

ジェットスキームの有理2重点上の ファイバーの既約成分の配置

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Yoshimune Koreeda (Hiroshima university) ジェットスキームの有理2重点上のファイバー October 27 2021 1 / 12

Overview

Setting

S : surface with rational double point singularity at 0

$m > 0$

S_m : " m -th jet scheme"

$\pi_m : S_m \rightarrow S$: "truncation map"

The following one-to-one correspondence is known (Mourada) :

Irred. comps. of $\pi_m^{-1}(0) \leftrightarrow$ Exceptional curves in a minimal resolution

Problem

Can one reconstruct the resolution graph from the informations of the jet scheme?

Overview

Main Theorem

For A_n - or D_4 -type singular surfaces, the following conditions are equivalent:

- The intersection of two distinct irreducible components of $\pi_m^{-1}(0)$ is "maximal".
- The corresponding exceptional curves on the minimal resolution intersect.

Today

$\rightsquigarrow$ Mainly talk about the case of the D_4 -type singular surface.

Yoshimune Koreeda (Hiroshima university) ジェットスキームの有理2重点上のファイバー October 27 2021 3 / 12

Preparation : Definition of jet schemes

Setting

$\mathbf{Sch}/\mathbb{C}$: the category of schemes over $\mathbb{C}$

$\mathbf{Set}$: the category of sets

$X \in \mathbf{Sch}/\mathbb{C}$: a scheme of finite type over $\mathbb{C}$

For $m \in \mathbb{Z}_{\geq 0}$, the functor

$$F_m^X : \mathbf{Sch}/\mathbb{C} \rightarrow \mathbf{Set}; Z \mapsto \mathrm{Hom}_Z(Z \times \mathrm{Spec} \mathbb{C}[t]/\langle t^{m+1} \rangle, Z \times X)$$

is a representable functor, and is represented by a scheme X_m of finite type over $\mathbb{C}$.

Definition(Jet scheme)

The scheme X_m is called the m -th jet scheme of X .

Yoshimune Koreeda (Hiroshima university) ジェットスキームの有理2重点上のファイバー October 27 2021 4 / 12

Yoshimune Koreeda (Hiroshima university) ジェットスキームの有理2重点上のファイバー October 27 2021 4 / 12

Preparation : Calculation of jet schemes

$$X = (f(x, y, z) = 0) \subset \mathbb{A}^3$$

The $m(\in \mathbb{Z}_{\geq 0})$ -th jet schemes are calculated as follows:

Let $\mathbf{x} = x_0 + x_1 t + x_2 t^2 + \cdots + x_m t^m$, $\mathbf{y} = y_0 + y_1 t + y_2 t^2 + \cdots + y_m t^m$,
 $\mathbf{z} = z_0 + z_1 t + z_2 t^2 + \cdots + z_m t^m$.

Expand $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv f^{(0)} + f^{(1)}t + \cdots + f^{(m)}t^m \pmod{t^{m+1}}$$

$$(f^{(0)}, \dots, f^{(m)}) \in \mathbb{C}[x_0, \dots, x_m, y_0, \dots, y_m, z_0, \dots, z_m].$$

Then the m -th jet scheme X_m of X is defined by

$$\langle f^{(0)}, \dots, f^{(m)} \rangle$$

in $\mathbb{A}^{3(m+1)}$.

$\pi_m : X_m \rightarrow X$: truncation morphism (the map given by "t = 0")

Preparation

For $X = \mathbb{A}^3$, the m -th jet scheme is $X_m = \mathbb{A}^{3(m+1)}$ and the truncation morphism is as follows:

$$\pi_m : X_m \rightarrow X; (a_0, \dots, a_m, b_0, \dots, b_m, c_0, \dots, c_m) \mapsto (a_0, b_0, c_0).$$

Definition(Singular fiber)

The inverse image of the singular point $\pi_m^{-1}(0)$, denoted by X_m^0 , is called the singular fiber.

Singular fiber

$S = \mathbf{V}(f) \subset \mathbb{A}^3$: surface with an A_n - or D_4 -type singular point at 0
 The singular fiber S_m^0 is defined by the following ideal in $(\mathbb{A}^3)_m \cong \mathbb{A}^{3(m+1)}$:

$$\langle x_0, y_0, z_0, f^{(0)}, \dots, f^{(m)} \rangle.$$

The following theorem was proven by H. Mourtada.

Theorem (H. Mourtada [1, Theorem 3.1, Theorem 3.2])

Suppose $n > 0$ and $m \gg 0$. The singular fiber S_m^0 decomposes into n irreducible components and their codimensions are 1 in S_m .

We consider the following problem about irreducible components of the singular fiber.

Problem

Fix the degree m of jets. When is the intersection of two distinct irreducible components of the singular fiber "maximal"?

Main result1

① For A_n -type singular surfaces,

Theorem(Main theorem 1) [2, Theorem 2.10, Corollary 2.14]

Suppose $m \geq 2n + 2$ and $S_m^0 = Z_m^1 \cup Z_m^2 \cup \cdots \cup Z_m^n$. For $1 \leq i < j \leq n$, the number of irreducible components $Z_m^i \cap Z_m^j$ is $n - (j - i) + 2$ and the codimension of $Z_m^i \cap Z_m^j$ in S_m is 2.

Moreover

$$Z_m^i \cap Z_m^j \text{ is maximal in } \{Z_m^{l_1} \cap Z_m^{l_2} \mid 1 \leq l_1 < l_2 \leq n\} \Leftrightarrow j - i = 1.$$

Method : Concrete calculation using the defining ideals of Z_m^i

② For a D_4 -type singular surface,

↪ The generators of defining ideal of irred. components are not known.

↪ Find some jets explicitly and determine the inclusion relations.

The irreducible components for D_4 -singularity (Mourtada)

Let $f = x^2 - y^2z + z^3$, $S = \mathbb{V}(f)$, $m \geq 5$ and

$R_m = \mathbb{C}[x_0, \dots, x_m, y_0, \dots, y_m, z_0, \dots, z_m]$.

The defining ideals of the irreducible components of S_m^0 are as follows:

$$I_m^0 := \langle x_0, x_1, x_2, \quad y_0, y_1 \quad z_0, z_1, \quad f^{(0)}, \dots, f^{(m)} \rangle$$

$$J_m^1 := \langle x_0, x_1, \quad y_0, \quad z_0, \quad z_1, \quad f^{(0)}, \dots, f^{(m)} \rangle$$

$$J_m^2 := \langle x_0, x_1, \quad y_0, \quad z_0, \quad y_1 - z_1, \quad f^{(0)}, \dots, f^{(m)} \rangle$$

$$J_m^3 := \langle x_0, x_1, \quad y_0, \quad z_0, \quad y_1 + z_1, \quad f^{(0)}, \dots, f^{(m)} \rangle$$

$I_m^0 \rightsquigarrow$ the defining ideal of an irreducible component

$J_m^i \rightsquigarrow$ the defining ideals of irreducible components in $y_1 \neq 0$

$$I_m^i := J_m^i \cdot (R_m)_{y_1} \cap R_m \quad (i = 1, 2, 3),$$

then

$$Z_m^i = \mathbb{V}(I_m^i) \quad (i = 0, 1, 2, 3).$$

Main result2

Theorem(Main theorem 2) [2, Theorem 3.17]

Suppose $m \geq 5$ and Z_m^i 's are as above. Then $S_m^0 = Z_m^0 \cup Z_m^1 \cup Z_m^2 \cup Z_m^3$ and the maximal elements of $\{Z_m^{i_1} \cap Z_m^{i_2} \mid 0 \leq i_1 < i_2 \leq 3\}$ are

$$Z_m^0 \cap Z_m^1, Z_m^0 \cap Z_m^2, Z_m^0 \cap Z_m^3.$$

We outline the proof of $Z_m^0 \cap Z_m^1 \not\subseteq Z_m^0 \cap Z_m^2$ i.e.

$Z_m^0 \cap Z_m^1 \supsetneq Z_m^0 \cap Z_m^1 \cap Z_m^2$, in the case $m \geq 6$.

Consider the following two elements.

$$h_1 = -4y_2^2z_2^2 + y_1^2z_3^2 + 4x_3^2z_2 - 4x_2x_3z_3$$

$$h_2 = -y_2^4 - 4y_2^3z_2 + 2y_2^2z_2^2 + 12y_2z_2^3 - 9z_2^4 + 4y_3^2z_1^2 - 8y_3z_1^2z_3 \\ + 4z_1^2z_3^2 + 8x_3^2y_2 - 8x_3^2z_2 - 8x_2x_3y_3 + 8x_2x_3z_3.$$

We can prove $h_1 \in I_m^1$ and $h_2 \in I_m^2$, and using these we can prove

$$y_2 \in \sqrt{I_m^0 + I_m^1 + I_m^2}.$$

On the other hand, we have $(0, t^2, 0) \in Z_m^0 \cap Z_m^1$ and so $y_2 \notin \sqrt{I_m^0 + I_m^1}$.

Main result2

For $m \gg 0$, we construct the graph as follows:

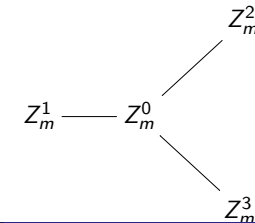
vertices : Irreducible components of S_m^0

edges : When two irreducible components are maximal in $\{Z_m^i \cap Z_m^j \mid 1 \leq i < j \leq n\}$.

Corollary [2, Corollary 2.16, Corollary 3.18]

For an A_n - or D_4 -type singularity, the graph constructed as above is isomorphic to the resolution graph of the minimal resolution of singularity.

In the D_4 case, the graph is as follows;



References

- [1] H. MOURTADA, Jet schemes of rational double point singularities, Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2014), 373–388.
- [2] Y. KOREEDA, On the configuration of the singular fibers of jet schemes of rational double points, arXiv:math/2012.08144, to appear in Communications in Algebra.