

Literal shuffle on ω -languages

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Abstract

We consider literal shuffle on ω -languages. First, we show that a duo (a family of ω -languages closed under ϵ -free morphisms and inverse ϵ -free morphisms) is closed under literal shuffle if and only if it is closed under intersection. Next we investigate the closure properties of some classes of the ω -regular languages under literal shuffle and shuffle. Last the relation between literal shuffle and shuffle are presented.

Key words: shuffle; literal shuffle; ω -regular language; duo

1 Introduction

The literal shuffle operation is introduced in [1] as a more constrained form of the shuffle operation. It models the synchronous behavior while the shuffle corresponds asynchronous one.

In this paper, we study literal shuffle on ω -languages. In section 2 basic definitions and notations are given. In section 3 we prove that a duo (a family of ω -languages closed under ϵ -free morphisms and inverse ϵ -free morphisms) is closed under literal shuffle if and only if it is closed under intersection. For languages of finite words, an analogous result for shuffle has been given: a trio is closed under shuffle if and only if it is closed under intersection [2]. We also investigate the closure properties of some subclasses of the class of ω -regular languages under literal shuffle and shuffle. Last we consider the relation between literal shuffle and shuffle.

2 Preliminaries

Let Σ be an alphabet. Σ^* denotes the set of all finite words over Σ , and Σ^ω denotes the set of all ω -words over Σ , i.e., the set of all mappings $\alpha : \{0, 1, 2, \dots\} \rightarrow \Sigma$. An ω -word is written by $\alpha = a_0a_1 \dots$ where $a_n = \alpha(n)$, ($n = 0, 1, \dots$). We call a subset of Σ^* (Σ^ω , resp.) a language (ω -language) over Σ .

A *deterministic finite automaton* (DA, for short) A over Σ is a 5-tuple $A = \langle S, \Sigma, \delta, s_0, F \rangle$, where S is a finite set of states, Σ is an alphabet, $\delta : S \times \Sigma \rightarrow S$ is a next state function, $s_0 \in S$ is an initial state, and $F \subseteq S$ is a set of accepting states.

The *run* $Run(A, \alpha)$ of a DA A on an ω -word α is an ω -word $\gamma \in S^\omega$ such

that $\gamma(0) = s_0$ and $\gamma(n+1) = \delta(\gamma(n), \alpha(n))$, ($n = 0, 1, \dots$). For a run γ of A , let

$$Ex(\gamma) = \{q \in S \mid q = \gamma(n) \text{ for some } n\},$$

$$Inf(\gamma) = \{q \in S \mid q = \gamma(n) \text{ for infinitely many } n\},$$

and define the following six types of acceptance of the DA A ,

$$E(A) = \{\alpha \mid Ex(Run(A, \alpha)) \cap F \neq \emptyset\},$$

$$E'(A) = \{\alpha \mid Ex(Run(A, \alpha)) \subseteq F\},$$

$$I(A) = \{\alpha \mid Inf(Run(A, \alpha)) \cap F \neq \emptyset\},$$

$$I'(A) = \{\alpha \mid Inf(Run(A, \alpha)) \subseteq F\},$$

$$L(A) = \{\alpha \mid F \subseteq Inf(Run(A, \alpha))\},$$

$$L'(A) = \{\alpha \mid F \not\subseteq Inf(Run(A, \alpha))\}.$$

The class of ω -languages of the form $E(A)$ ($E'(A)$, $I(A)$, $I'(A)$, $L(A)$, $L'(A)$, resp.) for some automaton A over Σ is denoted by $\mathbf{E}_\Sigma$ ($\mathbf{E}'_\Sigma$, $\mathbf{I}_\Sigma$, $\mathbf{I}'_\Sigma$, $\mathbf{L}_\Sigma$, $\mathbf{L}'_\Sigma$). All these classes are included in the class $\mathbf{R}_\Sigma$ of ω -regular languages over Σ (For the definition of ω -regular languages and the inclusion relations among these classes, see [6, 8, 9]).

Moreover, provided that a class $\mathbf{C}_\Sigma$ of ω -languages over Σ is defined for each alphabet Σ , we use the notation $(\mathbf{C}_\Sigma)$ for the family of all the classes $\mathbf{C}_\Sigma$. Note that a morphism $h : \Sigma^* \rightarrow \Delta^*$ can be extended to the mapping from Σ^ω to Δ^ω in the usual way. We say that a family $(\mathbf{C}_\Sigma)$ is closed under a morphism h if $h(X) = \{h(x) \mid x \in X\} \in \mathbf{C}_\Delta$ for any $X \in \mathbf{C}_\Sigma$, and $(\mathbf{C}_\Sigma)$ is closed under an inverse morphism h^{-1} if $h^{-1}(Y) = \{x \mid h(x) \in Y\} \in \mathbf{C}_\Sigma$ for any $Y \in \mathbf{C}_\Delta$.

A family (C_Σ) closed under ϵ -free morphisms and ϵ -free inverse morphisms is called a *duo*. Only three families (R_Σ) , (I'_Σ) and (E'_Σ) are duos among those mentioned above [5, 10].

For $\alpha, \beta \in \Sigma^\omega$, the shuffle $Sh(\alpha, \beta)$ and the literal shuffle $LSh(\alpha, \beta)$ are defined by

$$Sh(\alpha, \beta) = \{u_0 v_0 u_1 v_1 \dots \mid u_0 u_1 \dots = \alpha, v_0 v_1 \dots = \beta, u_0 \in \Sigma^*, u_i, v_i \in \Sigma^+\}$$

and

$$LSh(\alpha, \beta) = \alpha(0)\beta(0)\alpha(1)\beta(1)\dots$$

Moreover, we define for $X, Y \subseteq \Sigma^\omega$, $Sh(X, Y) = \cup \{Sh(\alpha, \beta) \mid \alpha \in X, \beta \in Y\}$ and $LSh(X, Y) = \{LSh(\alpha, \beta) \mid \alpha \in X, \beta \in Y\}$.

Then the following properties are easily obtained.

Lemma 1 For any $X, Y \subseteq \Sigma^\omega$,

1. $LSh(X, Y) = LSh(X, \Sigma^\omega) \cap LSh(\Sigma^\omega, Y)$.
2. $X \cap Y = h^{-1}(LSh(X, Y))$, where $h : \Sigma^\omega \rightarrow \Sigma^\omega$ is a morphism defined by $h(a) = aa$ for any $a \in \Sigma$.
3. $LSh(X^c, \Sigma^\omega) = LSh(X, \Sigma^\omega)^c$ and $LSh(\Sigma^\omega, X^c) = LSh(\Sigma^\omega, X)^c$, where $X^c = \Sigma^\omega - X$.

3 Closure properties under literal shuffle and shuffle

In this section we give a necessary and sufficient condition for a duo to be closed under literal shuffle, and investigate the closure properties for some subclasses of the ω -regular languages under literal shuffle and shuffle.

We say that a class C_Σ is closed under shuffle (literal shuffle, resp.) if $Sh(X, Y)$ ($LSh(X, Y)$) $\in C_\Sigma$ for any $X, Y \in C_\Sigma$. For a class C_Σ of ω -languages, we define $C_\Sigma^c = \{X^c \mid X \in C_\Sigma\}$. We note that $E'_\Sigma = E_\Sigma^c$, $L'_\Sigma = L_\Sigma^c$, and $I'_\Sigma = I_\Sigma^c$.

Lemma 2 *If (C_Σ) or (C_Σ^c) is a duo, then $LSh(X, \Sigma^\omega), LSh(\Sigma^\omega, X) \in C_\Sigma$ for any $X \in C_\Sigma$.*

Proof. Let $\Sigma' = \{\sigma' \mid \sigma \in \Sigma\}$ and $\# \notin \Sigma$. We define ϵ -free morphisms

$$h_1 : \Sigma^\omega \rightarrow (\Sigma \cup \{\#\})^\omega \text{ defined by } h_1(a) = a'\# \text{ for any } a \in \Sigma,$$

$$h_2 : \Sigma^\omega \rightarrow (\Sigma \cup \{\#\})^\omega \text{ defined by } h_2(a) = \#a' \text{ for any } a \in \Sigma,$$

$$g : (\Sigma \cup \Sigma')^\omega \rightarrow (\Sigma \cup \{\#\})^\omega \text{ defined by } g(a) = \# \text{ and } g(a') = a' \text{ for any } a \in \Sigma,$$

$$f : (\Sigma \cup \Sigma')^\omega \rightarrow \Sigma^\omega \text{ defined by } f(a) = f(a') = a \text{ for any } a \in \Sigma.$$

Then it is obvious that for any $X \in C_\Sigma$, $LSh(X, \Sigma^\omega) = f(g^{-1}(h_1(X)))$ and $LSh(\Sigma^\omega, X) = f(g^{-1}(h_2(X)))$. Hence, we have shown the lemma if (C_Σ) is a duo. If (C_Σ^c) is a duo, we can prove the lemma using Lemma 1.3. $\square$

Theorem 3 *Assume that (C_Σ) or (C_Σ^c) is a duo. Then C_Σ is closed under literal shuffle if and only if it is closed under intersection.*

Proof. If part is a direct consequence of Lemma 2 and Lemma 1.1.

Only if part is obtained directly from Lemma 1.2 and the observation that $U - h^{-1}(X) = h^{-1}(V - X)$ for any mapping $h : U \rightarrow V$ and $X \subseteq V$. $\square$

From this theorem, we have the following, since R_Σ , I_Σ , I'_Σ , E_Σ and E'_Σ are closed under intersection [8, 9].

Theorem 4 For any alphabet Σ , R_Σ , I_Σ , I'_Σ , E_Σ and E'_Σ are closed under literal shuffle.

Next, we show that L_Σ and L'_Σ are not closed under literal shuffle.

Theorem 5 L_Σ is not closed under literal shuffle, provided that Σ has at least two elements.

Proof. Let $X = LSh(L(A), L(A))$, where $A = \langle \{q_0, q_1, q_2, q_3\}, \{a, b\}, \delta, q_0, \{q_0\} \rangle$ is the automaton described in Fig. 1. Suppose that $X = L(M)$ with $M = \langle S, \Sigma, \tau, s_0, F \rangle$. It is obvious that $baba^\omega = LSh(bba^\omega, a^\omega) \in X$ and $b^\omega = LSh(b^\omega, b^\omega) \in X$. Since $F \subseteq Inf(Run(M, baba^\omega)) \cap Inf(Run(M, b^\omega))$, $\tau(s_0, baba^n) = \tau(s_0, b^m) \in F$ for some $n > 1$ and m because it is obvious that $F \neq \emptyset$. It means that $baba^n b^\omega \in X$. But, for any k , $baba^{2k+1} b^\omega = LSh(bba^k b^\omega, a^{k+2} b^\omega) \notin X$ and $baba^{2k+2} b^\omega = LSh(bba^{k+1} b^\omega, a^{k+2} b^\omega) \notin X$ since $\delta(q_0, bba^k) \neq \delta(q_0, a^{k+2})$ and $\delta(q_0, bba^{k+1}) \neq \delta(q_0, a^{k+2})$. $\square$

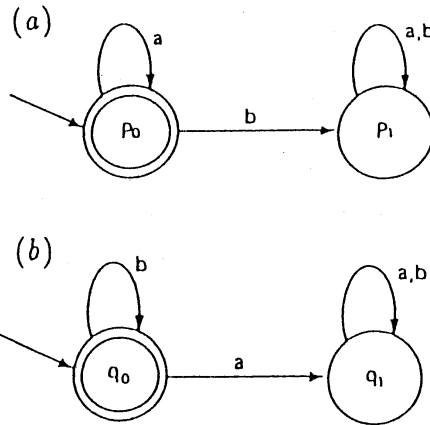
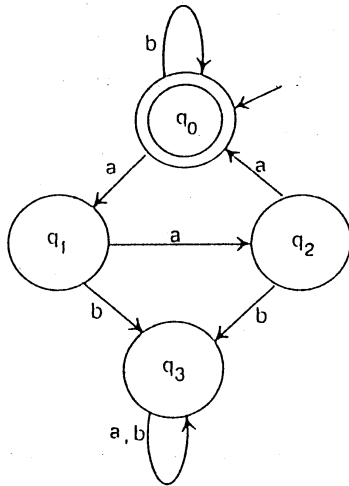


Fig.1. DA A in Theorem 5 Fig.2. (a) DA A_1 (b) DA A_2 in Theorem 6

Theorem 6 L'_Σ is not closed under literal shuffle, provided that Σ has at least two elements.

Proof. Let $\Sigma = \{a, b\}$, $X_1 = L'(A_1)$ and $X_2 = L'(A_2)$ where A_1 and A_2 are defined in Fig.2. Suppose that $X = LSh(X_1, X_2) = L'(A)$ with $A = \langle Q, \{a, b\}, \delta, s_0, F \rangle$.

Since a^ω and b^ω are not in X , they are in $L(A)$. It means that for some $n \geq 2$ and m , $\delta(s_0, a^n) = \delta(s_0, b^m) \in F$ and $a^n b^\omega \in L(A)$ because it is obvious that $F \neq \emptyset$. It contradicts the fact that $a^2 a^* b^\omega \subseteq LSh(aa^* b^\omega, aa^* b^\omega) \subseteq X$. $\square$

We also study the closure properties of ω -regular languages under shuffle.

Theorem 7 $E'_\Sigma, L_\Sigma, L'_\Sigma, I_\Sigma$ and I'_Σ are not closed under shuffle, provided that Σ has at least two elements.

Proof. Let $\Sigma = \{a, b\}$, and take ω -languages $a^\omega \in E'_\Sigma \cap L_\Sigma$, $b^\omega \cup b^* a^\omega \in E'_\Sigma$, and $b^* a^\omega \in L_\Sigma$. It is shown in [8, 9] that $Sh(a^\omega, b^* a^\omega) = \Sigma^* a^\omega \notin I_\Sigma$ and $Sh(a^\omega, b^\omega \cup b^* a^\omega) = (b^* a)^\omega \notin I'_\Sigma$. This completes the proof, since $E'_\Sigma \subseteq L'_\Sigma \subseteq I'_\Sigma$ and $L_\Sigma \subseteq I_\Sigma$. $\square$

Theorem 8 E_Σ and R_Σ are closed under shuffle.

Proof. Note that any $X \in E_\Sigma$ can be written as $X = R\Sigma^\omega$ for some regular language $R \subseteq \Sigma^*$ [8, 9], and $Sh(R_1\Sigma^\omega, R_2\Sigma^\omega) = Sh(R_1\Sigma^*, R_2\Sigma^*)\Sigma^\omega$. Thus the theorem for E_Σ is obtained from the fact that the class of regular languages are closed under shuffle.

It is proved in [7] that R_Σ is closed under shuffle. $\square$

The closure properties proved in this section are summarized in the following table.

	E_Σ	E'_Σ	I_Σ	I'_Σ	L_Σ	L'_Σ	R_Σ
Lsh	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\times$	$\times$	$\bigcirc$
Sh	$\bigcirc$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$

As shown in the above table, the closure results for E'_Σ , I_Σ and I'_Σ are different between shuffle and literal shuffle. We consider the relation between shuffle and literal shuffle. More precisely, we show that literal shuffle is represented by shuffle through ϵ -free morphisms and ϵ -free inverse morphisms. On the other hand, shuffle is represented by literal shuffle through ϵ -free morphisms and inverse morphisms (not necessarily ϵ -free).

Proposition 9 *Let $\Sigma' = \{a' \mid a \in \Sigma\}$ and define the ϵ -free morphisms*

$$h_1, h_2 : \Sigma^\omega \rightarrow (\Sigma \cup \Sigma')^\omega \text{ by } h_1(a) = a \text{ and } h_2(a) = a' \text{ for any } a \in \Sigma,$$

$$g : (\Sigma \times \Sigma')^\omega \rightarrow (\Sigma \cup \Sigma')^\omega \text{ by } g(\langle a, b' \rangle) = ab' \text{ for any } \langle a, b' \rangle \in (\Sigma \times \Sigma'),$$

$$f : (\Sigma \times \Sigma')^\omega \rightarrow \Sigma^\omega \text{ by } f(\langle a, b' \rangle) = ab \text{ for any } \langle a, b' \rangle \in (\Sigma \times \Sigma').$$

$$\text{Then for any } X, Y \subseteq \Sigma^\omega, LSh(X, Y) = f(g^{-1}(Sh(h_1(X), h_2(Y))))$$

Proof. It is immediate from the definition of the morphisms h_1, h_2, g and f . $\square$

Proposition 10 *Let $\# \notin \Sigma$ and define the morphisms*

$$h : (\Sigma \cup \{\#\})^\omega \rightarrow \Sigma^\omega \text{ by } h(a) = a \text{ and } h(\#) = \epsilon,$$

$$g : (\Sigma \times \Sigma \cup \Sigma \times \{\#\} \cup \{\#\} \times \Sigma)^\omega \rightarrow (\Sigma \cup \{\#\})^\omega \text{ by } g(\langle a, b \rangle) = ab \text{ for any } \langle a, b \rangle \in (\Sigma \times \Sigma \cup \Sigma \times \{\#\} \cup \{\#\} \times \Sigma),$$

$$f : (\Sigma \times \Sigma \cup \Sigma \times \{\#\} \cup \{\#\} \times \Sigma)^\omega \rightarrow \Sigma^\omega \text{ by } f(\langle a, b \rangle) = h(a)h(b) \text{ for any } \langle a, b \rangle \in (\Sigma \times \Sigma \cup \Sigma \times \{\#\} \cup \{\#\} \times \Sigma),$$

Then for any $X, Y \subseteq \Sigma^\omega$, $Sh(X, Y) = f(g^{-1}(LSh(h^{-1}(X), h^{-1}(Y))))$

Proof. It is immediate from the definition of the morphisms h , g and f . $\square$

Note that h^{-1} is an (not ϵ -free) inverse morphism while h_1 and h_2 are ϵ -free morphisms.

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