

## On the homomorphisms between scalar generalized Verma modules

松本久義 (Hisayosi Matumoto)  
 東京大学大学院数理科学研究科  
 Graduate School of Mathematical Sciences  
 University of Tokyo  
 3-8-1 Komaba, Tokyo  
 153-8914, JAPAN  
 e-mail: hisayosi@ms.u-tokyo.ac.jp

### § 0. Introduction

In this article, we consider the existence problem of homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [Boe 1985]). In [Matumoto 2003], we classified the homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras.

Here, we announce a classification of homomorphisms between scalar generalized Verma modules for ccertain non-maximal paraboplic subalgebras. The proof will appear elsewhere.

### § 1. Notations and Preliminaries

Let  $\mathfrak{g}$  be a complex reductive Lie algebra,  $U(\mathfrak{g})$  the universal enveloping algebra of  $\mathfrak{g}$ , and  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ . We denote by  $\Delta$  the root system with respect to  $(\mathfrak{g}, \mathfrak{h})$ . We fix some positive root system  $\Delta^+$  and let  $\Pi$  be the set of simple roots. Let  $W$  be the Weyl group of the pair  $(\mathfrak{g}, \mathfrak{h})$  and let  $\langle \cdot, \cdot \rangle$  be a non-degenerate invariant bilinear form on  $\mathfrak{g}$ . For  $w \in W$ , we denote by  $\ell(w)$  the length of  $w$  as usual. We also denote the inner product on  $\mathfrak{h}^*$  which is induced from the above form by the same symbols  $\langle \cdot, \cdot \rangle$ . For  $\alpha \in \Delta$ , we denote by  $s_\alpha$  the reflection in  $W$  with respect to  $\alpha$ . We denote by  $w_0$  the longest element of  $W$ . For  $\alpha \in \Delta$ , we define the coroot  $\check{\alpha}$  by  $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ , as usual. We call  $\lambda \in \mathfrak{h}^*$  dominant (resp. anti-dominant), if  $\langle \lambda, \check{\alpha} \rangle$  is not a negative (resp. positive) integer, for each  $\alpha \in \Delta^+$ . We call  $\lambda \in \mathfrak{h}^*$  regular, if  $\langle \lambda, \alpha \rangle \neq 0$ , for each  $\alpha \in \Delta$ . We denote by  $P$  the integral weight lattice, namely  $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ . If  $\lambda \in \mathfrak{h}^*$  is contained in  $P$ , we call  $\lambda$  an integral weight. We define  $\rho \in P$  by  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ . Put  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h} \ [H, X] = \alpha(H)X\}$ ,  $\mathfrak{u} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ ,  $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$ . Then  $\mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{g}$ . We denote by  $Q$  the root lattice, namely  $\mathbb{Z}$ -linear span of  $\Delta$ . We

also denote by  $Q^+$  the linear combination of  $\Pi$  with non-negative integral coefficients. For  $\lambda \in \mathfrak{h}^*$ , we denote by  $W_\lambda$  the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q\}.$$

We denote by  $\Delta_\lambda$  the set of integral roots.

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}\}.$$

It is well-known that  $W_\lambda$  is the Weyl group for  $\Delta_\lambda$ . We put  $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$ . This is a positive system of  $\Delta_\lambda$ . We denote by  $\Pi_\lambda$  the set of simple roots for  $\Delta_\lambda^+$  and denote by  $\Phi_\lambda$  the set of reflection corresponding to the elements in  $\Pi_\lambda$ . So,  $(W_\lambda, \Phi_\lambda)$  is a Coxeter system. We denote by  $Q_\lambda$  the integral root lattice, namely  $Q_\lambda = \mathbb{Z}\Delta_\lambda^+$  and put  $Q_\lambda^+ = N\Pi_\lambda$ .

Next, we fix notations for a parabolic subalgebra (which contains  $\mathfrak{b}$ ). Hereafter, through this article we fix an arbitrary subset  $\Theta$  of  $\Pi$ . Let  $\bar{\Theta}$  be the set of the elements of  $\Delta$  which are written by linear combinations of elements of  $\Theta$  over  $\mathbb{Z}$ . Put  $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall \alpha \in \Theta \ \alpha(H) = 0\}$ ,  $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha \in \bar{\Theta}} \mathfrak{g}_\alpha$ ,  $\mathfrak{n}_\Theta = \sum_{\alpha \in \Delta^+ \setminus \bar{\Theta}} \mathfrak{g}_\alpha$ ,  $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$ . Then  $\mathfrak{p}_\Theta$  is a parabolic subalgebra of  $\mathfrak{g}$  which contains  $\mathfrak{b}$ . Conversely, for an arbitrary parabolic subalgebra  $\mathfrak{p} \supseteq \mathfrak{b}$ , there exists some  $\Theta \subseteq \Pi$  such that  $\mathfrak{p} = \mathfrak{p}_\Theta$ . We denote by  $W_\Theta$  the Weyl group for  $(\mathfrak{l}_\Theta, \mathfrak{h})$ .  $W_\Theta$  is identified with a subgroup of  $W$  generated by  $\{s_\alpha \mid \alpha \in \Theta\}$ . We denote by  $w_\Theta$  the longest element of  $W_\Theta$ . Using the invariant non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$ , we regard  $\mathfrak{a}_\Theta^*$  as a subspace of  $\mathfrak{h}^*$ . It is known that there is a unique nilpotent (adjoint) orbit (say  $\mathcal{O}_{\mathfrak{p}_\Theta}$ ) whose intersection with  $\mathfrak{n}_\Theta$  is Zariski dense in  $\mathfrak{n}_\Theta$ .  $\mathcal{O}_{\mathfrak{p}_\Theta}$  is called the Richardson orbit with respect to  $\mathfrak{p}_\Theta$ . We denote by  $\bar{\mathcal{O}}_{\mathfrak{p}_\Theta}$  the closure of  $\mathcal{O}_{\mathfrak{p}_\Theta}$  in  $\mathfrak{g}$ . Put  $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta \rho)$  and  $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta \rho)$ . Then,  $\rho^\Theta \in \mathfrak{a}_\Theta^*$ .

Define

$$\begin{aligned} P_\Theta^{++} &= \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \ \langle \lambda, \check{\alpha} \rangle \in \{1, 2, \dots\}\} \\ {}^\circ P_\Theta^{++} &= \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \ \langle \lambda, \check{\alpha} \rangle = 1\} \end{aligned}$$

We easily have

$${}^\circ P_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For  $\mu \in \mathfrak{h}^*$  such that  $\mu + \rho \in P_\Theta^{++}$ , we denote by  $\sigma_\Theta(\mu)$  the irreducible finite-dimensional  $\mathfrak{l}_\Theta$ -representation whose highest weight is  $\mu$ . Let  $E_\Theta(\mu)$  be the representation space of  $\sigma_\Theta(\mu)$ . We define a left action of  $\mathfrak{n}_\Theta$  on  $E_\Theta(\mu)$  by  $X \cdot v = 0$  for all  $X \in \mathfrak{n}_\Theta$  and  $v \in E_\Theta(\mu)$ . So, we regard  $E_\Theta(\mu)$  as a  $U(\mathfrak{p}_\Theta)$ -module.

For  $\mu \in P_\Theta^{++}$ , we define a generalized Verma module ([Lepowsky 1977]) as follows.

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all  $\lambda \in \mathfrak{h}^*$ , we write  $M(\lambda) = M_\emptyset(\lambda)$ .  $M(\lambda)$  is called a Verma module. For  $\mu \in P_\Theta^{++}$ ,  $M_\Theta(\mu)$  is a quotient module of  $M(\mu)$ . Let  $L(\mu)$  be the unique highest weight  $U(\mathfrak{g})$ -module with the highest weight  $\mu - \rho$ . Namely,  $L(\mu)$  is a unique irreducible quotient of  $M(\mu)$ . For  $\mu \in P_\Theta^{++}$ , the canonical projection of  $M(\mu)$  to  $L(\mu)$  is factored by  $M_\Theta(\mu)$ .

$\dim E_\Theta(\mu - \rho) = 1$  if and only if  $\mu \in {}^\circ P_\Theta^{++}$ . If  $\mu \in {}^\circ P_\Theta^{++}$ , we call  $M_\Theta(\mu)$  a scalar generalized Verma module.

## § 2. Reductions of the problem

We retain the notation of §1. In particular,  $\Theta$  is a subset of  $\Pi$ .

### 2.1 Basic results of Lepowsky

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

**Theorem 2.1.1.** ([Lepowsky 1976])

Let  $\mu, \nu \in {}^\circ P_{\Theta}^{++}$ .

- (1)  $\dim \operatorname{Hom}_{U(\mathfrak{g})}(M_{\Theta}(\mu), M_{\Theta}(\nu)) \leq 1$ .
- (2) Any non-zero homomorphism of  $M_{\Theta}(\mu)$  to  $M_{\Theta}(\nu)$  is injective.

Hence, the existence problem of homomorphisms between scalar generalized Verma modules is reduce to the following problem.

**Problem 1** Let  $\mu, \nu \in {}^\circ P_{\Theta}^{++}$ . When is  $M_{\Theta}(\mu) \subseteq M_{\Theta}(\nu)$  ?

### 2.2 Reduction to the integral infinitesimal character setting

Since the both  $\nu \in W\mu$  and  $\nu - \mu \in Q^+$  are necessary condition for the above problem, we can reformulate our problem as follows.

**Problem 2** Let  $\lambda \in {}^\circ P_{\Theta}^{++}$  be dominant. Let  $x, y \in W_{\lambda}$  be such that  $x\lambda, y\lambda \in {}^\circ P_{\Theta}^{++}$ . When is  $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$  ?

We fix  $\lambda \in {}^\circ P_{\Theta}^{++}$ . Then, we can construct a suralgebra  $\mathfrak{g}'$  of  $\mathfrak{h}$  such that the corresponding Coxeter system is  $(W_{\lambda}, \Phi_{\lambda})$ . Since  $\Theta \subseteq \Pi_{\lambda}$  holds, we can construct corresponding parabolic subalgebra  $\mathfrak{p}'_{\Theta}$  of  $\mathfrak{g}'$ . For  $\mu \in P_{\Theta}^{++}$ , we denote by  $M'_{\Theta}(\mu)$  the corresponding generalized Verma module of  $\mathfrak{g}'$ . We consider the category  $\mathcal{O}$  in the sense of [Bernstein-Gelfand-Gelfand 1976] corresponding to our particular choice of positive root system. More precisely, we denote by  $\mathcal{O}$  (respectively  $\mathcal{O}'$ ) "the category  $\mathcal{O}$ " for  $\mathfrak{g}$  (respectively  $\mathfrak{g}'$ ). We denote by  $\mathcal{O}_{\lambda}$  (respectively,  $\mathcal{O}'_{\lambda}$ ) the full subcategory of  $\mathcal{O}$  (respectively  $\mathcal{O}'$ ) consisting of the objects with a generalized infinitesimal character  $\lambda$ . Soergel's celebrated theorem ([Soergel 1990] Theorem 11) says that there is a Category equivalence between  $\mathcal{O}_{\lambda}$  and  $\mathcal{O}'_{\lambda}$ . Under the equivalence a Verma module  $M(x\lambda)$  ( $x \in W_{\lambda}$ ) corresponds to  $M'(x\lambda)$ . From Lepowsky's generalized BGG resolutions of the generalized Verma modules and their rigidity, we easily see  $M_{\Theta}(x\lambda)$  corresponds to  $M'_{\Theta}(x\lambda)$  under Soergel's category equivalence. So, we have the following lemma as a corollary of Soergel's theorem.

**Lemma 2.2.1.** Let  $\lambda \in \mathfrak{h}^*$  be dominant. Let  $x, y \in W_{\lambda}$  be such that  $x\lambda, y\lambda \in {}^\circ P_{\Theta}^{++}$ . Then, the following two conditions are equivalent.

- (1)  $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$ .
- (2)  $M'_{\Theta}(x\lambda) \subseteq M'_{\Theta}(y\lambda)$ .

This lemma tells us that we may reduce Problem 2 to the case that  $\lambda$  is integral.

We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$

### § 3. Excellent parabolic subalgebras

#### 3.1 $\theta$ -acceable positive roots

Hereafter, we fix a subset  $\Theta$  of  $\Pi$ . For  $\alpha \in \Delta$ , we put

$$\Delta(\alpha) = \{\beta \in \Delta \mid \exists c \in \mathbb{R} \ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\},$$

$$\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+,$$

$$U_\alpha = \mathbb{C}S + \mathbb{C}\alpha \subseteq \mathfrak{h}^*.$$

Then  $(U_\alpha, \Delta(\alpha), \langle \cdot, \cdot \rangle)$  is a subroot system of  $(\mathfrak{h}^*, \Delta, \langle \cdot, \cdot \rangle)$ . The set of simple roots for  $\Delta^+(\alpha)$  is denoted by  $\Pi(\alpha)$ . If  $\alpha|_{\mathfrak{a}_\Theta} = 0$ , then  $S = \Pi(\alpha)$ . If  $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ , then  $\Pi(\alpha)$  is written as  $S \cup \{\tilde{\alpha}\}$ . If  $\alpha \in \Delta$  satisfies  $\alpha|_{\mathfrak{a}_\Theta} \neq 0$  and  $\alpha = \tilde{\alpha}$ , then we call  $\alpha$   $\Theta$ -reduced. For  $\alpha \in \Delta^+$ , we denote by  $W_\Theta(\alpha)$  the Weyl group of  $(\mathfrak{h}^*, \Delta(\alpha))$ . Clearly,  $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$ . We denote by  $w^\alpha$  the longest element of  $W_\Theta(\alpha)$ . We call  $\alpha \in \Delta$   $\Theta$ -acceptable iff  $w^\alpha w_\Theta = w_\Theta w^\alpha$ . We denote by  $\Delta^\Theta$  (resp.  $\Delta_r^\Theta$ ) the set of  $\Theta$ -acceptable roots (resp.  $\Theta$ -reduced  $\Theta$ -acceptable roots). Put  $(\Delta^\Theta)^+ = \Delta^+ \cap \Delta^\Theta$  and  $(\Delta_r^\Theta)^+ = \Delta^+ \cap \Delta_r^\Theta$ . For  $\alpha \in \Delta^\Theta$ , we define

$$\sigma_\alpha = w^\alpha w_\Theta = w_\Theta w^\alpha.$$

Clearly,  $\sigma_\alpha^2 = 1$ ,  $\sigma_\alpha = \sigma_{\tilde{\alpha}}$ . If  $\alpha|_{\mathfrak{a}_\Theta} = 0$ , then  $\sigma_\alpha = 1$ . If  $\alpha \in \Delta$  is orthogonal to all the elements in  $\Theta$ , then we can easily see  $\alpha$  is  $\Theta$ -reduced and  $s_\alpha = \sigma_\alpha$ . For  $\alpha \in \Delta$ , we put

$$V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha \rangle = 0\}.$$

For  $\alpha \in \Delta^\Theta$ , we put  $\hat{\alpha} = \tilde{\alpha}|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^*$ . We can easily see:

**Lemma 3.1.1.** *Let  $\alpha \in \Delta_r^\Theta$ . Then, we have*

- (1)  $\sigma_\alpha$  preserves  $\mathfrak{a}_\Theta^*$ .
- (2)  $\sigma_\alpha \in W(\Theta)$ . In particular,  $\sigma_\alpha \rho_\Theta = \rho_\Theta$ .
- (3)  $\sigma_\alpha \hat{\alpha} = -\hat{\alpha}$ .
- (4)  $\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$  is the reflection with respect to  $V_\alpha$ .

#### 3.2 Excellent parabolic subalgebras

We retain the notations in the previous section.

Let  $\Theta \subseteq \Pi$ .

A parabolic subalgebra  $\mathfrak{p}_\Theta$  is called excellent, if all the roots are  $\Theta$ -acceptable.

**Remark** If  $\mathfrak{p}_\Theta$  is a complexified minimal parabolic subalgebra of a real form of  $\mathfrak{g}$  such that the  $m$ -part of the Langlands decomposition of  $\mathfrak{p}_\Theta$  is semisimple, then all the roots are  $\Theta$ -acceptable and  $\sigma_\alpha$  is a reflection with respect to a restricted root  $\hat{\alpha}$  for each  $\alpha \in \Delta_r^\Theta$ .

If  $\mathfrak{g}$  is a classical algebra, we can classify excellent parabolic subalgebras of  $\mathfrak{g}$  as follows.

(1) Let  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$  (the case of  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$  is similar) and let  $k$  be a positive integer dividing  $n$ . We consider the following parabolic subalgebras.

$\mathfrak{p}(A_{n-1,k})$  : a parabolic subalgebra of  $\mathfrak{g}$  whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{n/k}.$$

Then,  $\mathfrak{p}(A_{n-1,k})$  is excellent. Conversely any excellent parabolic subalgebra is conjugate to  $A_{n,k}$  for some  $k$ .

(2) Let  $\mathfrak{g}$  be a complex simple Lie algebra of the type  $X_n$ . Here,  $X$  means one of  $B$ ,  $C$ , and  $D$ . Let  $k$  and  $\ell$  be positive integers such that  $k$  divides  $n - \ell$ .

We consider the following parabolic subalgebras.

$\mathfrak{p}(X_{n,k,\ell})$  : a parabolic subalgebra of  $\mathfrak{g}$  whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{(n-\ell)/k} \oplus X_\ell.$$

Here,  $X_\ell$  means that the complex simple Lie algebra of the type  $X_\ell$ .  $X_0$  means the zero Lie algebra.

$\mathfrak{p}(X_{n,k,\ell})$  is excellent unless  $X = D$ ,  $\ell = 0$ , and  $k$  is an odd number greater than 1. Any excellent parabolic subalgebra is conjugate to one of such  $\mathfrak{p}(X_{n,k,\ell})$ s.

## § 4. Main result

### 4.1 Elementary homomorphisms

Here, we review some notion in [Matumoto 1993] §3. Hereafter,  $\mathfrak{g}$  means a reductive Lie algebra over  $\mathbb{C}$  and retain the notations in §1-3. We fix a subset  $\Theta$  of  $\Pi$  and  $\alpha \in \Delta$ .

We denote by  $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$  the fundamental weight for  $\alpha$  with respect to the basis  $\Pi(\alpha) = \Theta \cup \{\alpha\}$ . Namely  $\omega_\alpha$  satisfies that  $\langle \omega_\alpha, \beta \rangle = 0$  for  $\beta \in \Theta$ ,  $\langle \beta, \check{\alpha} \rangle = 1$ , and  $\omega_\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$ . Here,  $\mathfrak{c}(\mathfrak{g}(\alpha))$  is the center of  $\mathfrak{g}(\alpha)$ . We see that there is some positive real number  $a$  such that  $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta}$ , since  $\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$ . Hence, we have  $V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \omega_\alpha \rangle = 0\}$ .

For  $\alpha \in (\Delta_r^\Theta)^+$ , we define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta \in \Delta(\alpha)} \mathfrak{g}_\beta, \quad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then,  $\mathfrak{g}(\alpha)$  is a reductive Lie subalgebra of  $\mathfrak{g}$  whose root system is  $\Delta(\alpha)$  and  $\mathfrak{p}_\Theta(\alpha)$  is a maximal parabolic subalgebra of  $\mathfrak{g}(\alpha)$ .

Put  $\rho(\alpha) = \frac{1}{2} \sum_{\beta \in \Delta^+(\alpha)} \beta$ . For  $\nu \in \mathfrak{a}_\Theta^*$ , we denote by  $\mathbb{C}_\nu$  the one-dimensional  $U(\mathfrak{p}_\Theta(\alpha))$ -module corresponding to  $\nu$ . For  $\nu \in \mathfrak{a}_\Theta^*$  we define a generalized Verma module for  $\mathfrak{g}(\alpha)$  as follows.

$$M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu - \rho(\alpha)}.$$

Then, we have:

**Theorem 4.1.1.** ([Matumoto 2003]) Let  $\nu$  be an arbitrary element in  $V_\alpha$  and let  $c$  be either 1 or  $\frac{1}{2}$ . Assume that  $M_\Theta^{g(\alpha)}(\rho_\Theta - nc\omega_\alpha) \subseteq M_\Theta^{g(\alpha)}(\rho_\Theta + nc\omega_\alpha)$  for all  $n \in \mathbb{N}$ . Then, we have  $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$  for all  $n \in \mathbb{N}$ .

We call the above homomorphism of  $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha)$  into  $M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$  an elementary homomorphism.

The following working hypothesis is proposed in [Matumoto 2003].

**Working Hypothesis** An arbitrary nontrivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.

The working hypothesis in the case of the Verma modules is nothing but the result of Bernstein-Gelfand-Gelfand.

I would like to propose :

**Conjecture** For an excellent parabolic subalgebra, the above working hypothesis is affirmative.

## 4.2 Main result

Now, we state our main result.

**Theorem 4.2.1.** Let  $\mathfrak{g}$  be a classical Lie algebra and let  $\mathfrak{p}_\Theta$  be one of the following cases.

- (a)  $\mathfrak{p}(A_{n-1,k}) \quad (k|n),$
- (b)  $\mathfrak{p}(B_{n,2k,m}) \quad (k \leq m),$
- (c)  $\mathfrak{p}(B_{n,2k+1,m}) \quad (k \geq m),$
- (d)  $\mathfrak{p}(C_{n,2k,m}) \quad (k \leq m),$
- (e)  $\mathfrak{p}(C_{n,2k+1,m}) \quad (k \geq m),$
- (f)  $\mathfrak{p}(D_{n,2k-1,m}) \quad (k \leq m),$
- (g)  $\mathfrak{p}(D_{n,2k,m}) \quad (k \geq m).$

Then, any homomorphism between scalar generalized Verma modules with integral infinitesimal characters is a composition of elementary homomorphisms.

## References

- J. Bernstein, I. M. Gelfand, and S. I. Gelfand, Structure of representations generated by vectors of highest weight, *Funct. Anal. Appl.* **5** (1971), 1-8.
- B. Boe, Homomorphism between generalized Verma modules, *Trans. Amer. Math. Soc.* **288** (1985), 791-799.
- J. Lepowsky, Conical vectors in induced modules, *Trans. Amer. Math. Soc.* **208** (1975), 219-272.
- J. Lepowsky, Existence of conical vectors in induced modules, *Ann. of Math.* **102** (1975), 17-40.
- H. Matumoto, On the existence of homomorphisms between scalar generalized Verma modules, in: *Contemporary Mathematics*, 145, 259-274, Amer. Math. Soc., Providence, RI, 1993.

H. Matumoto, The homomorphisms between scalar generalized Verma modules associated to maximal parabolic subalgebras, preprint 2003, arXiv math.RT/0309454.

W. Soergel, Kategorie  $\mathcal{O}$ , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, J. Amer. Math. Soc. **3** (1990), 421–445.