

HEIGHT INEQUALITY FOR CURVES OVER FUNCTION FIELDS

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0. INTRODUCTION

The geometric case of the height inequality (cf. [V3]) was discussed at the conference. By the geometric case, we mean that the global field in the question is a function field of one variable over complex number field $\mathbb{C}$, instead of a number field which is a finite extension of $\mathbb{Q}$. Hence in our geometric case, problem is algebro-geometric nature. Since we consider geometry over $\mathbb{C}$, our problem is also complex analytic nature.

Our method belongs to the second view point. We use techniques of classical function theory such as Ahlfors' theory of covering surfaces, area-length method to prove the height inequality for curves in the geometric case, which is the main result of our discussion.

1. NOTATIONS

Let B be a smooth, projective, connected curve over $\mathbb{C}$. Let k be the function field of B . Let $S \subset B$ be a finite set of points which will be fixed throughout. Let X be a smooth, projective, geometrically connected variety over k and $D \subset X$ be an effective divisor. Let L be a line bundle on X .

Following P. Vojta [V3], we define the functions

$$h_{L,k}(P), N_{k,S}(D, P), N_{k,S}^{(1)}(D, P), m_{k,S}(D, P), d_k(P)$$

as follows.

First, take a model of X over B , i.e., smooth variety $\mathfrak{X}$ projective over B such that the generic fiber is X . Then by taking the normalization of the Zariski closure of $P \in \mathfrak{X}(\bar{k}) = X(\bar{k})$, we can associate the following commutative diagram.

$$\begin{array}{ccc} B' & \xrightarrow{f_P} & \mathfrak{X} \\ p \downarrow & & \downarrow \pi \\ B & \xlongequal{\quad} & B \end{array}$$

Here B' is the curve whose function field is isomorphic to $k(P)$.

Let $\mathfrak{D} \subset \mathfrak{X}$ and $\mathfrak{L}$ be an extension of $D \subset X$ and L to $\mathfrak{X}$, respectively.

Put

$$\begin{aligned} h_{\mathfrak{L},k}(P) &= \frac{1}{\deg p} \deg f_P^* \mathfrak{L}, \\ N_{k,S}(\mathfrak{D}, P) &= \frac{1}{\deg p} \sum_{x \in B' \setminus p^{-1}(S)} \text{ord}_x f_P^* \mathfrak{D} \quad (P \in X(\bar{k}) \setminus D), \\ N_{k,S}^{(1)}(\mathfrak{D}, P) &= \frac{1}{\deg p} \sum_{x \in B' \setminus p^{-1}(S)} \min(1, \text{ord}_x f_P^* \mathfrak{D}) \quad (P \in X(\bar{k}) \setminus D) \end{aligned}$$

and

$$m_{k,S}(\mathfrak{D}, P) = \frac{1}{\deg p} \sum_{x \in p^{-1}(S)} \text{ord}_x f_P^* \mathfrak{D} \quad (P \in X(\bar{k}) \setminus D).$$

If we replace the models $\mathfrak{X}$, $\mathfrak{D}$ and $\mathfrak{L}$ to other models $\mathfrak{X}'$, $\mathfrak{D}'$ and $\mathfrak{L}'$, we have

$$\begin{aligned} h_{\mathfrak{L},k}(P) &= h_{\mathfrak{L}',k}(P) + O(1), \quad N_{k,S}(\mathfrak{D}, P) = N_{k,S}(\mathfrak{D}', P) + O(1), \\ N_{k,S}^{(1)}(\mathfrak{D}, P) &= N_{k,S}^{(1)}(\mathfrak{D}', P) + O(1), \quad m_{k,S}(\mathfrak{D}, P) = m_{k,S}(\mathfrak{D}', P) + O(1), \end{aligned}$$

where $O(1)$ are bounded terms independent of $P \in X(\bar{k})$. Hence we write as

$$\begin{aligned} h_{L,k}(P) &= h_{\mathfrak{L},k}(P) + O(1), \quad N_{k,S}(D, P) = N_{k,S}(\mathfrak{D}, P) + O(1), \\ N_{k,S}^{(1)}(D, P) &= N_{k,S}^{(1)}(\mathfrak{D}, P) + O(1), \quad m_{k,S}(D, P) = m_{k,S}(\mathfrak{D}, P) + O(1). \end{aligned}$$

Finally, put

$$d_k(P) = \frac{1}{\deg p} \deg(\text{ram } p),$$

where $\text{ram } p \subset B'$ is the ramification divisor of p .

2. MAIN CONJECTURE

Ofcourse, we have equality

$$(2.1) \quad h_{L(D),k}(P) = N_{k,S}(D, P) + m_{k,S}(D, P) + O(1),$$

where $L(D)$ is the line bundle associated to D . Our problem is that *What happens if we replace the right hand side of (2.1) by the term $N_{k,S}^{(1)}(D, P)$?*

In this case, we can't hope any equality. Instead, we hope the inequality like

$$(2.2) \quad h_{K_X(D),k} \leq N_{k,S}^{(1)}(D, P) + d_k(P) + (\text{small error term}),$$

where K_X is the canonical line bundle on X .

Heuristic proof of (2.2):

1. We only consider k rational points $P \in X(k)$ for simplicity. Let $\mathcal{M}$ be the connected component of the moduli space of sections of $\pi : \mathfrak{X} \rightarrow B$ containing the section $f_P : B \rightarrow \mathfrak{X}$.
2. For integers $k \geq 0$, put

$$\mathcal{M}_k = \{f' \in \mathcal{M} : \deg f'^*\mathfrak{D} - \#\text{supp}(f'^*\mathfrak{D}) \geq k\}.$$

Then $\mathcal{M}_k \subset \mathcal{M}$ is a Zariski closed subset and form a sequence

$$\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots.$$

3. For a generic $f' \in \mathcal{M}$, $f'(B)$ and $\mathfrak{D}$ would intersect transversely. Hence we hope

$$\deg f'^*\mathfrak{D} = \#\text{supp}(f'^*\mathfrak{D}),$$

which implies $\mathcal{M}_1 \subsetneq \mathcal{M}_0 = \mathcal{M}$ and $\text{codim}(\mathcal{M}_1, \mathcal{M}_0) \geq 1$.

4. More generally, we hope $\text{codim}(\mathcal{M}_{k+1}, \mathcal{M}_k) \geq 1$ for $k \geq 0$.
5. Hence, for $k = \dim \mathcal{M} + \varepsilon$, we hope “ $\mathcal{M}_k = \emptyset$ ”, which implies

$$\deg f_P^*\mathfrak{D} - \dim \mathcal{M} \leq \#\text{supp}(f_P^*\mathfrak{D}) + \varepsilon.$$

6. By the equality “ $\dim \mathcal{M} \doteq -h_{K_X, k}(P)$ ”, which seems to be true, and the fact $\#S < \infty$ we get

$$h_{K_X(D), k}(P) \leq N_{k, S}^{(1)}(D, P) + \varepsilon + O(1)$$

as desired.

Unfortunately, the above inequality (2.2) is not correct in general, and it seems very difficult to justify the above argument.

The precise conjecture is

Conjecture ([V3]). *Let X be a smooth projective variety over k , let D be a normal crossings divisor on X , let L be a big line bundle on X , let $r \in \mathbb{Z}_{>0}$ and let $\varepsilon > 0$. Then there exists a proper Zariski closed subset $Z = Z(k, S, X, D, L, r, \varepsilon) \subsetneq X$ such that*

$$h_{K_X(D), k}(P) \leq N_{k, S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{L, k}(P) + O_\varepsilon(1)$$

for all $P \in X(\bar{k}) \setminus Z$ with $[k(P) : k] < r$.

Remark 2.3. (1) Using Arakelov geometry, the number field case of the above conjecture can be formulated in the same manner (see [V3]).

(2) When X is a curve, Z is a union of points. Hence $P \in Z$ satisfies $h_{K_X(D), k}(P) < O_\varepsilon(1)$, which means that we don't need Z in this case.

3. MAIN RESULT

We can prove the one dimensional case of above conjecture.

Theorem . *Let X be a smooth projective curve over k , let D be a reduced divisor on X , let L be a big line bundle on X and let $\varepsilon > 0$. Then we have*

$$(3.1) \quad h_{K_X(D),k}(P) \leq N_{k,S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{L,k}(P) + O_\varepsilon(1)$$

for all $P \in X(\bar{k}) \setminus D$.

Remark 3.2. (1) *In our case, we don't need r in above conjecture.*

(2) *When (X, D) has splitting, i.e., there exists (X_0, D_0) on $\mathbb{C}$ such that $(X, D) = (X_0 \otimes_{\mathbb{C}} k, D_0 \otimes_{\mathbb{C}} k)$, our theorem is an easy consequence of Hurwitz's formula.*

The following corollary directly follows from our theorem (cf. [V1], [V2]).

Corollary . *Let X be a smooth projective curve over k and let $\varepsilon > 0$. Then we have*

$$h_{K_X,k}(P) \leq (1 + \varepsilon)d_k(P) + O_\varepsilon(1)$$

for all $P \in X(\bar{k})$.

4. ABOUT PROOF

Our proof is based on Ahlfors' theory of covering surfaces [A], which is an important theory in classical complex analysis. Roughly speaking, main result of Ahlfors' theory is kind of Hurwitz's formula for non-proper covering of surfaces.

First, we reduce the general case of our theorem to the special case that $X = \mathbb{P}_k^1$ and $D = (P_1) + \cdots + (P_q)$ where P_i are distinct k -rational points of $\mathbb{P}_k^1$. This reduction step is algebraic; using a ramified cover and the ramification formula.

Then this special case is equivalent to the following; *Let $a_1, \dots, a_q$ be distinct rational functions on B , let $\varepsilon > 0$. Then there is a positive constant $C(\varepsilon) > 0$ such that for all covering $\pi : Y \rightarrow B$ and rational function f on Y such that $f \neq a_i \circ \pi$, we have*

$$(4.1) \quad (q - 2 - \varepsilon) \deg f \leq \sum_{i=1}^q \#\{z \in Y; a_i \circ \pi(z) = f(z)\} \\ + \deg(\text{ram } \pi) + C(\varepsilon) \deg \pi.$$

To prove (4.1), we first divide B by sufficiently small, finite Jordan domains Δ_λ such that

$$B = \bigcup_{\lambda: \text{finite}} \overline{\Delta_\lambda}, \quad \Delta_\lambda \cap \Delta_{\lambda'} = \emptyset \text{ for } \lambda \neq \lambda'.$$

If each Δ_λ is small enough, then the move of rational functions a_i on Δ_λ are very small, hence the situation is close to the constant case. As already mentioned above, if rational functions a_i are constant, then (4.1) can be proved by Hurwitz's formula. In our case, since Δ_λ is non-compact, we use Ahlfors' theory instead of Hurwitz's formula to prove localized inequality of (4.1) on Δ_λ . Then we sum all these localized inequality over λ to obtain (4.1). In this part, we also need so-called area-length method, which is an important technique in complex analysis.

Our inequality (4.1) is an algebraic analogue of a long standing conjecture, called defect relation for small functions, in one dimensional value distribution theory. And above proof is a modification of an argument in [Y1].

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