field discriminant of the octic field K . Then we have $D_K = 2^{12}(dmn\ell)^4$ and an integral basis of K is :*

$$Z_K = \mathbb{Z} \left[1, \sqrt{mn}, \sqrt{dn}, \frac{\sqrt{dm} + \sqrt{dn}}{2}, \frac{1 + \sqrt{d_1m_1n_1\ell}}{2}, \frac{\sqrt{mn} + \sqrt{d_1m_2n_2\ell}}{2}, \frac{\sqrt{dn} + \sqrt{d_2m_1n_2\ell}}{2}, \frac{\sqrt{dm} + \sqrt{dn} + e_1\sqrt{d_2m_2n_1\ell} + e_2\sqrt{d_2m_1n_2\ell}}{4} \right]$$

where $e_i = \pm 1$ ($i = 1, 2$), $e_1 \equiv d_1m_1$, $e_2 \equiv d_1n_1 \pmod{4}$.

§3. Non-monogenic field

It is known that in the case of $d_1m_1n_1 = 1$ that is, there exist a quartic subfield L and a quadratic k of K with $(D_L, D_k) = 1$, the fields K are non-monogenic except for the cyclotomic field $\mathbb{Q}(\zeta_{24})$ of conductor 24 [MN], where D_F means the discriminant of an algebraic number field F over $\mathbb{Q}$. From now on, we consider the case of $d_1m_1n_1 \geq 1$ and as an application of Theorem 1, we can slightly generalize Proposition 5 in [MN], whose proof was done using the relative different with respect to K over a suitable quadratic subfield. We assume that K is *monogenic*.

Let

$$\xi = b_1\sqrt{mn} + b_2\sqrt{dn} + b_3\frac{\sqrt{dm} + \sqrt{dn}}{2} + b_4\frac{1 + \sqrt{d_1m_1n_1\ell}}{2} + b_5\frac{\sqrt{mn} + \sqrt{d_1m_2n_2\ell}}{2} + b_6\frac{\sqrt{dn} + \sqrt{d_2m_1n_2\ell}}{2} + b_7\frac{\sqrt{dm} + \sqrt{dn} + e_1\sqrt{d_2m_2n_1\ell} + e_2\sqrt{d_2m_1n_2\ell}}{4}$$

be a generator of a power integral basis of Z_K . Now we calculate a factor $(\xi - \xi^\sigma)(\xi - \xi^\sigma)^\rho$

of the discriminant $d_{K/Q}(\xi) = \Delta^2 [1, \xi, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6, \xi^7]$ of a number ξ ;

$$\begin{aligned}
& (\xi - \xi^\sigma)(\xi - \xi^{\sigma^2})^\rho \\
&= \left\{ (2b_2 + b_3 + b_6 + \frac{b_7}{2})\sqrt{dn} + (b_3 + \frac{b_7}{2})\sqrt{dm} + (b_6 + \frac{b_7e_2}{2})\sqrt{d_2m_1n_2\ell} + \frac{b_7e_1\sqrt{d_2m_2n_1\ell}}{2} \right\} \\
&\times \left\{ (2b_2 + b_3 + b_6 + \frac{b_7}{2})\sqrt{dn} + (b_3 + \frac{b_7}{2})\sqrt{dm} - (b_6 + \frac{b_7e_2}{2})\sqrt{d_2m_1n_2\ell} - \frac{b_7e_1\sqrt{d_2m_2n_1\ell}}{2} \right\} \\
&= \left\{ (2b_2 + b_3 + b_6 + \frac{b_7}{2})\sqrt{dn} + (b_3 + \frac{b_7}{2})\sqrt{dm} \right\}^2 - \left\{ (b_6 + \frac{b_7e_2}{2})\sqrt{d_2m_1n_2\ell} + \frac{b_7e_1\sqrt{d_2m_2n_1\ell}}{2} \right\}^2 \\
&= \left\{ (2b_2 + b_3 + b_6)^2 + (2b_2b_7 + b_3b_7 + b_6b_7) + \frac{b_7^2}{4} \right\} dn + (b_3^2 + b_3b_7 + \frac{b_7^2}{4}) dm \\
&- (b_6^2 + b_6b_7e_2 + \frac{b_7^2e_2^2}{4}) d_2m_1n_2\ell - \frac{b_7^2e_1^2d_2m_2n_1\ell}{4} \\
&+ \left\{ (2^2b_2b_3 + 2b_3^2 + 2b_3b_6 + 2b_3b_7 + 2b_2b_7 + b_6b_7 + \frac{b_7^2}{2})d - (b_6b_7e_1d_2\ell + \frac{b_7^2e_2e_1d_2\ell}{2}) \right\} \sqrt{mn},
\end{aligned}$$

namely, this factor is an integer of the quadratic field $k_1 = \mathbf{Q}(\sqrt{mn})$ of the fixed field by the subgroup $\langle \sigma, \rho \rangle$ in G . Then we denote it by $\eta_{11} = B + C(\sqrt{mn})$. Thus we obtain

$$\begin{aligned}
B/d_2 &\equiv \left\{ b_3^2 + b_6^2 + b_3b_7 + \frac{b_7^2}{4} \right\} d_1n + \left(b_3^2 + b_3b_7 + \frac{b_7^2}{4} \right) d_1m \\
&- \left(b_6^2 + b_6b_7 + \frac{b_7^2}{4} \right) m_1n_2\ell - \frac{b_7^2m_2n_1\ell}{4} \\
&\equiv \frac{b_7^2}{4} (d_1(m+n) - (m_1n_2 + m_2n_1)\ell) \\
&\equiv \frac{\{d_1(m+n) - (d_1n + 4k + d_1m + 4k)\}}{4} \equiv 0 \pmod{2},
\end{aligned}$$

by $d_1m_1n_1\ell \equiv 1 + 4k \pmod{8}$ and $m+n \equiv 0 \pmod{4}$, since $m_1n_2\ell \cdot 1 \equiv d_1m_1^2n_1n_2\ell^2 + 4m_1n_2\ell k \equiv d_1n + 4k \pmod{8}$ and $m_2n_1\ell \cdot 1 \equiv d_1m_1m_2n_1^2\ell^2 + 4m_2n_1\ell k \equiv d_1m + 4k \pmod{8}$.

$$\begin{aligned}
C/d_2 &\equiv (b_6b_7 + \frac{b_7^2}{2})d_1 - (b_6b_7e_1\ell + \frac{b_7^2e_2e_1\ell}{2}) \\
&\equiv b_6b_7(d_1 - e_1\ell) + \frac{b_7^2}{2}(d_1 - e_2e_1\ell) \equiv 0 \pmod{2}
\end{aligned}$$

by $e_1 \equiv d_1m_1$, $e_2 \equiv d_1n_1 \pmod{4}$, since $d_1 - e_2e_1\ell \equiv d_1 - d_1^2m_1n_1\ell \equiv d_1(1 - d_1m_1n_1\ell) \equiv 0 \pmod{4}$. So we can write $\eta_{11} = (\xi - \xi^\sigma)(\xi - \xi^{\sigma^2})^\rho = 2d_2E_1$ for an integer $E_1 = B_1 + C_1\sqrt{mn}$ in $k_1 = \mathbf{Q}(\sqrt{mn})$. By the same computation, we obtain $\eta_{12} = (\xi - \xi^\rho)(\xi - \xi^{\rho^2})^\sigma = \ell E_2$, $\eta_{13} = (\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma^2\rho})^\rho = d_1E_3$ for units E_j in k_1 ($j = 2, 3$). By the assumption that Z_K is generated by ξ , we have

$$d_{K/Q}(\xi) = \pm N_K(\mathfrak{d}(\xi)) = \pm D_K,$$

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where $\mathfrak{d}(\alpha)$, $N_K(\alpha)$ and $N_K(\mathfrak{a})$ means the different of a number, norm of α and an ideal $\mathfrak{a}$ with respect to $K/\mathbf{Q}$, respectively[Wa]. Then, because η_{1j} is a partial factor of $d_{K/\mathbf{Q}}(\xi)$, the integers E_j should be units in $k_1 = \mathbf{Q}(\sqrt{mn})$. Here the following is our basic identity:

$$(\xi - \xi^\sigma)(\xi - \xi^\sigma)^\rho - (\xi - \xi^\rho)(\xi - \xi^\rho)^\sigma - (\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho})^\rho = 0$$

for $(\xi - \xi^\sigma)(\xi - \xi^\sigma)^\rho = \eta_{11}$, $(\xi - \xi^\rho)(\xi - \xi^\rho)^\sigma = \eta_{12}$ and $(\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho})^\rho = \eta_{13}$. Then we have the equation

$$2d_2E_1 - \ell E_2 - d_1E_3 = 0 \quad \text{in } k_1 = \mathbf{Q}(\sqrt{D_1}), \quad D_1 = m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2,$$

where E_1, E_2 and E_3 are units in k_1 .

In the same way, we obtain seven equations corresponding to each of the seven quadratic subfields k_j of K .

Proposition 2. *If $K = \mathbf{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1\ell})$ is monogenic, then the following simultaneous equations hold:*

$$(1) \quad \ell E_{11} + 2d_2E_{12} + d_1E_{13} = 0 \quad \text{in } k_1 = \mathbf{Q}(\sqrt{D_1}), \quad D_1 = m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2,$$

$$(2) \quad \ell E_{21} + 2m_2E_{22} + m_1E_{23} = 0 \quad \text{in } k_2 = \mathbf{Q}(\sqrt{D_2}), \quad D_2 = d_1 \cdot 2d_2 \cdot n_1 \cdot 2n_2,$$

$$(3) \quad \ell E_{31} + 2n_2E_{32} + n_1E_{33} = 0 \quad \text{in } k_3 = \mathbf{Q}(\sqrt{D_3}), \quad D_3 = d_1 \cdot 2d_2 \cdot m_1 \cdot 2m_2,$$

$$(4) \quad 2d_2E_{41} + 2m_2E_{42} + 2n_2E_{43} = 0 \quad \text{in } k_4 = \mathbf{Q}(\sqrt{D_4}), \quad D_4 = d_1 \cdot m_1 \cdot n_1 \cdot \ell,$$

$$(5) \quad 2d_2E_{51} + m_1E_{52} + n_1E_{53} = 0 \quad \text{in } k_5 = \mathbf{Q}(\sqrt{D_5}), \quad D_5 = d_1 \cdot 2m_2 \cdot 2n_2 \cdot \ell,$$

$$(6) \quad d_1E_{61} + 2m_2E_{62} + n_1E_{63} = 0 \quad \text{in } k_6 = \mathbf{Q}(\sqrt{D_6}), \quad D_6 = 2d_2 \cdot m_1 \cdot 2n_2 \cdot \ell,$$

$$(7) \quad d_1E_{71} + m_1E_{72} + 2n_2E_{73} = 0 \quad \text{in } k_7 = \mathbf{Q}(\sqrt{D_7}), \quad D_7 = 2d_2 \cdot 2m_2 \cdot n_1 \cdot \ell,$$

where each E_{ij} is a unit in the corresponding quadratic subfield k_i of K and each D_i the field discriminant of k_i , respectively.

For the case of a real quadratic field, the following lemma holds:

Lemma 2. *Let E_j be a power $\varepsilon_0^j = \frac{u_j + v_j\sqrt{D}}{2}$ of the fundamental unit $\varepsilon_0 = \frac{u + v\sqrt{D}}{2} > 1$ in a real quadratic field $\mathbf{Q}(\sqrt{D})$ with the field discriminant D and $\bar{\alpha} = \alpha^\gamma$ for α in $\mathbf{Q}(\sqrt{D})$ and $\gamma (\neq 1)$ in $\text{Gal}(\mathbf{Q}(\sqrt{D})/\mathbf{Q})$. Let*

$$\begin{cases} a + bE_j + cE_k = 0, \\ a + b\bar{E}_j + c\bar{E}_k = 0 \end{cases} \quad (*)$$

for $abc \neq 0$. Denote the matrix

$$\begin{pmatrix} 1 & E_j & E_k \\ 1 & \bar{E}_j & \bar{E}_k \end{pmatrix}$$

attached to the equation (*) by A and the rank of A by r_D . Then we have a solution (a, b, c) of rational integers :

$$\begin{cases} a \pm b \pm c = 0 & \text{for } r_D = 1, \\ \frac{a}{u_k v_j - u_j v_k} = \frac{b}{2v_k} = \frac{c}{-2v_j} & \text{for } r_D = 2 \end{cases}$$

with $E_i = \frac{u_i + v_i \sqrt{D}}{2}$.

Proof. This lemma means that the integral solutions should be on the plane for the rank $r_D = 1$ of the coefficient matrix A and on the line i.e. the intersection of two planes for $r_D = 2$, respectively.

First, we consider the case of $r_D = 1$, then for

$$\begin{cases} E_i = \frac{u_i + v_i \sqrt{D}}{2}, \\ \overline{E}_i = \frac{u_i - v_i \sqrt{D}}{2}, \end{cases}$$

$E_i, \overline{E}_i$ should be a rational number. Then we have $E_j = u_j = \pm 1$ and $E_k = u_k = \pm 1$. Hence $a \pm b \pm c = 0$. Second, we assume $r_D = 2$. Then we have

$$a : b : c = \left| \frac{E_j}{E_j} \frac{E_k}{E_k} \right| : \left| \frac{E_k}{E_k} 1 \right| : \left| 1 \frac{E_j}{E_j} \right| = u_k v_j - u_j v_k : 2v_k : -2v_j.$$

Hence

$$\frac{a}{u_k v_j - u_j v_k} = \frac{b}{2v_k} = \frac{c}{-2v_j}.$$

□

In the case of any octic field $\mathbf{Q}(\sqrt{m_1 m_2 n_1 n_2}, \sqrt{d_1 d_2 n_1 n_2}, \sqrt{d_1 m_1 n_1 \ell})$, by the following lemma, we can deduce to evaluate the rank r_D of a quadratic field $\mathbf{Q}(\sqrt{D})$ for a few cases with respect to the order of values $d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell$ in the set of seven parameters.

Lemma 3. Let denote the set $\{d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$ by D . Then it holds that:

- (1) For one parameter s in D , there exist only four quadratic subfields k_j whose discriminants D_j are divisible by s .
- (2) For two parameters s, t in D , there exist only two quadratic subfields k_j whose discriminants D_j are divisible by st .
- (3) Let s, t, u be three parameters in D , such that stu is a divisor of the field discriminant of D_j of k_j . Then there exists only one quadratic subfield k_j whose discriminant D_j is divisible by stu .

Proof. (1) We can confirm the claim (1) for each of $\binom{\#D}{1} = 7$ parameter in D from seven equations in Proposition 2, such that there exist just four fields k_1, k_3, k_4, k_6 whose discriminant is divisible by m_1 .

(2) We can do the claim (2) of $\binom{\#D}{2} = 21$ pairs of parameters in D by the same way as in (1). For instance, there exist just two fields k_3, k_7 whose discriminants are divisible by d_2m_2 .

(3) We assume that $D_i = stua$ and $D_j = stub$. Then we have $D_iD_j = (stu)^2ab$. However, the quadratic subfield $\mathbb{Q}(\sqrt{ab})$ does not coincide with any $k_j (1 \leq j \leq 7)$. $\square$

Remark 1. We can confirm that the number of triplets (s, t, u) within the order of parameters in D is equal to $28 = 7 \times 1 \times \binom{4}{3} < \binom{\#D}{3} = 35$ such that each of stu is a divisor of the field discriminant D_j of k_j .

Next, we prepare the key lemma for the proof of Theorem 2.

Lemma 4. For the set $D = \{a, b, c, d, e, f, g\}$ of seven positive rational integers, assume that $a > b \geq c > \max\{d, e, f, g\}$ and $d > f$ or $a > b > c \geq \max\{d, e, f, g\}$ and $d > f$. Then

(1) For the field $\mathbb{Q}(\sqrt{bcst})$, where $s, t \in D \setminus \{a, b, c\}$ and units E_i in $\mathbb{Q}(\sqrt{bcst})$, the rank r_{bcst} of the equations

$$\begin{cases} a + uE_j + vE_k = 0, \\ a + u\overline{E_j} + v\overline{E_k} = 0, \end{cases}$$

with $\{u, v\} = D \setminus \{a, b, c, s, t\}$ is equal to 1.

(2) For the field $\mathbb{Q}(\sqrt{astu})$, where $s, t, u \in D \setminus \{a, b, c\}$ and units E_i in $\mathbb{Q}(\sqrt{astu})$, the rank r_{astu} of the equations

$$\begin{cases} b + cE_j + vE_k = 0, \\ b + c\overline{E_j} + v\overline{E_k} = 0, \end{cases}$$

with $\{v\} = D \setminus \{a, b, c, s, t, u\}$ is equal to 1.

Sketch of Idea. Our idea for the proof of this lemma is as follows. For the quadratic subfield k including the coefficients of the simultaneous equation (*), if the field discriminant D_k is divisible by the biggest parameter (case (1)) or the second and the third ones (case (2)), since the fundamental unit (> 1) of k is relatively big, the ratios for the line in Lemma 2 would not be permitted. Thus the ranks of the coefficient matrix for both cases should be equal to one, respectively, namely any integral solution of (*) lies on the plane [PMN].

$\square$

Finally, we show the following main theorem, which is a generalization of a prototype[PMN].

Theorem 2. *Let $K = \mathbf{Q}(\sqrt{a_1}, \dots, \sqrt{a_r})$ be the 2-elementary abelian extensions over $\mathbf{Q}$ whose degree 2^r is greater than 8 or real octic ones for square free integers $a_1, \dots, a_r$. Then the fields K are non-monogenic.*

Sketch of Proof. By Proposition 1, it is enough to consider an octic field K . Let $(2) = \mathfrak{L}_1^e \cdots \mathfrak{L}_g^e$ be the prime ideal decomposition of a rational prime 2 in K . For the ramification index of 2, if $e \leq 1$, then by Lemma 1 and the relative degree f of a prime 2 is at most 2, we have $1 \cdot 2^1 < 8$ or $1 \cdot 2^2 \leq 8 + 1 - 1$ for $e = 1$ and $2 \cdot 2^1 \leq 8$ or $2 \cdot 2^2 \leq 8 + 2 - 1$ for $e = 2$, namely K is non-monogenic. Then in the case of $e \geq 3$, we can deduce that the type of an octic field K is $K = \mathbf{Q}(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$, where $a_1 = mn \equiv 3, a_2 = dn \equiv 2, a_3 = d_1 m_1 n_1 \ell \equiv 1 \pmod{4}$, for $d = d_1 d_2, m = m_1 m_2, n = n_1 n_2$ and $dmn\ell$ is square free. Put $D = \{d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$. We denote again by $\{a, b, c, d, e, f, g\}$ any transposition on the seven parameters in D . Without loss of generality, we may assume that $a > b > c \geq \max\{d, e, f, g\}$. Using Lemma 4, it is enough for us to consider the following two cases.

Case (I). The field K includes $k_{j_1} = \mathbf{Q}(\sqrt{abct})$ for some $t \in D \setminus \{a, b, c\}$, for instance, $t = d$.

Case (II). The field K does not include the field $\mathbf{Q}(\sqrt{abcs})$ for any $s \in D \setminus \{a, b, c\}$.

In the case (I), we can deduce that the four parameters a, b, c, d with $c \geq d$ must lie on suitable two planes and in the case (II), a, b, e, g with $e > g$ do on four planes, respectively. However, the order of the parameters would be destroyed. Then we can prove that any real octic fields K does not have a power integral basis[PNM]. $\square$

Remark 2. Recently, in [PNM] we proved that all the 2-elementary abelian fields K with degree $[K : \mathbf{Q}] \geq 8$ are non-monogenic except for the field $\mathbf{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{-3}) = \mathbf{Q}(\zeta_{24})$.

Problem. For a primitive element ξ in K , let $\text{Ind}(\xi)$, $\tilde{m}(K)$ and $m(K)$ be the index $\sqrt{\left| \frac{dK(\xi)}{dK} \right|}$ of an element ξ , the minimum index $\min_{\xi \in K} \{\text{Ind}(\xi)\}$ of K and the field index $\gcd\{\text{Ind}(\xi)\}_{\xi \in K}$ of K , respectively. Let the fields K run through all the real octic fields whose Galois groups are 2-elementary abelian. Then evaluate the values of

$$\inf_K \tilde{m}(K) \quad \text{and} \quad \inf_K m(K),$$

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respectively.

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