

A new characterization of ℓ_p by an L_p -function

Aoi Honda ^a Yoshiaki Okazaki ^a Hiroshi Sato ^b

^a *Kyushu Institute of Technology, 680-4, Kawazu, Izuka 820-8502, Japan*

^b *Kyushu University, Faculty of Mathematics, Fukuoka 812-8581, Japan*

Abstract

In this talk, we shall show that the classical sequence space ℓ_p ($1 < p < +\infty$) is completely determined by one function $f(x) (\neq 0) \in L_p(\mathbb{R})$ which satisfies the p -integrability condition.

We introduce a new sequence space $\Lambda_p(f)$ defined by an L_p -function $f (\neq 0)$ for $1 \leq p < +\infty$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\},$$

where

$$\Psi_p(a : f) := \left(\sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p dx \right)^{\frac{1}{p}}.$$

We shall give a characterization for $\Lambda_p(f) = \ell_p$. We shall also discuss the linear and topological properties of $\Lambda_p(f)$.

1 Introduction

Let $f (\neq 0)$ be an L_p -function on the real line $\mathbb{R}$.

For $1 \leq p < +\infty$ and for a real sequence $a = \{a_n\} \in \mathbb{R}^\infty$, we set

$$\Psi_p(a : f) := \left(\sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p dx \right)^{\frac{1}{p}},$$

and define $\Lambda_p(f)$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\}.$$

By the triangular inequality of L_p -norm and by the translation invariance of the Lebesgue measure, we have

$$\Psi_p(a - b : f) \leq \Psi_p(a : f) + \Psi_p(b : f),$$

which implies that $\Lambda_p(f)$ is an additive subgroup of $\mathbb{R}^\infty$.

Define a metric on $\Lambda_p(f)$ by

$$d_p(a, b) := \Psi_p(a - b : f).$$

Then $(\Lambda_p(f), d_p(a, b))$ becomes a topological group. The space $\mathbb{R}_0^\infty$, the direct sum, is a dense subset of $(\Lambda_p(f), d_p(a, b))$.

2 Relations between $\Lambda_p(f)$ and ℓ_p

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbb{R}$ and the p -integral defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p dx$$

is finite. In particular $I_2(\sqrt{f})$, for probability density function $f(x)$, coincides with the Shepp's integral (Shepp[3]).

Theorem 1 ([2]) Let $1 \leq p < +\infty$ and let $f(\neq 0)$ be an L_p -function on $\mathbb{R}$. Then $\Lambda_p(f) \subset \ell_p$

Proof. Assume that $\Psi_p(a; f) < +\infty$ for $a = \{a_k\} \in \mathbb{R}^\infty$. Without loss of generality, we may assume $a_k \neq 0$ for every k .

First we shall prove $\{a_k\}$ is bounded. If there is a subsequence $\{a_{k'}\}$ such that $|a_{k'}| \rightarrow +\infty$, then $\Psi_p(a; f) < +\infty$ implies

$$0 = \lim_{k'} \left(\int_{-\infty}^{+\infty} |f(x - a_{k'}) - f(x)|^p dx \right)^{1/p} = 2^{1/p} \|f\|_{L_p}$$

which contradicts to $\|f\|_{L_p} > 0$.

Next we shall prove that $\{a_k\}$ converges to 0. Assume that there is a subsequence $a_{k'}$ such that $a_{k'} \rightarrow a_0 \neq 0$. Then we have

$$0 = \lim_{k'} \int_{-\infty}^{+\infty} |f(x - a_{k'}) - f(x)|^p dx = \int_{-\infty}^{+\infty} |f(x - a_0) - f(x)|^p dx,$$

which implies $f(x - a_0) = f(x)$, a.e.(dx). This contradicts to the integrability of $f(x)$.

Finally, we shall prove

$$\rho := \inf_k \int_{-\infty}^{+\infty} \left| \frac{f(x - a_k) - f(x)}{a_k} \right|^p dx > 0.$$

Assume that there exists a subsequence $a_{k'}$ such that

$$\int_{-\infty}^{+\infty} \left| \frac{f(x - a_{k'}) - f(x)}{a_{k'}} \right|^p dx \rightarrow 0$$

Then it follows that

$$\frac{f(x - a_{k'}) - f(x)}{a_{k'}} \rightarrow 0 \text{ in } L_p(\mathbb{R}).$$

Consequently, $f(x)$ is absolutely continuous with $f'(x) = 0, a.e.(dx)$, that implies $f = 0$, which is a contradiction.

Therefore we have

$$+\infty > \sum_k \int_{-\infty}^{+\infty} \left| \frac{f(x - a_k) - f(x)}{a_k} \right|^p dx |a_k|^p \geq \rho \sum_k |a_k|^p,$$

which proves the theorem.

Theorem 2 ([2]) Let $1 < p < +\infty$ and $f(\neq 0)$ be a non-negative integrable function on $\mathbb{R}$. Then $\Lambda_p(f) = \ell_p$ if and only if $I_p(f) < +\infty$.

Proof. Assume $\Psi_p(a; f) < +\infty$ for every $a = \{a_k\} \in \ell_p$. We set

$$\psi(a) := \int_{-\infty}^{+\infty} \left| f(x - a) - f(x) \right|^p dx,$$

$$u_n := 2^{-\frac{n}{p}},$$

and

$$F_n(x) := \frac{f(x - u_n) - f(x)}{u_n}.$$

Then we shall show

$$K := \sup_N 2^N \psi(u_N) = \sup_N \int_{-\infty}^{+\infty} \left| F_N(x) \right|^p dx < +\infty.$$

Assume, on the contrary, that for every n there exists $N(n) > n$ satisfying

$$2^{N(n)} \psi(u_{N(n)}) > 2^n.$$

Then for the sequence

$$\mathbf{a}_0 := (\widehat{u_{N(1)}}, \widehat{\cdots}, \widehat{u_{N(1)}}), \cdots, (\widehat{u_{N(n)}}, \widehat{\cdots}, \widehat{u_{N(n)}}), \cdots),$$

we have $\mathbf{a}_0 \in \ell_p$ and $\Psi_p(\mathbf{a}_0; f) = +\infty$, which is a contradiction.

Since $L_p(\mathbb{R}, dx)$, $1 < p < +\infty$, is a separable reflexible Banach space, each bounded closed ball is compact and metrizable with respect to the weak topology. So that there exists a subsequence $\{F_{n_j}(x)\}$ and $h(x) \in L_p(\mathbb{R}, dx)$ such that $\{F_{n_j}(x)\}$ converges weakly to $h(x)$.

Consequently, $f(x)$ is absolutely continuous, $f'(x) = -h(x)$, a.e.(dx), and we have

$$I_p(f) = \int_{-\infty}^{+\infty} |f'(x)|^p dx = \int_{-\infty}^{+\infty} |h(x)|^p dx < +\infty.$$

Conversely, assume $I_p(f) < +\infty$. Then by the mean value theorem and Fubini's theorem, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} |f(x - a_k) - f(x)|^p dx &= |a_k|^p \int_{-\infty}^{+\infty} \left| \int_0^1 f'(x - ta_k) dt \right|^p dx \\ &\leq |a_k|^p \int_{-\infty}^{+\infty} dx \int_0^1 |f'(x - ta_k)|^p dt = |a_k|^p \int_{-\infty}^{+\infty} |f'(x)|^p dx = I_p(f) |a_k|^p, \end{aligned}$$

which implies

$$\sum_k \int_{-\infty}^{+\infty} |f(x - a_k) - f(x)|^p dx \leq I_p(f) \sum_{k=1}^{+\infty} |a_k|^p < +\infty.$$

3 Linearity of $\Lambda_p(f)$

We say $f(x)$ is an N-modal function if there exist $a_n, n = 1, 2, \dots, 2N + 1$ such that

$$-\infty = a_1 < a_2 < \cdots < a_{2N} < a_{2N+1} = +\infty,$$

$f(x)$ is non-decreasing on the interval $[a_{2k-1}, a_{2k}]$, and

$f(x)$ is non-increasing on the interval $[a_{2k}, a_{2k+1}]$.

Let $\alpha = \alpha(f) := \frac{1}{2} \min\{a_{k+1} - a_k\}$ if $N \geq 2$ and $\alpha := +\infty$ if $N = 1$.

Lemma 3 Let $f(x) : [-2a, 2a] \rightarrow [0, +\infty)$ be a function such that $f(x)$ is non-decreasing on $[-2a, 0]$ and is non-increasing on $[0, 2a]$, where $a \geq 0$. Then for every $t \in [0, 1]$, we have

$$\int_0^a |f(x-ta) - f(x)|^p dx \leq \int_a^{2a} |f(x-a) - f(x)|^p dx + 3 \int_0^a |f(x-a) - f(x)|^p dx.$$

Proof. Let u be the x -coordinate of the cross point of $f(x)$ and of $f(x-ta)$ and v be the x -coordinate of the cross point of $f(x-ta)$ and of $f(x-a)$. Then we have $0 \leq u \leq ta \leq v \leq a$. We have

$$\begin{aligned} \int_0^{ta} |f(x-ta) - f(x)|^p dx &= \left(\int_0^u + \int_u^{ta} \right) \\ &\leq \int_0^u (f(x) - f(x-a))^p dx + \int_u^{ta} (f(x-ta) - f(x+a-ta))^p dx \\ &\leq \int_0^{ta} |f(x-a) - f(x)|^p dx + \int_{u-ta}^0 |f(x) - f(x+a)|^p dx \\ &= \int_0^{ta} |f(x-a) - f(x)|^p dx + \int_{u-ta+a}^a |f(x-a) - f(x)|^p dx \\ &\leq 2 \int_0^a |f(x-a) - f(x)|^p dx, \end{aligned}$$

where we have used the facts

$$f(x-a) \leq f(x-ta) \leq f(x) \text{ on } [0, u] \text{ and}$$

$$f(x+a-ta) \leq f(x) \leq f(x-ta) \text{ on } [u, ta].$$

On the other hand we have

$$\begin{aligned} \int_{ta}^a |f(x-ta) - f(x)|^p dx &= \left(\int_{ta}^v + \int_v^a \right) \\ &\leq \int_{ta}^v (f(x-ta) - f(x+a-ta))^p dx + \int_v^a (f(x-a) - f(x))^p dx \\ &= \int_0^{v-ta} (f(x) - f(x+a))^p dx + \int_v^a (f(x-a) - f(x))^p dx \\ &= \int_a^{v-ta+a} (f(x-a) - f(x))^p dx + \int_v^a (f(x-a) - f(x))^p dx \\ &\leq \int_a^{2a} |f(x-a) - f(x)|^p dx + \int_0^a |f(x-a) - f(x)|^p dx, \end{aligned}$$

where we have used the facts

$$f(x+a-ta) \leq f(x) \leq f(x-ta) \text{ on } [ta, v], \text{ and}$$

$$f(x) \leq f(x-ta) \leq f(x-a) \text{ on } [v, a].$$

Consequently we have the inequality of Lemma 3.

Lemma 4 Let $f(x) : [-2a, 2a] \rightarrow [0, +\infty)$ be a function such that $f(x)$ is non-increasing on $[-2a, 0]$ and is non-decreasing on $[0, 2a]$, where $a \geq 0$. Then for every $t \in [0, 1]$, we have

$$\int_0^a |f(x-ta) - f(x)|^p dx \leq \int_{-a}^0 |f(x-a) - f(x)|^p dx + 3 \int_0^a |f(x-a) - f(x)|^p dx.$$

Proof. Let u be the x -coordinate of the cross point of $f(x)$ and of $f(x-ta)$ and v be the x -coordinate of the cross point of $f(x-ta)$ and of $f(x-a)$. Then we have $0 \leq u \leq ta \leq v \leq a$. We have

$$\begin{aligned} \int_0^{ta} |f(x-ta) - f(x)|^p dx &= \left(\int_0^u + \int_u^{ta} \right) \\ &\leq \int_0^u (f(x-a) - f(x))^p dx + \int_u^{ta} (f(x-a-ta) - f(x-ta))^p dx \\ &\leq \int_0^{ta} |f(x-a) - f(x)|^p dx + \int_{u-ta}^0 |f(x-a) - f(x)|^p dx \\ &\leq \int_0^a |f(x-a) - f(x)|^p dx + \int_{-a}^0 |f(x-a) - f(x)|^p dx, \end{aligned}$$

where we have used the facts

$$f(x) \leq f(x-ta) \leq f(x-a) \text{ on } [0, u] \text{ and}$$

$$f(x-ta) \leq f(x) \leq f(x-a-ta) \text{ on } [u, ta].$$

On the other hand we have

$$\begin{aligned} \int_{ta}^a |f(x-ta) - f(x)|^p dx &= \left(\int_{ta}^v + \int_v^a \right) \\ &\leq \int_{ta}^v (f(x-a-ta) - f(x-ta))^p dx + \int_v^a (f(x) - f(x-a))^p dx \\ &= \int_0^{v-ta} (f(x-a) - f(x))^p dx + \int_v^a (f(x) - f(x-a))^p dx \\ &\leq 2 \int_0^a |f(x-a) - f(x)|^p dx, \end{aligned}$$

where we have used the facts

$$f(x-ta) \leq f(x) \leq f(x-a-ta) \text{ on } [ta, v], \text{ and}$$

$$f(x-a) \leq f(x-ta) \leq f(x) \text{ on } [v, a].$$

Consequently we have the inequality of Lemma 4.

Theorem 5 Let $f(x)$ be a non-negative integrable N -modal function. Then for every $0 \leq a \leq \alpha$ and every $0 \leq t \leq 1$, we have

$$\int_{-\infty}^{+\infty} |f(x - ta) - f(x)|^p dx \leq 5 \int_{-\infty}^{+\infty} |f(x - a) - f(x)|^p dx.$$

Proof. On the subset

$$S := [a_1, a_2] \cup [a_2 + a, a_3] \cup [a_3 + a, a_4] \cup \cdots \cup [a_{2N} + a, a_{2N+1}]$$

we have

$$f(x - a) \leq f(x - ta) \leq f(x) \text{ for } x \in [a_1, a_2] \text{ or } x \in [a_{2k-1}, a_{2k}], \text{ and}$$

$$f(x) \leq f(x - ta) \leq f(x - a) \text{ for } x \in [a_{2k}, a_{2k+1}] \text{ or } x \in [a_{2N} + a, a_{2N+1}],$$

which implies

$$\int_S |f(x - ta) - f(x)|^p dx \leq \int_S |f(x - a) - f(x)|^p dx.$$

By applying Lemma1 and Lemma2 for the function $g(x) = f(x + a_k)$, we have

$$\begin{aligned} & \int_{a_{2k}}^{a_{2k}+a} |f(x - ta) - f(x)|^p dx \\ & \leq \int_{a_{2k}}^{a_{2k}+2a} |f(x - a) - f(x)|^p dx + 3 \int_{a_{2k}}^{a_{2k}+a} |f(x - a) - f(x)|^p dx, \text{ and} \\ & \int_{a_{2k+1}}^{a_{2k+1}+a} |f(x - ta) - f(x)|^p dx \\ & \leq \int_{a_{2k+1}-a}^{a_{2k+1}} |f(x - a) - f(x)|^p dx + 3 \int_{a_{2k+1}}^{a_{2k+1}+a} |f(x - a) - f(x)|^p dx. \end{aligned}$$

Consequently we have the inequality.

Theorem 6 Let $f(x)$ be a non-negative integrable N-modal function. Then $\Lambda_p(f)$ is a linear space.

Proof. Let $\{a_n\} \in \Lambda_p(f)$. We shall show that $t\{a_n\} \in \Lambda_p(f)$ for every $0 \leq t \leq 1$. Without loss of generality, we may assume $a_n \geq 0$. Since $\Lambda_p(f) \subset \ell_p$, there exists K such that $a_n \leq \alpha$ for every $n \geq K$. The finite sequence $t(a_1, \dots, a_{K-1}, 0, 0, \dots)$ belongs to $\Lambda_p(f)$ and the sequence $t(0, 0, \dots, 0, a_K, a_{K+1}, \dots)$ belongs to $\Lambda_p(f)$ by Theorem 1, so that $t\{a_n\} \in \Lambda_p(f)$.

4 Completeness of $\Lambda_p(f)$

Theorem 7 ([1]) Let $f(\neq 0)$ be an L_p -function. Then $\Lambda_p(f)$ is complete with respect to d_p for $1 \leq p < +\infty$.

Proof. Let $\mathbf{a}^{(k)} \in \Lambda_p(f), k = 1, 2, \dots$, be a Cauchy sequence in d_p . Then for every $\varepsilon > 0$, there exists N such that

$$(*) \quad \sum_n \int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)} + a_n^{(l)}) - f(x) \right|^p dx \leq \varepsilon^p.$$

for $k, l \geq N$. For any fixed n , we have

$$\int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)} + a_n^{(l)}) - f(x) \right|^p dx \rightarrow 0, \text{ as } k, l \rightarrow +\infty.$$

Then it follows that $a_n^{(k)} - a_n^{(l)} \rightarrow 0$ as $k, l \rightarrow +\infty$, that is, $\{a_n^{(k)}\}$ is a Cauchy sequence (see the proof of Theorem 2.)

Let $\mathbf{a}_n^{(0)} := \lim_k a_n^{(k)}$. Then we shall show $\mathbf{a}^{(k)} \rightarrow \mathbf{a}^{(0)} := \{\mathbf{a}_n^{(0)} \mid n = 1, 2, \dots\}$ in d_p . In the inequality (*), taking $\liminf_{l \rightarrow +\infty}$, by the Fatou's Lemma, we have

$$\begin{aligned} \varepsilon^p &\geq \sum_n \liminf_{l \rightarrow +\infty} \int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)}) - f(x - a_n^{(l)}) \right|^p dx \\ &= \sum_n \int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)}) - f(x - a_n^{(0)}) \right|^p dx = d_p(\mathbf{a}^{(k)}, \mathbf{a}^{(0)})^p, \end{aligned}$$

which shows $\mathbf{a}^{(k)} \rightarrow \mathbf{a}^{(0)}$ with respect to d_p .

References

- [1] A. Honda, Y. Okazaki and H. Sato, A class of sequence spaces defined by a non-negative integrable function, ISBFS 2006.
- [2] A. Honda, Y. Okazaki and H. Sato, An L_p -function determines ℓ_p , preprint.
- [3] L. A. Shepp, Distinguishing a sequence of random variables from a translate of itself, Ann. Math. Statist. 1965, 36, 1107–1112.