

On the Chern character of $SO(n)$

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In [6] and [7] we described the Chern character homomorphism $ch: K^*(G) \rightarrow H^{**}(G; \mathbb{Q})$ for $G = Spin(2n+1)$ and $SO(2n+1)$. The purpose of this paper is to study ch for $G = SO(2n)$ and $Spin(2n)$, where $n \geq 1$.

§1. Representation rings.

In this section, for later use, we quote from [3] and [8] some results on the complex representation rings of classical Lie groups that concern us.

Let G be a compact, connected Lie group. Its representation ring $R(G)$ is the Grothendieck construction of the semiring of isomorphism classes $[V]$ of G -modules V over $\mathbb{C}$. It has an augmentation

$$\varepsilon: R(G) \rightarrow \mathbb{Z}$$

which assigns to each class $[V]$ the dimension of V . Let T be a maximal torus of G . The Weyl group $W(G) = N(T)/T$ of G acts on T and therefore on $R(T)$. The inclusion $i: T \rightarrow G$ induces a monomorphism $i^*: R(G) \rightarrow R(T)$ and its image $i^*(R(G))$ coincides with the subring $R(T)^{W(G)}$ of elements left element-wise fixed by $W(G)$. Thus, through i^* , $R(G)$ can be regarded as a subring of $R(T)$. Besides, $R(G)$ is a λ -ring with operations

$$\lambda^k: R(G) \rightarrow R(G) \quad \text{for } k \geq 0$$

induced by the exterior powers of G -modules over $\mathbb{C}$. Their properties needed in the sequel are: $\lambda^0(x) = 1$ for all $x \in R(G)$; if $\varepsilon(x) = n$, then $\varepsilon(\lambda^k(x)) = \binom{n}{k}$ and $\lambda^k(x) = 0$ for $k > n$.

Let T be the maximal torus of diagonal matrices in the unitary group $U(n)$. If $\alpha_1, \dots, \alpha_n$ denote the standard 1-dimensional representations of T , then

$$(1.1) \quad R(T) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}] / (\alpha_1 \alpha_1^{-1} - 1, \dots, \alpha_n \alpha_n^{-1} - 1).$$

Put $\lambda_1 = [\mathbb{C}^n] \in R(U(n))$ and let $\lambda_k = \lambda^k(\lambda_1)$. Then

$$(1.2) \quad R(U(n)) = \mathbb{Z}[\lambda_1, \dots, \lambda_{n-1}, \lambda_n, \lambda_n^{-1}] / (\lambda_n \lambda_n^{-1} - 1)$$

and, as a subring of $R(T)$, the relation

$$(1.3) \quad \prod_{i=1}^n (1 + \alpha_i t) = \sum_{k=0}^n \lambda_k t^k$$

holds (in the polynomial ring $R(T)[t]$).

For the rotation groups $SO(n)$, there is a fibre bundle

$$(1.4) \quad SO(n) \xrightarrow{j_n} SO(n+1) \xrightarrow{q_n} SO(n+1)/SO(n) = S^n.$$

Let

$$(1.5) \quad j'_n: U(n) \rightarrow SO(2n)$$

be the real restriction mapping. Then $j'_n(T)$ becomes a maximal torus of $SO(2n)$, which we denote by T again. Further, $j_{2n}(T)$ becomes a maximal torus of $SO(2n+1)$, we denote by T also. Put $\mu'_1 = [(R^{2n+1})^{\mathbb{C}}] \in R(SO(2n+1))$ and let $\mu'_k = \lambda^k(\mu'_1)$. Then

$$(1.6) \quad R(SO(2n+1)) = \mathbb{Z}[\mu'_1, \mu'_2, \dots, \mu'_n]$$

and, as a subring of $R(T)$ (which is just as in (1.1)), the relation

$$(1.7) \quad (1+t) \prod_{i=1}^n (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n+1} \mu'_k t^k$$

holds. Note that $\mu'_{2n+1-k} = \mu'_k$ for $k = 0, 1, \dots, 2n+1$; in fact,

$$\begin{aligned} \sum_{k=0}^{2n+1} \mu'_{2n+1-k} t^k &= (t+1) \prod_{i=1}^n (t + \alpha_i)(t + \alpha_i^{-1}) \\ &= (t+1) \prod_{i=1}^n (\alpha_i^{-1}t + 1) \alpha_i \alpha_i^{-1} (\alpha_i t + 1) \\ &= (1+t) \prod_{i=1}^n (1 + \alpha_i^{-1}t)(1 + \alpha_i t) = \sum_{k=0}^{2n+1} \mu'_k t^k. \end{aligned}$$

For $n \geq 3$, the spinor group $\text{Spin}(n)$ is a universal covering group of $\text{SO}(n)$:

$$(1.8) \quad S^0 = \{1, -1\} \longrightarrow \text{Spin}(n) \xrightarrow{p_n} \text{SO}(n).$$

Then $p_n^{-1}(T)$ is a maximal torus of $\text{Spin}(n)$, which we denote by $\tilde{T}$. For this torus $\tilde{T}$ of $\text{Spin}(2n+1)$, there are 1-dimensional representations $\alpha_1, \dots, \alpha_n$ of $\tilde{T}$ such that

$$(1.9) \quad R(\tilde{T}) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}, (\alpha_1 \cdots \alpha_n)^{1/2}] / \\ (\alpha_1 \alpha_1^{-1} - 1, \dots, \alpha_n \alpha_n^{-1} - 1, ((\alpha_1 \cdots \alpha_n)^{1/2})^2 - \alpha_1 \cdots \alpha_n)$$

and $p_{2n+1}: \tilde{T} \rightarrow T$ induces the obvious inclusion $R(T) \rightarrow R(\tilde{T})$ under the descriptions of (1.1) and (1.9). Set $\mu'_k = p_{2n+1}^*(\mu_k) \in R(\text{Spin}(2n+1))$. Let Δ_{2n+1} be the spin representation of dimension 2^n . Then

$$(1.10) \quad R(\text{Spin}(2n+1)) = \mathbb{Z}[\mu'_1, \mu'_2, \dots, \mu'_{n-1}, \Delta_{2n+1}]$$

and in $R(\tilde{T})$,

$$(1.11) \quad \Delta_{2n+1} = \prod_{i=1}^n (\alpha_i^{1/2} + \alpha_i^{-1/2}) = \sum_{\varepsilon_i = \pm 1} \alpha_1^{\varepsilon_1/2} \cdots \alpha_n^{\varepsilon_n/2}$$

Moreover in $R(\text{Spin}(2n+1))$, the following relation holds:

$$(1.12) \quad \Delta_{2n+1}^2 = \sum_{k=0}^n \mu'_k.$$

Therefore, $p_{2n+1}^*: R(\text{SO}(2n+1)) \rightarrow R(\text{Spin}(2n+1))$ is given by

$$(1.13) \quad p_{2n+1}^*(\mu'_k) = \mu_k \quad (k = 1, \dots, n-1),$$

$$p_{2n+1}^*(\mu'_n) = \Delta_{2n+1}^2 - \sum_{k=0}^{n-1} \mu'_k.$$

Put $\mu_1 = [(R^{2n})^C] \in R(SO(2n))$ and let $\mu_k = \lambda^k(\mu_1)$. In particular, μ_n can be halved, that is, there are two representations μ_n^+, μ_n^- of $SO(2n)$ such that

$$(1.14) \quad \mu_n = \mu_n^+ + \mu_n^- \quad \text{and} \quad \varepsilon(\mu_n^+) = \varepsilon(\mu_n^-) = \frac{1}{2} \binom{2n}{n}.$$

Then

$$(1.15) \quad R(SO(2n)) = Z[\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n^+, \mu_n^-] / (\gamma_n)$$

where

$$\gamma_n = (\mu_n^+ + \sum_{i \geq 1} \mu_{n-2i})(\mu_n^- + \sum_{i \geq 1} \mu_{n-2i}) - (\mu_{n-1} + \sum_{j \geq 1} \mu_{n-1-2j}).$$

Here the summations in the right side end at $\dots + \mu_4 + \mu_2 + 1$ or $\dots + \mu_3 + \mu_1$. As a subring of $R(T)$ (which is just as in (1.1)), the relations

$$(1.16) \quad \prod_{i=1}^n (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n} \mu_k t^k$$

and

$$(1.17) \quad \begin{aligned} \mu_n^+ &= \sum_{\prod \varepsilon_i = 1} \alpha_1^{\varepsilon_1} \dots \alpha_n^{\varepsilon_n} + \sum'' \alpha_1^{\varepsilon_1} \dots \hat{\alpha}_i^{\varepsilon_i} \dots \hat{\alpha}_j^{\varepsilon_j} \dots \alpha_n^{\varepsilon_n}, \\ \mu_n^- &= \sum_{\prod \varepsilon_i = -1} \alpha_1^{\varepsilon_1} \dots \alpha_n^{\varepsilon_n} + \sum'' \alpha_1^{\varepsilon_1} \dots \hat{\alpha}_i^{\varepsilon_i} \dots \hat{\alpha}_j^{\varepsilon_j} \dots \alpha_n^{\varepsilon_n} \end{aligned}$$

hold, where the notation $\hat{\alpha}^{\varepsilon}$ means the replacement of α^{ε} by 1 and the number of $\hat{\alpha}$ in the summation $\sum''$ is even and positive.

Set $\mu_k = p_{2n}^*(\mu_k) \in R(\text{Spin}(2n))$. Let $\Delta_{2n}^+, \Delta_{2n}^-$ be the half spin representations, each of dimension 2^{n-1} . Then

$$(1.18) \quad R(\text{Spin}(2n)) = Z[\mu_1, \mu_2, \dots, \mu_{n-2}, \Delta_{2n}^+, \Delta_{2n}^-]$$

and in $R(\tilde{T})$ (which is just as in (1.9)),

$$(1.19) \quad \Delta_{2n}^+ = \sum_{\prod \varepsilon_i = 1} \alpha_1^{\varepsilon_1/2} \dots \alpha_n^{\varepsilon_n/2}, \quad \Delta_{2n}^- = \sum_{\prod \varepsilon_i = -1} \alpha_1^{\varepsilon_1/2} \dots \alpha_n^{\varepsilon_n/2}.$$

Moreover in $R(\text{Spin}(2n))$, the following relations hold:

$$(1.20) \quad \begin{aligned} \Delta_{2n}^+ \Delta_{2n}^- &= \mu_{n-1} + \sum_{q \geq 1} \mu_{n-1-2q}, \\ \Delta_{2n}^{+2} &= \mu_n^+ + \sum_{k \geq 1} \mu_{n-2k}, \\ \Delta_{2n}^{-2} &= \mu_n^- + \sum_{k \geq 1} \mu_{n-2k}. \end{aligned}$$

Therefore, $p_{2n}^*: R(\text{SO}(2n)) \rightarrow R(\text{Spin}(2n))$ is given by

$$(1.21) \quad \begin{aligned} p_{2n}^*(\mu_k) &= \mu_k \quad (k = 1, \dots, n-2), \\ p_{2n}^*(\mu_{n-1}) &= \Delta_{2n}^+ \Delta_{2n}^- - \sum_{q \geq 1} \mu_{n-1-2q}, \\ p_{2n}^*(\mu_n^+) &= \Delta_{2n}^{+2} - \sum_{k \geq 1} \mu_{n-2k}, \\ p_{2n}^*(\mu_n^-) &= \Delta_{2n}^{-2} - \sum_{k \geq 1} \mu_{n-2k}. \end{aligned}$$

The following propositions are immediate consequences of the above results.

Proposition 1.1. (1) $j_{2n}^*: R(\text{SO}(2n+1)) \rightarrow R(\text{SO}(2n))$ satisfies

$$j_{2n}^*(\mu_1') = \mu_1 + 1.$$

(2) $j_{2n-1}^*: R(\text{SO}(2n)) \rightarrow R(\text{SO}(2n-1))$ satisfies

$$j_{2n-1}^*(\mu_1) = \mu_1' + 1.$$

Proposition 1.2. (1) $\tilde{j}_{2n}^*: R(\text{Spin}(2n+1)) \rightarrow R(\text{Spin}(2n))$ satisfies

$$\begin{aligned} \tilde{j}_{2n}^*(\mu_1') &= \mu_1 + 1, \\ \tilde{j}_{2n}^*(\Delta_{2n+1}) &= \Delta_{2n}^+ + \Delta_{2n}^-. \end{aligned}$$

(2) $\tilde{j}_{2n-1}^*: R(\text{Spin}(2n)) \rightarrow R(\text{Spin}(2n-1))$ satisfies

$$\begin{aligned} \tilde{j}_{2n-1}^*(\mu_1) &= \mu_1' + 1, \\ \tilde{j}_{2n-1}^*(\Delta_{2n}^+) &= \tilde{j}_{2n-1}^*(\Delta_{2n}^-) = \Delta_{2n-1}. \end{aligned}$$

Proposition 1.3. $j_n'^*: R(SO(2n)) \rightarrow R(U(n))$ satisfies

$$j_n'^*(u_k) = \lambda_n^{-1} \sum_{i=0}^k \lambda_i \lambda_{n-k+i} \quad (k = 1, \dots, n).$$

Proposition 1.4. In $R(T)$,

$$\Delta_{2n}^{+2} - \Delta_{2n}^{-2} = \lambda_n^{-1} \left(\sum_{i=0}^n \lambda_i \right) \left(\sum_{j=0}^n (-1)^{n-j} \lambda_j \right).$$

§2. Cohomology rings

In this section we fix some notations concerning the integral cohomology of our groups G .

For $G = U(n)$, by Borel's transgression theorem, there exist elements $x_{2i-1} \in H^{2i-1}(U(n); \mathbb{Z})$, $i = 1, \dots, n$, such that

$$H^*(U(n); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1, x_3, \dots, x_{2n-1})$$

and

$$PH^*(U(n); \mathbb{Z}) = \mathbb{Z}\{x_1, x_3, \dots, x_{2n-1}\}$$

where P denotes the primitive module functor. Thus

$$(2.1) \quad H^*(U(n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1, x_3, \dots, x_{2n-1}).$$

$H^*(SO(n); \mathbb{Z})$ has 2-torsion only. For $G = SO(2n+1)$, by Poincaré duality and Borel's transgression theorem, there exist elements $x_{4i-1} \in H^{4i-1}(SO(2n+1); \mathbb{Z})$, $i = 1, \dots, n$, such that

$$H^*(SO(2n+1); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(x_3, x_7, \dots, x_{4n-1})$$

and

$$PH^*(SO(2n+1); \mathbb{Q}) = \mathbb{Q}\{x_3, x_7, \dots, x_{4n-1}\}.$$

Thus

$$(2.2) \quad H^*(SO(2n+1); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3, x_7, \dots, x_{4n-1}).$$

$H^*(\text{Spin}(n); \mathbb{Z})$ has 2-torsion if and only if $n \geq 7$. For $G = \text{Spin}(2n+1)$, by similar reasons, there exist elements $\tilde{x}_{4i-1} \in H^{4i-1}(\text{Spin}(2n+1); \mathbb{Z})$, $i = 1, \dots, n$, such that

$$H^*(\text{Spin}(2n+1); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-1})$$

and

$$PH^*(\text{Spin}(2n+1); \mathbb{Q}) = \mathbb{Q}\{\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-1}\}.$$

Thus

$$(2.3) \quad H^*(\text{Spin}(2n+1); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-1}).$$

For $G = \text{SO}(2n)$, there exist elements $x_{4i-1} \in H^{4i-1}(\text{SO}(2n); \mathbb{Z})$, $i = 1, \dots, n-1$, and $x'_{2n-1} \in H^{2n-1}(\text{SO}(2n); \mathbb{Z})$ such that

$$H^*(\text{SO}(2n); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(x_3, x_7, \dots, x_{4n-5}, x'_{2n-1})$$

and

$$(2.4) \quad PH^*(\text{SO}(2n); \mathbb{Q}) = \mathbb{Q}\{x_3, x_7, \dots, x_{4n-5}, x'_{2n-1}\}.$$

Thus

$$(2.5) \quad H^*(\text{SO}(2n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3, x_7, \dots, x_{4n-5}, x'_{2n-1}).$$

For $G = \text{Spin}(2n)$, there exist elements $\tilde{x}_{4i-1} \in H^{4i-1}(\text{Spin}(2n); \mathbb{Z})$, $i = 1, \dots, n-1$, and $\tilde{x}'_{2n-1} \in H^{2n-1}(\text{Spin}(2n); \mathbb{Z})$ such that

$$H^*(\text{Spin}(2n); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1})$$

and

$$PH^*(\text{Spin}(2n); \mathbb{Q}) = \mathbb{Q}\{\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1}\}.$$

Thus

$$(2.6) \quad H^*(\text{Spin}(2n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1}).$$

The following two propositions are easy.

Proposition 2.1. $j_{2n-1}^*: H^*(\text{SO}(2n); \mathbb{Z}) \rightarrow H^*(\text{SO}(2n-1); \mathbb{Z})$ satisfies

$$j_{2n-1}^*(x_{4i-1}) = x_{4i-1} \quad (i = 1, \dots, n-1),$$

$$j_{2n-1}^*(x'_{2n-1}) = 0.$$

Proposition 2.2. $\tilde{j}_{2n-1}^*: H^*(\text{Spin}(2n); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n-1); \mathbb{Z})$ satisfies

$$\begin{aligned}\tilde{j}_{2n-1}^*(\tilde{x}_{4i-1}) &= \tilde{x}_{4i-1} \quad (i = 1, \dots, n-1), \\ \tilde{j}_{2n-1}^*(\tilde{x}'_{2n-1}) &= 0.\end{aligned}$$

Proposition 2.3. $j_n^*: H^*(\text{SO}(2n); \mathbb{Z}) \rightarrow H^*(\text{U}(n); \mathbb{Z})$ satisfies

$$j_n^*(x'_{2n-1}) = x_{2n-1}.$$

Furthermore, when $n = 2m$, not only $j_{2m}^*(x'_{4m-1}) = x_{4m-1}$, but also $j_{2m}^*(x_{4m-1}) = 0$.

Proof. Since

$$\begin{aligned}H^*(\text{SO}(2n)/\text{U}(n); \mathbb{Z}) &= \mathbb{Z}[e_2, e_4, \dots, e_{2n-2}]/ \\ &\quad (e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i} : k = 1, \dots, n-1) \\ &= \Delta_{\mathbb{Z}}(e_2, e_4, \dots, e_{2n-2})\end{aligned}$$

where $\Delta_R(\)$ denotes the algebra over a ring R having a simple system of generators, and $e_{2i} \in H^{2i}(\text{SO}(2n)/\text{U}(n); \mathbb{Z})$ (see [4, Chapter 3, Theorem 6.11]), the result follows from the spectral sequence argument for the integral cohomology of the fibre bundle

$$\text{U}(n) \xrightarrow{j_n'} \text{SO}(2n) \xrightarrow{q_n'} \text{SO}(2n)/\text{U}(n).$$

§3. K-rings

In this section we collect some results on the complex K-theory of our groups G .

Let

$$\mathcal{B}: R(G) \rightarrow K^{-1}(G)$$

be the map of [2]; then $\mathcal{B}$ has the following properties:

(3.1)(1) For each $\rho_1, \rho_2 \in R(G)$,

$$\mathcal{B}(\rho_1 + \rho_2) = \mathcal{B}(\rho_1) + \mathcal{B}(\rho_2);$$

(2) If $n \in R(G)$ is the class of a trivial G -module of dimension n , then $\mathcal{B}(n) = 0$;

(3) For each $\rho_1, \rho_2 \in R(G)$,

$$\mathcal{B}(\rho_1 \rho_2) = \varepsilon(\rho_2) \mathcal{B}(\rho_1) + \varepsilon(\rho_1) \mathcal{B}(\rho_2).$$

As will be seen below, the $\mathbb{Z}/(2)$ -graded K -rings $K^*(G)$ can be described by using this map $\mathcal{B}$. Indeed, Hodgkin's theorem [2, Theorem A] says that if G is a compact connected Lie group with $\pi_1(G)$ torsion-free, then $K^*(G)$ is torsion-free and has the structure of a Hopf algebra over $\mathbb{Z}$; more precisely, if

$$R(G) = \mathbb{Z}[\rho_1, \rho_2, \dots, \rho_q],$$

then

$$K^*(G) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\rho_1), \mathcal{B}(\rho_2), \dots, \mathcal{B}(\rho_q))$$

where each $\mathcal{B}(\rho_i)$ is primitive. Therefore, for $G = U(n)$, it follows from (1.2) and (3.1) that $\mathcal{B}(\lambda_n^{-1}) = -\mathcal{B}(\lambda_n)$ and

$$(3.2) \quad K^*(U(n)) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\lambda_1), \dots, \mathcal{B}(\lambda_{n-1}), \mathcal{B}(\lambda_n)).$$

Similarly, for $G = \text{Spin}(2n+1)$, it follows from (1.10) that

$$(3.3) \quad K^*(\text{Spin}(2n+1)) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\mu'_1), \dots, \mathcal{B}(\mu'_{n-1}), \mathcal{B}(\Delta_{2n+1})).$$

For $G = \text{SO}(2n+1)$, by (1.6), $\mathcal{B}(\mu'_1), \dots, \mathcal{B}(\mu'_n) \in K^{-1}(\text{SO}(2n+1))$. According to [1], there exist other two elements $\varepsilon_{2n+1} \in K^{-1}(\text{SO}(2n+1))$ and $\varepsilon_{2n+1} \in K^0(\text{SO}(2n+1))$ such that

$$(3.4) \quad p_{2n+1}^*(\varepsilon_{2n+1}) = 2\mathcal{B}(\Delta_{2n+1})$$

and

$$K^*(\text{SO}(2n+1)) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\mu'_1), \dots, \mathcal{B}(\mu'_{n-1}), \varepsilon_{2n+1}) \otimes$$

$$(\mathbb{Z}\{1\} \oplus \mathbb{Z}/(2^n)\{\varepsilon_{2n+1}\})/(\varepsilon_{2n+1} \otimes \varepsilon_{2n+1}).$$

Thus

$$(3.5) \quad K^*(SO(2n+1))/\text{Tor} = \Lambda_{\mathbb{Z}}(\beta(u'_1), \dots, \beta(u'_{n-1}), \varepsilon_{2n+1}).$$

Similarly, for $G = \text{Spin}(2n)$, it follows from (1.18) that

$$(3.6) \quad K^*(\text{Spin}(2n)) = \Lambda_{\mathbb{Z}}(\beta(u_1), \dots, \beta(u_{n-2}), \beta(\Delta_{2n}^+), \beta(\Delta_{2n}^-)).$$

For $G = SO(2n)$, by (1.15), $\beta(u_1), \dots, \beta(u_{n-1}), \beta(u_n^+), \beta(u_n^-) \in K^{-1}(SO(2n))$. According to [1], there exist other three elements $\delta_{2n}, \varepsilon_{2n} \in K^{-1}(SO(2n))$ and $\varepsilon_{2n} \in K^0(SO(2n))$ such that

$$(3.7) \quad p_{2n}^*(\delta_{2n}) = \beta(\Delta_{2n}^+) - \beta(\Delta_{2n}^-), \quad p_{2n}^*(\varepsilon_{2n}) = 2\beta(\Delta_{2n}^+)$$

and

$$K^*(SO(2n)) = \Lambda_{\mathbb{Z}}(\beta(u_1), \dots, \beta(u_{n-2}), \delta_{2n}, \varepsilon_{2n}) \otimes (\mathbb{Z}\{1\} \oplus \mathbb{Z}/(2^{n-1})\{\varepsilon_{2n}\})/(\varepsilon_{2n} \otimes \varepsilon_{2n}).$$

Thus

$$(3.8) \quad K^*(SO(2n))/\text{Tor} = \Lambda_{\mathbb{Z}}(\beta(u_1), \dots, \beta(u_{n-2}), \delta_{2n}, \varepsilon_{2n}).$$

Using these results, we can deduce the following from the results of §1.

Proposition 3.1. (1) In $K^*(\text{Spin}(2n+1))$,

$$\beta(u'_n) = 2^{n+1}\beta(\Delta_{2n+1}) - \sum_{k=1}^{n-1} \beta(u'_k).$$

(2) In $K^*(\text{Spin}(2n))$,

$$\beta(u_{n-1}) = 2^{n-1}\beta(\Delta_{2n}^+) + 2^{n-1}\beta(\Delta_{2n}^-) - \sum_{q=1}^{[(n-2)/2]} \beta(u_{n-1-2q}).$$

Proposition 3.2. (1) In $K^*(SO(2n+1))/\text{Tor}$,

$$\beta(u'_n) = 2^n \varepsilon_{2n+1} - \sum_{k=1}^{n-1} \beta(u'_k).$$

(2) In $K^*(SO(2n))/\text{Tor}$,

$$\begin{aligned}
\mathcal{B}(\mu_{n-1}) &= -2^{n-1}\delta_{2n} + 2^{n-1}\varepsilon_{2n} - \sum_{q=1}^{[(n-2)/2]} \mathcal{B}(\mu_{n-1-2q}), \\
\mathcal{B}(\mu_n^+) &= 2^{n-1}\varepsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \mathcal{B}(\mu_{n-2k}), \\
\mathcal{B}(\mu_n^-) &= -2^n\delta_{2n} + 2^{n-1}\varepsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \mathcal{B}(\mu_{n-2k}).
\end{aligned}$$

Proposition 3.3. (1) $\tilde{j}_{2n}^*: K^*(\text{Spin}(2n+1)) \rightarrow K^*(\text{Spin}(2n))$ satisfies

$$\begin{aligned}
\tilde{j}_{2n}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1), \\
\tilde{j}_{2n}^*(\mathcal{B}(\Delta_{2n+1}^+)) &= \mathcal{B}(\Delta_{2n}^+) + \mathcal{B}(\Delta_{2n}^-), \\
(2) \quad \tilde{j}_{2n-1}^*: K^*(\text{Spin}(2n)) &\rightarrow K^*(\text{Spin}(2n-1)) \text{ satisfies} \\
\tilde{j}_{2n-1}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1'), \\
\tilde{j}_{2n-1}^*(\mathcal{B}(\Delta_{2n}^+)) &= \tilde{j}_{2n-1}^*(\mathcal{B}(\Delta_{2n}^-)) = \mathcal{B}(\Delta_{2n-1}).
\end{aligned}$$

Proposition 3.4. (1) $j_{2n}^*: K^*(\text{SO}(2n+1))/\text{Tor} \rightarrow K^*(\text{SO}(2n))/\text{Tor}$ satisfies

$$\begin{aligned}
j_{2n}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1), \\
j_{2n}^*(\varepsilon_{2n+1}) &= -2\delta_{2n} + 2\varepsilon_{2n}, \\
(2) \quad j_{2n-1}^*: K^*(\text{SO}(2n))/\text{Tor} &\rightarrow K^*(\text{SO}(2n-1))/\text{Tor} \text{ satisfies} \\
j_{2n-1}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1'), \\
j_{2n-1}^*(\delta_{2n}) &= 0, \quad j_{2n-1}^*(\varepsilon_{2n}) = \varepsilon_{2n-1}.
\end{aligned}$$

Proposition 3.5. $j_n'^*: K^*(\text{SO}(2n))/\text{Tor} \rightarrow K^*(U(n))$ satisfies

$$\begin{aligned}
j_n'^*(\mathcal{B}(\mu_1)) &= \mathcal{B}(\lambda_1) + \mathcal{B}(\lambda_{n-1}) - n\mathcal{B}(\lambda_n), \\
j_n'^*(\delta_{2n}) &= \sum_{k=1}^n (-1)^{n-k} \mathcal{B}(\lambda_k), \\
j_n'^*(\varepsilon_{2n}) &= \sum_{k=1}^{n-1} (1+(-1)^{n-k}) \mathcal{B}(\lambda_k) - 2(2^{n-2}-1)\mathcal{B}(\lambda_n).
\end{aligned}$$

§4. The Chern character homomorphisms

In this section we prove our main results.

Let $\phi: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ be a function defined by

$$(4.1) \quad \phi(n, k, q) = \sum_{j=1}^k (-1)^{j-1} \binom{n}{k-j} j^{q-1} \quad \text{for } n, k, q \in \mathbb{N}.$$

Then, following Method I of [5, pp. 464-466] and using Lemma 1 of [5], we have

Proposition 4.1. In the notations of (3.2) and (2.1), ch :

$K^*(U(n)) \rightarrow H^{**}(U(n); \mathbb{Q})$ is given by

$$\text{ch}(\beta(\lambda_k)) = \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \phi(n, k, i) x_{2i-1} \quad (k \geq 1).$$

Let $\mathcal{P} = \{2^i \mid i = 0, 1, 2, \dots\}$. For each $n \in \mathbb{N}$ there is a unique integer $s(n)$ such that $2^{s(n)-1} < n \leq 2^{s(n)}$. Let

$$q(n, i) = \begin{cases} 2 & \text{if } n \notin \mathcal{P}, i = 2^{s(n)-1} \\ & \text{or } n \in \mathcal{P}, i = 2^{s(n)} \\ 1 & \text{otherwise} \end{cases}$$

Then Theorem 1 of [6] is

Theorem 4.2. In the notations of (3.3) and (2.3), $\text{ch}: K^*(\text{Spin}(2n+1)) \rightarrow H^{**}(\text{Spin}(2n+1); \mathbb{Q})$ is given by

$$\begin{aligned} \text{ch}(\beta(\tilde{\mu}'_k)) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2^{q(n,i)}}{(2i-1)!} \phi(2n+1, k, 2i) \tilde{x}_{4i-1}, \\ \text{ch}(\beta(\Delta_{2n+1})) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2^{q(n,i)}}{(2i-1)!} \left(\frac{1}{2^{n+1}} \sum_{k=1}^n \phi(2n+1, k, 2i) \right) \tilde{x}_{4i-1}. \end{aligned}$$

While, Theorem 5.3 of [7] is

Theorem 4.3. In the notations of (3.5) and (2.2), $\text{ch}: K^*(SO(2n+1))/\text{Tor} \rightarrow H^{**}(SO(2n+1); \mathbb{Q})$ is given by

$$\begin{aligned}\text{ch}(\beta(u'_k)) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2}{(2i-1)!} \phi(2n+1, k, 2i) x_{4i-1}, \\ \text{ch}(\varepsilon_{2n+1}) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2}{(2i-1)!} \left(\frac{1}{2^n} \sum_{k=1}^n \phi(2n+1, k, 2i) \right) x_{4i-1}.\end{aligned}$$

Corollary 4.4. (1) $p_{2n+1}^*: H^*(SO(2n+1); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n+1); \mathbb{Z})$ satisfies

$$p_{2n+1}^*(x_{4i-1}) = 2^{q(n,i)-1} \tilde{x}_{4i-1} \quad (i = 1, \dots, n).$$

(2) $p_{2n}^*: H^*(SO(2n); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n); \mathbb{Z})$ satisfies

$$\begin{aligned}p_{2n}^*(x_{4i-1}) &= 2^{q(n-1,i)-1} \tilde{x}_{4i-1} \quad (i = 1, \dots, n-1), \\ p_{2n}^*(x'_{2n-1}) &= \tilde{x}'_{2n-1}.\end{aligned}$$

Theorem 4.5. In the notations of (3.8) and (2.5), $\text{ch}: K^*(SO(2n))/\text{Tor} \rightarrow H^{**}(SO(2n); \mathbb{Q})$ is given by

$$\begin{aligned}\text{ch}(\beta(u_k)) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2}{(2i-1)!} \phi(2n, k, 2i) x_{4i-1} + \\ &\quad \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \phi(2n, k, n) x'_{2n-1}, \\ \text{ch}(\delta_{2n}) &= x'_{2n-1}, \\ \text{ch}(\varepsilon_{2n}) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2}{(2i-1)!} \left(\frac{1}{2^{n-1}} \sum_{q=0}^{[(n-2)/2]} \phi(2n, n-1-2q, 2i) \right) x_{4i-1} + \\ &\quad \left(1 + \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \right) \left(\frac{1}{2^{n-1}} \sum_{q=0}^{[(n-2)/2]} \phi(2n, n-1-2q, n) \right) x'_{2n-1}.\end{aligned}$$

Proof. Since $\beta(u_1)$ is primitive in the Hopf algebra $K^*(SO(2n))/\text{Tor}$ (see [1]), by (2.4) we may set

$$(4.2) \quad \text{ch}(\beta(u_1)) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1}$$

for some $a_i, a' \in \mathbb{Q}$. Let us compute these coefficients. Apply j_{2n-1}^* to (4.2). Then the left hand side is

$$\begin{aligned} j_{2n-1}^* \text{ch}(\mathcal{B}(u_1)) &= \text{ch}(j_{2n-1}^*(\mathcal{B}(u_1))) \\ &= \text{ch}(\mathcal{B}(u'_1)) && \text{by Proposition 3.4(2)} \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2}{(2i-1)!} x_{4i-1} && \text{by Theorem 4.3,} \end{aligned}$$

and the right hand side is

$$j_{2n-1}^* \left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1} \right) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition 2.1. Hence $a_i = (-1)^{i-1} 2 / (2i-1)!$ for $i = 1, \dots, n-1$. Apply j_n^* to (4.2). Then the left hand side is

$$\begin{aligned} j_n^* \text{ch}(\mathcal{B}(u_1)) &= \text{ch}(j_n^*(\mathcal{B}(u_1))) \\ &= \text{ch}(\mathcal{B}(\lambda_1) + \mathcal{B}(\lambda_{n-1}) - n\mathcal{B}(\lambda_n)) && \text{by Proposition 3.5} \\ &= \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} (1 + \phi(n, n-1, i) - n\phi(n, n, i)) x_{2i-1} \end{aligned}$$

by Proposition 4.1. Since $\phi(n, k, 1) = \binom{n-1}{k-1}$, we have

$$1 + \phi(n, n-1, 1) - n\phi(n, n, 1) = 1 + (n-1) - n \cdot 1 = 0$$

If $2 \leq i \leq n$, then $\phi(n, n-k, i) = (-1)^i \phi(n, k, i)$ and $\phi(n, k, i) = 0$ for $k \geq n$ by (4.14) and (4.15) of [7]. Therefore

$$\begin{aligned} 1 + \phi(n, n-1, i) - n\phi(n, n, i) &= 1 + (-1)^i \phi(n, 1, i) - 0 \\ &= 1 + (-1)^i. \end{aligned}$$

Consequently, for $i = 1, \dots, n$,

$$1 + \phi(n, n-1, i) - n\phi(n, n, i) = 1 + (-1)^i.$$

Thus

$$j_n^* \text{ch}(\mathcal{B}(u_1)) = \dots + \frac{(-1)^{n-1} (1 + (-1)^n)}{(n-1)!} x_{2n-1}.$$

On the other hand, the right hand side is

$$j_n^* \left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1} \right) = \dots + a' x_{2n-1}$$

by Proposition 2.3. Hence $a' = (-1)^{n-1}(1+(-1)^n)/(n-1)!$. This proves the first equality for $k = 1$, and that for $k > 1$ follows from it and Lemma 1 of [5].

Next we set

$$(4.3) \quad \text{ch}(\delta_{2n}) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1}$$

for some $a_i, a' \in \mathbb{Q}$. Let us compute these coefficients. Apply j_{2n-1}^* to (4.3). Then the left hand side is

$$j_{2n-1}^* \text{ch}(\delta_{2n}) = \text{ch} j_{2n-1}^*(\delta_{2n}) = \text{ch}(0) = 0$$

by Proposition 3.4(2), and the right hand side is

$$j_{2n-1}^* \left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1} \right) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition 2.1. Hence $a_i = 0$ for $i = 1, \dots, n-1$. Apply

j_n^* to (4.3). Then the left hand side is

$$\begin{aligned} j_n^* \text{ch}(\delta_{2n}) &= \text{ch}(j_n^*(\delta_{2n})) \\ &= \text{ch} \left(\sum_{k=1}^n (-1)^{n-k} \beta(\lambda_k) \right) && \text{by Proposition 3.5} \\ &= \sum_{k=1}^n (-1)^{n-k} \text{ch}(\beta(\lambda_k)) \\ &= \sum_{k=1}^n (-1)^{n-k} \left(\sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \phi(n, k, i) x_{2i-1} \right) && \text{by Proposition 4.1} \\ &= \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \left(\sum_{k=1}^n (-1)^{n-k} \phi(n, k, i) \right) x_{2i-1}. \end{aligned}$$

But

$$\begin{aligned} \sum_{k=1}^n (-1)^{n-k} \phi(n, k, i) &= \sum_{k=1}^n (-1)^{n-k} \left(\sum_{j=1}^k (-1)^{j-1} \binom{n}{k-j} j^{i-1} \right) && \text{by (4.1)} \\ &= \sum_{j=1}^n \sum_{k=j}^n (-1)^{n-k+j-1} \binom{n}{k-j} j^{i-1} \\ &= \sum_{j=1}^n \sum_{\ell=0}^{n-j} (-1)^{n-\ell-1} \binom{n}{\ell} j^{i-1} \\ &= (-1)^{n-1} \sum_{j=1}^n \left(\sum_{\ell=0}^{n-j} (-1)^{\ell} \binom{n}{\ell} \right) j^{i-1} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-1} \sum_{j=1}^n (-1)^{n-j} \binom{n-1}{n-j} j^{i-1} \\
&= \sum_{j=1}^n (-1)^{j-1} \binom{n-1}{n-j} j^{i-1} \\
&= \Phi(n-1, n, i) \\
&= \begin{cases} 0 & \text{if } i = 1, \dots, n-1 \\ (-1)^{n-1} (n-1)! & \text{if } i = n \end{cases} \quad \text{by [7].}
\end{aligned}$$

Thus

$$j_n'^* \text{ch}(\delta_{2n}) = \frac{(-1)^{n-1}}{(n-1)!} (-1)^{n-1} (n-1)! x_{2n-1} = x_{2n-1}.$$

On the other hand, the right hand side is

$$j_n'^* \left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1} \right) = a' x_{2n-1}$$

by Proposition 2.3. Hence $a' = 1$. This proves the second equality.

The third equality is obtained from the first and second equalities by using the relation

$$2^{n-1} \varepsilon_{2n} = 2^{n-1} \delta_{2n} + \sum_{q=0}^{[(n-2)/2]} \beta(u_{n-1-2q})$$

of Proposition 3.2(2).

Corollary 4.6. $j_n'^*: H^*(SO(2n); \mathbb{Z}) \rightarrow H^*(U(n); \mathbb{Z})$ satisfies

$$j_n'^*(x_{4i-1}) = \begin{cases} (-1)^i x_{4i-1} & \text{if } i = 1, \dots, [(n-1)/2] \\ 0 & \text{if } i = [(n-1)/2] + 1, \dots, n-1. \end{cases}$$

Corollary 4.7. $j_{2n}^*: H^*(SO(2n+1); \mathbb{Z}) \rightarrow H^*(SO(2n); \mathbb{Z})$ satisfies:

(i) if $n = 2m + 1$,

$$x_{4i-1} \quad (i = 1, \dots, 2m)$$

$$j_{4m+2}^*(x_{4i-1}) = \begin{cases} 0 & (i = 2m+1) \end{cases};$$

(ii) if $n = 2m$,

$$j_{4m}^*(x_{4i-1}) = \begin{cases} x_{4i-1} & (i = 1, \dots, 2m-1 \text{ and } i \neq m) \\ x_{4m-1} + (-1)^m x'_{4m-1} & (i = m) \\ 0 & (i = 2m) \end{cases}$$

Corollary 4.8. In the notations of (3.6) and (2.6), $\text{ch}: K^*(\text{Spin}(2n)) \rightarrow H^{**}(\text{Spin}(2n); \mathbb{Q})$ is given by

$$\begin{aligned} \text{ch}(\beta(u_k)) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{q(n-1,i)}}{(2i-1)!} \phi(2n, k, 2i) \tilde{x}_{4i-1} + \\ &\quad \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \phi(2n, k, n) \tilde{x}'_{2n-1}, \end{aligned}$$

$$\text{ch}(\beta(\Delta_{2n}^+)) =$$

$$\begin{aligned} &\sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{q(n-1,i)}}{(2i-1)!} \left(\frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, 2i) \right) \tilde{x}_{4i-1} + \\ &\left(\frac{1}{2} + \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \right) \left(\frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, n) \right) \tilde{x}'_{2n-1}, \end{aligned}$$

$$\text{ch}(\beta(\Delta_{2n}^-)) =$$

$$\begin{aligned} &\sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{q(n-1,i)}}{(2i-1)!} \left(\frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, 2i) \right) \tilde{x}_{4i-1} + \\ &\left(\frac{-1}{2} + \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \right) \left(\frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, n) \right) \tilde{x}'_{2n-1}. \end{aligned}$$

Corollary 4.9. $\tilde{j}_{2n}^*: H^*(\text{Spin}(2n+1); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n); \mathbb{Z})$ satisfies:

(i) if $n = 2m + 1$,

$$\tilde{j}_{4m+2}^*(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & (i = 1, \dots, 2m) \\ 0 & (i = 2m+1) \end{cases};$$

(ii) if $n = 2m$,

$$\tilde{j}_{4m}^*(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & (i = 1, \dots, 2m-1 \text{ and } i \neq m) \\ \tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & (i = m \text{ and } m \notin \mathcal{P}) \\ 2\tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & (i = m \text{ and } m \in \mathcal{P}) \\ 0 & (i = 2m) \end{cases}$$

Remark. Let $\phi_t: N \times N \rightarrow Z$ be a function defined by

$$\phi_t(n, q) = \sum_{k=1}^n \phi(n, k, q) \quad \text{for } n, q \in N.$$

Then $\phi_t(n, 1) = 2^{n-1}$, $\phi_t(n, 2i) = 2\phi_t(n-1, 2i)$ and $\phi_t(n, 2i+1) = 0$ for $i = 1, \dots, [(n-1)/2]$. With this notation we find that

$$2 \sum_{k=1}^n \phi(2n+1, k, 2i) = \phi_t(2n+1, 2i) \quad \text{for } i = 1, \dots, n$$

and

$$\begin{aligned} 4 \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, 2i) &= \phi_t(2n, 2i) = \\ 2\phi(2n, n, 2i) + 4 \sum_{k=1}^{[(n-1)/2]} \phi(2n, n-2k, 2i) &\quad \text{for } i = 1, \dots, n-1 \end{aligned}$$

(cf. Theorems 4.3 and 4.5).

Finally, we display a list of $\text{ch}: K^*(G) \rightarrow H^{**}(G; \mathbb{Q})$ for $G = SO(n)$ and $\text{Spin}(n)$ with $3 \leq n \leq 9$:

$SO(3)$

$$\text{ch}(\varepsilon_3) = x_3$$

$SO(4)$

$$\text{ch}(\delta_4) = x'_3$$

$$\text{ch}(\varepsilon_4) = x_3$$

$SO(5)$

$$\text{ch}(\mathcal{B}(u'_1)) = 2x_3 - \frac{1}{3}x_7$$

$\text{Spin}(3)$

$$\text{ch}(\mathcal{B}(\Delta_3)) = \tilde{x}_3$$

$\text{Spin}(4)$

$$\text{ch}(\mathcal{B}(\Delta_4^+)) = \tilde{x}_3$$

$$\text{ch}(\mathcal{B}(\Delta_4^-)) = \tilde{x}_3 - \tilde{x}'_3$$

$\text{Spin}(5)$

$$\text{ch}(\mathcal{B}(u'_1)) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7$$

$$\text{ch}(\varepsilon_5) = 2x_3 + \frac{1}{6}x_7$$

SO(6)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2x_3 - \frac{1}{3}x_7$$

$$\text{ch}(\delta_6) = x'_5$$

$$\text{ch}(\varepsilon_6) = 2x_3 + x'_5 + \frac{1}{6}x_7$$

SO(7)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2x_3 - \frac{1}{3}x_7 + \frac{1}{60}x_{11}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 10x_3 + \frac{1}{3}x_7 - \frac{5}{12}x_{11}$$

$$\text{ch}(\varepsilon_7) = 4x_3 + \frac{1}{3}x_7 + \frac{1}{30}x_{11}$$

Spin(7)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 10\tilde{x}_3 + \frac{2}{3}\tilde{x}_7 - \frac{5}{12}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\Delta_7)) = 2\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11}$$

SO(8)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2x_3 - \frac{1}{3}x_7 - \frac{1}{3}x'_7 + \frac{1}{60}x_{11}$$

$$\text{ch}(\mathcal{B}(\mu_2)) = 12x_3 - \frac{2}{5}x_{11}$$

$$\text{ch}(\delta_8) = x'_7$$

$$\text{ch}(\varepsilon_8) = 4x_3 + \frac{1}{3}x_7 + \frac{4}{3}x'_7 + \frac{1}{30}x_{11}$$

Spin(8)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7 - \frac{1}{3}\tilde{x}'_7 + \frac{1}{60}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\mu_2)) = 12\tilde{x}_3 - \frac{2}{5}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\Delta_8^+)) = 2\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{2}{3}\tilde{x}'_7 + \frac{1}{60}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\Delta_8^-)) = 2\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 - \frac{1}{3}\tilde{x}'_7 + \frac{1}{60}\tilde{x}_{11}$$

SO(9)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2x_3 - \frac{1}{3}x_7 + \frac{1}{60}x_{11} - \frac{1}{2520}x_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 14x_3 - \frac{1}{3}x_7 - \frac{23}{60}x_{11} + \frac{119}{2520}x_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_3)) = 42x_3 + 3x_7 - \frac{3}{20}x_{11} - \frac{1071}{2520}x_{15}$$

$$\text{ch}(\mathcal{B}(\Delta_5)) = \tilde{x}_3 + \frac{1}{6}\tilde{x}_7$$

Spin(6)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2\tilde{x}_3 + \tilde{x}'_5 - \frac{2}{3}\tilde{x}_7$$

$$\text{ch}(\mathcal{B}(\Delta_6^+)) = \tilde{x}_3 + \frac{1}{2}\tilde{x}'_5 + \frac{1}{6}\tilde{x}_7$$

$$\text{ch}(\mathcal{B}(\Delta_6^-)) = \tilde{x}_3 - \frac{1}{2}\tilde{x}'_5 + \frac{1}{6}\tilde{x}_7$$

$$\text{ch}(\varepsilon_9) = 8x_3 + \frac{2}{3}x_7 + \frac{1}{15}x_{11} + \frac{17}{2520}x_{15}$$

Spin(9)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2\tilde{x}_3 - \frac{1}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11} - \frac{1}{1260}\tilde{x}_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 14\tilde{x}_3 - \frac{1}{3}\tilde{x}_7 - \frac{23}{60}\tilde{x}_{11} + \frac{119}{1260}\tilde{x}_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_3)) = 42\tilde{x}_3 + 3\tilde{x}_7 - \frac{3}{20}\tilde{x}_{11} - \frac{1071}{1260}\tilde{x}_{15}$$

$$\text{ch}(\Delta_9) = 4\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{1}{30}\tilde{x}_{11} + \frac{17}{2520}\tilde{x}_{15}.$$

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