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Elementary equivalence of rational function fields

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Abstract

Let $K|k$ be a function field over a field k. We suppose that k is an algebraically closed field or a finite extension of a prime field. We prove that $K \equiv k(x)$ implies $K \cong k(t)$, where t is an indeterminate. For the case that k is algebraically closed, this was proven by Duret (1992), and for the case that k is a finite extension of a prime field, by Scanlon (2008). However we give a simple unified proof for both cases.

1 Introduction

It appears to be an interesting question, whether *for finitely generated fields, the elementary equivalence is the same as the isomorphism.* In the beginning of 1980's Sabbagh asked the following question: Let K and L be function fields over $\mathbb{Q}$ with $\mathrm{td}(K|\mathbb{Q}) = 1$ and $\mathrm{td}(L|\mathbb{Q}) = 2$. Is it then possible that K and L are elementarily equivalent?

The answer is no by the results of Pop (2002): The transcendence degree and transcendence bases of a function fields over number fields are definable in the language of rings. It is already known in the positive characteristic case and in the geometric case (that is, if the base field is algebraically closed). Furthernore, he showed the followings:

Let K and L be function fields over prime fields with $K \equiv L$. Then

1. *there are embeddings $K \to L$ and $L \to K$.*
2. *Furthermore, if one of them is of general type then they are isomorphic.*

We say that $K|\kappa$ is of general type if it is the function field of a projective smooth variety over κ of general type. It is known that smooth hypersurfaces of dimension n with degree $d > n + 2$ are of general type. Roughly speaking, almost all varieties are of general type. We note that rational function fields and elliptic function fields are not of general type.

Non-general case remained an open question. However Scanlon (2008) proved elementary equivalence implies isomorphism for all infinite function fields over prime fields by using biinterpretability of such fields.

For geometric case, that is, for function fields over algebraically cosed fields, Duret (1992) proved the folowing, using the facts on plane algebraic curves.

Let K and L be function fields over an algebraically closed field κ with transcendence degree 1. *Suppose $K \equiv L$.*

1. *If one of them has genus different from* 1, *then $K \cong L$.*
2. *If κ has characteristic* 0 *and one of them is an elliptic function field with no complex multiplication, then $K \cong L$.*

Later Pop (2002) proved the followings:

Let $K|\kappa$ and $L|\lambda$ be elementarily equivalent function fields over algebraically closed fields κ, respectively λ. Suppose $K \equiv L$. Then

1. $\mathrm{td}(K|\kappa)$ *equals* $\mathrm{td}(L|\lambda)$.
2. *Suppose $K|\kappa$ is of general type. Then there exist function subfields $K_0 \hookrightarrow K|\kappa_0$ and $L_0|\lambda_0 \hookrightarrow L|\lambda$ such that $K = K_0\kappa$ and $L = L_0\lambda$, and $K_0|\kappa_0 \cong L_0|\lambda_0$ as function fields.*
   In particular, if $\kappa \cong \lambda$ are isomorphic, then $K|\kappa \cong L|\lambda$ are isomorphic as function fields.

We note that it remains open whether or not $K \equiv L$ implies that $\kappa \cong \lambda$. (Note $\kappa \equiv \lambda$.) It is unknown whether $\mathbb{Q}^{alg}(X) \equiv \mathbb{C}(X)$. (It is known that $\mathbb{Q}^{alg}[X] \not\equiv \mathbb{C}[X]$.)

2 Rational function fields in one indeterminate

We begin with the following simple lemma.

Lemma 1 *Let $K|k$ be a function field $K = k(x, y)$ with $f(x, y) = 0$ for an irreducible polynomial $f(X, Y)$ over k.*

We let $\deg_X(f) = m$ and $\deg_Y(f) = n$. ($\deg_X(f)$ and $\deg_Y(f)$ denote the degree of $f(X, Y)$ with respect to X and Y respectively.) Let t be an indeterminate. Then

$K \cong k(t)$ iff there exist polynomials $p, q, r, s \in k[X]$ with $\deg(p), \deg(q) \leq n$ and $\deg(r), \deg(s) \leq m$ such that $f(p/q, r/s) = 0$ as polynomials.

Proof. The converse follows from Lüroth's theorem. We suppose that $K \cong k(t)$. Let $x \leftrightarrow \alpha = p(t)/q(t)$ and $y \leftrightarrow \beta = r(t)/s(t)$, where p and q (respectively r and s) are polynomials in $k[t]$ and have no common factors. We have $k(p/q, r/s) = k(t)$. Of course we have that $f(p/q, r/s) = 0$ as polynomials.

We know that $q(X) - \alpha p(X)$ is an irreducible polynomial over $k(\alpha)$, hence $[k(t) : k(\alpha)] = \max(\deg(p), \deg(q))$. Since $[K : k(x)] = n$, we have $[k(t) : k(\alpha)] = n$, hence $\deg(p), \deg(q) \leq n$. Similarly we have $\deg(r), \deg(s) \leq m$. □

Proposition 2 *Let k be an algebraically closed field or a finite extension of a prime field. Let t be an indeterminate. Let K be a function field over k. Then $K \equiv k(t)$ implies $K \cong k(x)$.*

Proof. Suppose $K \equiv k(t)$. We easily have $\mathrm{ch}(K) = \mathrm{ch}(k(t))$. Since the transcendence degree and transcendence bases of k are definable, we have that $\mathrm{td}(K|k) = 1$ and the constant field k is definable.

Let $K = k(x, y)$ and $f(x, y) = 0$ for some irreducible polynomial $f(X, Y)$ with $\deg_X(f) = m$ and $\deg_Y(f) = n$. Suppose $K \ncong k(x)$. By the lemma, $f(X, Y)$ has no rational parametrisations, $p(X)/q(X), r(X)/s(X)$ with $\deg(p), \deg(q) \leq n$ and $\deg(r), \deg(s) \leq m$.

If K is defined over a finite extension of prime field, we have $k(t) \models \exists X, Y f(X, Y) = 0$. In general it does not hold. So quantifying the coefficients, we have that in $k(t)$, there are α, β and a polynomial $f'(X, Y)$ with $\deg_X(f') = m$ and $\deg_Y(f') = n$ such that $f'(\alpha, \beta) = 0$ holds and $f'(X, Y)$ has no rational parametrisations, $p(X)/q(X), r(X)/s(X)$ with $\deg(p), \deg(q) \leq n$ and $\deg(r), \deg(s) \leq m$. By Lüroth's theorem, there exists $t' \in k(t)$ such that $k(t') = k(\alpha, \beta)$. If we write $\alpha = p(t')/q(t')$ and $\beta = r(t')/s(t')$ then we have $\deg(p), \deg(q) > n$ or $\deg(r), \deg(s) > m$. From the proof of the lemma, we have a contradiction. □

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