

Two-fold ground states of the Pauli-Fierz Hamiltonian including spin

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Abstract

The Pauli-Fierz Hamiltonian describes an interaction between a low energy electron and photons. Existence of ground states has been established. The purpose of this talk is to show that its ground states is *exactly* two-fold in a weak coupling region.

1 The Pauli-Fierz Hamiltonian

This is a joint work¹ with Herbert Spohn². The Hamiltonian in question is the Pauli-Fierz Hamiltonian in nonrelativistic QED with spin, which will be denoted by H acting on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}.$$

Here $L^2(\mathbb{R}^3; \mathbb{C}^2)$ denotes the Hilbert space for the electron with spin σ , where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli spin 1/2 matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\mathcal{F}$ is the symmetric Fock space for the photons given by $\mathcal{F} = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3 \times \{1, 2\}))_{\text{sym}}^n$. Here $(\cdots)_{\text{sym}}^n$ denotes the n -fold symmetric tensor product of $(\cdots)$ with $(\cdots)_{\text{sym}}^0 = \mathbb{C}$.

The photons live in $\mathbb{R}^3$ and have helicity ± 1 . The Fock vacuum is denoted by Ω . The photon field is represented in $\mathcal{F}$ by the two-component Bose field $a(k, j)$, $j = 1, 2$, with commutation relations

$$[a(k, j), a^*(k', j')] = \delta_{jj'} \delta(k - k'),$$

¹ [12].

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$$[a(k, j), a(k', j')] = 0, \quad [a^*(k, j), a^*(k', j')] = 0.$$

The energy of the photons is given by

$$H_f = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk,$$

i.e., H_f restricted to $(L^2(\mathbf{R}^3 \times \{1, 2\}))_{\text{symm}}^n$ is the multiplication by $\sum_{j=1}^n \omega(k_j)$, and the momentum of the photons is

$$P_f = \sum_{j=1,2} \int k a^*(k, j) a(k, j) dk.$$

Throughout units are such that $\hbar = 1$, $c = 1$. Physically $\omega(k) = |k|$. The case is somewhat singular and we assume that ω is continuous, rotation invariant, and that (1) $\inf_{k \in \mathbf{R}^3} \omega(k) \geq \omega_0 > 0$, (2) $\omega(k_1) + \omega(k_2) \geq \omega(k_1 + k_2)$, (3) $\lim_{|k| \rightarrow \infty} \omega(k) = \infty$. A typical example is

$$\omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}, \quad m_{\text{ph}} > 0.$$

For a recent result of the massless case see [3]. The quantized transverse vector potential is defined through

$$A_\varphi(x) = \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e_j(k) (a^*(k, j) e^{-ikx} + a(k, j) e^{ikx}) dk.$$

Here e_1 and e_2 are polarization vectors which together with $\hat{k} = k/|k|$ form a standard basis in $\mathbf{R}^3$. $\varphi : \mathbf{R}^3 \rightarrow \mathbf{R}$ is a form factor which ensures an ultraviolet cutoff. It is assumed to be $\varphi(Rx) = \varphi(x)$ for an arbitrary rotation R , continuous, bounded with some decay at infinity, and normalized as $\int \varphi(x) dx = 1$. We will work with the Fourier transform $\hat{\varphi}(k) = (2\pi)^{-3/2} \int \varphi(x) e^{-ikx} dx$. It satisfies (1) $\hat{\varphi}(Rk) = \hat{\varphi}(k)$, (2) $\bar{\hat{\varphi}} = \hat{\varphi}$ for notational simplicity, (3) $\hat{\varphi}(0) = (2\pi)^{-3/2}$, and (4) the decay

$$\int (\omega(k)^{-2} + \omega(k)^{-1} + 1 + \omega(k)) |\hat{\varphi}(k)|^2 dk < \infty.$$

The quantized magnetic field is correspondingly

$$B_\varphi(x) = i \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} (k \times e_j(k)) (a^*(k, j) e^{-ikx} - a(k, j) e^{ikx}) dk.$$

With these preparation the Pauli-Fierz Hamiltonian, including spin, is defined by

$$H = \frac{1}{2} (-i\nabla_x \otimes 1 - eA_\varphi(x))^2 + 1 \otimes H_f - \frac{e}{2} \sigma \otimes B_\varphi(x). \quad (1.1)$$

Since obvious from the context we will drop the tensor notation $\otimes$.

2 Invariances

2.1 Total momentum

Let us define the total momentum by $P_{\text{total}} = -i\nabla_x + P_f$. We see that

$$[P_{\text{total}}, H] = 0. \quad (2.1)$$

(2.1) immediately implies that H has no ground state. Instead of H we consider the Hamiltonian with a fixed total momentum as follows. By (2.1), we see that (1.1) is decomposable with respect to the spectrum of P_{total} ,

$$H = \int_{\mathbb{R}^3}^{\oplus} H_p dp,$$

where

$$H_p = \frac{1}{2}(p - P_f - eA_\varphi)^2 - \frac{e}{2}\sigma B_\varphi + H_f, \quad (2.2)$$

acting on $\mathbb{C}^2 \otimes \mathcal{F}$. Here $A_\varphi = A_\varphi(0)$ and $B_\varphi = B_\varphi(0)$. The total momentum $p \in \mathbb{R}^3$ is regarded as a parameter. Recently an adiabatic perturbation of the Hamiltonian (2.2) has been studied in [16]. We define

$$H_{p0} = \frac{1}{2}(p - P_f)^2 + H_f,$$

and $H_{Ip} = H_p - H_{p0}$. We have $\|H_{Ip}\psi\| \leq c_*(e)\|(H_{p0} + 1)\psi\|$, where

$$c_*(e) = c_* \left\{ |e| \left\{ \int \left(\frac{1}{\omega(k)^2} + \omega(k) \right) |\hat{\varphi}(k)|^2 dk \right\}^{1/2} + e^2 \int \left(\frac{1}{\omega(k)^2} + 1 \right) |\hat{\varphi}(k)|^2 dk \right\}$$

with some constant c_* . Then $|e| < e_*$ with a certain $e_* > 0$ implies $c_*(e) < 1$. In particular H_p is self-adjoint on $D(H_f) \cap D(P_f^2)$ for all $p \in \mathbb{R}^3$ and bounded from below, for $|e| < e_*$. The ground state energy of H_p is

$$E(p) = \inf \sigma(H_p) = \inf_{\psi \in D(H_p), \|\psi\|=1} (\psi, H_p \psi).$$

If $E(p)$ is an eigenvalue, the corresponding spectral projection is denoted by P_p . $\text{Tr} P_p$ is identical with the multiplicity of ground states. The bottom of the continuous spectrum is denoted by $E_c(p)$. Under our assumptions one knows that

$$E_c(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k)).$$

See [4, 5, 17]. Thus it is natural to set

$$\Delta(p) = E_c(p) - E(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k) - E(p)).$$

2.2 Total angular momentum

Let $\vec{n} \in \mathbb{R}^3$ be a unit vector. It follows that, for $\theta \in \mathbb{R}$,

$$e^{i(\theta/2)\vec{n}\cdot\theta}\sigma_\mu e^{-i(\theta/2)\vec{n}\cdot\theta} = (R\sigma)_\mu, \quad \mu = 1, 2, 3,$$

where $R = (R_{\mu\nu})_{1 \leq \mu, \nu \leq 3} = R(\vec{n}, \theta) \in \text{SO}(3)$ presents the rotation around $\vec{n}$ through an angle θ , and $(R\sigma)_\mu = \sum_{\nu=1,2,3} R_{\mu\nu}\sigma_\nu$. We define the field angular momentum relative to the origin by

$$J_f = \sum_{j=1,2} \int (k \times (-i\nabla_k)) a^*(k, j) a(k, j) dk$$

and the helicity by

$$S_f = i \int \hat{k} \{a^*(k, 2)a(k, 1) - a^*(k, 1)a(k, 2)\} dk.$$

Let $a^\sharp(f, j) = \int a^\sharp(k, j) f(k) dk$. It holds that

$$[a(f, 1), S_f] = -ia(\hat{k}f, 2), \quad [a(f, 2), S_f] = ia(\hat{k}f, 1),$$

$$[a^*(f, 1), S_f] = -ia^*(\hat{k}f, 2), \quad [a^*(f, 2), S_f] = ia^*(\hat{k}f, 1).$$

One sees that

$$e^{i\theta\vec{n}\cdot(J_f+S_f)} H_f e^{-i\theta\vec{n}\cdot(J_f+S_f)} = H_f,$$

$$e^{i\theta\vec{n}\cdot(J_f+S_f)} P_f e^{-i\theta\vec{n}\cdot(J_f+S_f)} = R P_f,$$

$$e^{i\theta\vec{n}\cdot(J_f+S_f)} A_\varphi e^{-i\theta\vec{n}\cdot(J_f+S_f)} = R A_\varphi.$$

Define the total angular momentum by

$$J_{\text{total}} = J_f + S_f + \frac{1}{2}\sigma.$$

It follows that

$$e^{i\theta\vec{n}\cdot J_{\text{total}}} H_{Rp} e^{-i\theta\vec{n}\cdot J_{\text{total}}} = \frac{1}{2} \{ (R\sigma) \cdot (Rp - R P_f - e R A_\varphi) \}^2 + H_f = H_p.$$

In particular $E(p) = E(Rp)$. Moreover taking $\vec{n} = \hat{p} = p/|p|$ we have

$$e^{i\theta\hat{p}\cdot J_{\text{total}}} H_p e^{-i\theta\hat{p}\cdot J_{\text{total}}} = H_p.$$

Formally we may say that H_p has a “field angular momentum+helcity+SU(2)” symmetry. It is easily seen that $\sigma(\hat{p} \cdot (J_f + S_f)) = \mathbb{Z}$ and $\sigma(\hat{p} \cdot \sigma) = \{-1, 1\}$. Thus

$$\sigma(\hat{p} \cdot J_{\text{total}}) = \mathbb{Z} + \frac{1}{2},$$

which is independent of p . Thus $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and H_p are decomposable as

$$\mathbb{C}^2 \otimes \mathcal{F} = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} \mathcal{H}(z),$$

and

$$H_p = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} H_p(z).$$

As our main result we state

Theorem 2.1 *Suppose $|e| < e_0$ with some constant e_0 given in (3.3), and $\Delta(p) > 0$. Then H_p has two orthogonal ground states, $\psi_{\pm}$, with $\psi_{\pm} \in \mathcal{H}(\pm 1/2)$.*

We emphasize that all our estimates on the allowed ranges for p and e do *not* depend on m_{ph} if we take $\omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}$.

3 A proof of Theorem 2.1

In what follows $\psi_p = \begin{pmatrix} \psi_{p+} \\ \psi_{p-} \end{pmatrix}$ denotes an *arbitrary* ground state of H_p . The number operator is defined by

$$N_{\text{f}} = \sum_{j=1,2} \int a^*(k, j) a(k, j) dk.$$

The following lemma is shown in [15]

Lemma 3.1 *Suppose $\Delta(p) > 0$. Then*

$$(\psi_p, N_{\text{f}} \psi_p) \leq 2e^2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk \|\psi_p\|^2.$$

We set

$$\theta(p) = 2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk.$$

Let P_{Ω} be the projection onto $\{\mathbb{C}\Omega\}$.

Lemma 3.2 *Suppose that $\Delta(p) > 0$ and $e^2 < 1/\theta(p)$. Then $(\psi_p, P_{\Omega} \psi_p) > 0$.*

Proof: Since $P_{\Omega} + N_{\text{f}} \geq 1$, we have

$$(\psi_p, P_{\Omega} \psi_p) \geq \|\psi_p\|^2 - \|N_{\text{f}}^{1/2} \psi_p\|^2 > (1 - e^2 \theta(p)) \|\psi_p\|^2.$$

Thus the lemma follows. □

Let $\varphi_+ = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}$ and $\varphi_- = \begin{pmatrix} 0 \\ \Omega \end{pmatrix}$, which are the ground states of H_{p0} with $p = (0, 0, 1)$ and $\varphi_{\pm} \in \mathcal{H}(\pm 1/2)$. Let us denote by P the projection onto $\{c_1\varphi_1 + c_2\varphi_2, c_1, c_2 \in \mathbb{C}\}$.

Let $\{\phi_i\}$ be a base of the space spanned by ground states of H_p and $\{\psi_j\}$ that of the complement.

Lemma 3.3 *Suppose $e^2 < 1/(3\theta(p))$. Then $\text{Tr} P_p \leq 2$.*

Proof: For $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, since $(\psi, P\psi) = |(\Omega, \psi_+)|^2 + |(\Omega, \psi_-)|^2 = (\psi, (1 \otimes P_{\Omega})\psi)$, we have $(\psi, (P + 1 \otimes N_f)\psi) = (\psi, 1 \otimes (P_{\Omega} + N_f)\psi) \geq \|\psi\|^2$. Hence $P + N_f \geq 1$. Then

$$\begin{aligned} \text{Tr}(P_p(1 - P)) &= \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p(1 - P)\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, (1 - P)\phi) \\ &\leq \sum_{\phi \in \{\phi_i\}} (\phi, N_f\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, P_p N_f\phi) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p N_f\phi) = \text{Tr}(P_p N_f). \end{aligned}$$

Thus $\text{Tr}(P_p(1 - P)) \leq \text{Tr}(P_p N_f)$. It follows that

$$\text{Tr}(P_p P) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p P\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, P_p \phi) \leq 2.$$

Thus $\text{Tr}(P_p P) \leq 2$. Moreover we have $\text{Tr}(P_p N_f) \leq e^2 \theta(p) \text{Tr} P_p$, since

$$\begin{aligned} \text{Tr}(P_p N_f) &= \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p N_f \phi) = \sum_{\phi \in \{\phi_i\}} (\phi, N_f \phi) \\ &\leq e^2 \theta(p) \sum_{\phi \in \{\phi_i\}} (\phi, \phi) = e^2 \theta(p) \text{Tr} P_p. \end{aligned}$$

Then $\text{Tr} P_p - \text{Tr}(P_p P) = \text{Tr} P_p(1 - P) \leq \text{Tr}(P_p N_f) \leq e^2 \theta(p) \text{Tr} P_p$. Hence it follows that $(1 - e^2 \theta(p)) \text{Tr} P_p \leq \text{Tr}(P_p P) \leq 2$. We have

$$\text{Tr} P_p \leq \frac{2}{1 - e^2 \theta(p)} < 3.$$

Thus the lemma follows. $\square$

We say that $\psi \in \mathcal{F}$ is real, if $\psi^{(n)}(k_1, j_1, \dots, k_n, j_n)$ is a real-valued function on $L^2(\mathbb{R}^{3n} \times \{1, 2\}^n)$ for all $n \geq 0$. The set of real ψ is denoted by $\mathcal{F}_{\text{real}}$. We define the set of reality-preserving operators $\mathcal{O}_{\text{real}}(\mathcal{F})$ as follows:

$$\mathcal{O}_{\text{real}}(\mathcal{F}) = \{A|A : \mathcal{F}_{\text{real}} \cap D(A) \longrightarrow \mathcal{F}_{\text{real}}\}.$$

It is seen that H_f and P_f are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Since, for all $k \in \mathbb{R}$ and $z \in \mathbb{R}^3$,

$$\begin{aligned} & ((H_{p0} + z)^k \psi)^{(n)}(k_1, j_1, \dots, k_n, j_n) \\ &= \left(\frac{1}{2} \left(p - \sum_{i=1}^n k_i \right)^2 + \sum_{i=1}^n \omega(k_i) + z \right)^k \psi^{(n)}(k_1, j_1, \dots, k_n, j_n), \end{aligned}$$

$(H_{p0} + z)^k$ is also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover A_φ and iB_φ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$.

Lemma 3.4 Suppose $|e| < e_*$. Let $x \in \mathbb{C}^2$. Then there exists $a(t) \in \mathbb{R}$ independent of x such that for $t \geq 0$

$$(x \otimes \Omega, e^{-t(H_p - E(p))} x \otimes \Omega)_{\mathcal{H}} = a(t)(x, x)_{\mathbb{C}^2}. \quad (3.1)$$

Proof: Note that $\|H_{Ip}(1 + H_{p0})^{-1}\| < 1$ for $|e| < e_*$. Then, by spectral theory, one has

$$\begin{aligned} e^{-t(H_p - E(p))} &= \lim_{n \rightarrow \infty} \left(1 + \frac{t}{n} (H_p - E(p)) \right)^{-n} \\ &= \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \left\{ \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2} \left(\sum_{k=0}^m \left(-\frac{t}{n} \widetilde{H_{Ip}} \right)^k \right) \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2} \right\}^n. \end{aligned}$$

Here

$$\begin{aligned} \widetilde{H_{Ip}} &= \widetilde{H_{IIP}} + i\sigma \cdot \widetilde{B}, \\ \widetilde{B} &= \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2} (iB_\varphi) \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2}, \\ \widetilde{H_{IIP}} &= \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2} (H_{IIP} - E(p)) \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2}, \\ H_{IIP} &= -e(p - P_f) \cdot A_\varphi + \frac{e^2}{2} A_\varphi^2. \end{aligned}$$

It is seen that

$$\widetilde{H_{Ip}}^2 = \widetilde{H_{IIP}} \widetilde{H_{IIP}} - \widetilde{B} \cdot \widetilde{B} + i\sigma \cdot (\widetilde{H_{IIP}} \widetilde{B} + \widetilde{B} \widetilde{H_{IIP}} - \widetilde{B} \wedge \widetilde{B}) = M + i\sigma \cdot L.$$

Here both of $M = \widetilde{H_{IIP}} \widetilde{H_{IIP}} - \widetilde{B} \cdot \widetilde{B}$ and $L = \widetilde{H_{IIP}} \widetilde{B} + \widetilde{B} \widetilde{H_{IIP}} - \widetilde{B} \wedge \widetilde{B}$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover

$$\widetilde{H_{Ip}}^3 = \widetilde{H_{IIP}} M - \widetilde{B} L + i\sigma \cdot (\widetilde{B} M + \widetilde{H_{IIP}} L - \widetilde{B} \wedge L),$$

where both of $\widetilde{H_{IIP}} M - \widetilde{B} L$ and $\widetilde{B} M + \widetilde{H_{IIP}} L - \widetilde{B} \wedge L$ are also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Thus, repeating above procedure, one obtains

$$\sum_{k=0}^m \left(-\frac{t}{n} \widetilde{H_{Ip}} \right)^k = a_m + i\sigma \cdot b_m,$$

where a_m and b_m are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Hence there exist $a_{nm} \in \mathcal{O}_{\text{real}}(\mathcal{F})$ and $b_{nm} \in \mathcal{O}_{\text{real}}(\mathcal{F})$ such that

$$\left\{ \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2} \left(\sum_{k=0}^m \left(-\frac{t}{n} \widehat{H_{1p}} \right)^k \right) \left(1 + \frac{t}{n} H_{0p} \right)^{-1/2} \right\}^n = a_{nm} + i\sigma \cdot b_{nm}.$$

Finally

$$(x \otimes \Omega, e^{-t(H_p - E(p))} x \otimes \Omega) = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (x, x)(\Omega, a_{nm}\Omega) + i \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (x, \sigma x)(\Omega, b_{nm}\Omega).$$

Since the left-hand side is real, the second term of the right-hand side vanishes and $a(t) = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (\Omega, a_{nm}\Omega)$ exists, which establishes the desired result. $\square$

Lemma 3.5 Suppose $|e| < e_*$ and $|e| < 1/\sqrt{\theta(p)}$. Then there exists $a > 0$ such that

$$PP_pP = aP.$$

Proof: Note that $P_p = s - \lim_{t \rightarrow \infty} e^{-t(H_p - E(p))}$. Thus by Lemma 3.4,

$$(x \otimes \Omega, P_p x \otimes \Omega) = \lim_{t \rightarrow \infty} (x \otimes \Omega, e^{-t(H_p - E(p))} x \otimes \Omega) = \lim_{t \rightarrow \infty} a(t)(x, x)$$

for all $x \in \mathbb{C}^2$. Since by Lemma 3.2, $(x \otimes \Omega, P_p x \otimes \Omega) \neq 0$ for some $x \in \mathbb{C}^2$, $\lim_{t \rightarrow \infty} a(t)$ exists and it does not vanish. For arbitrary $\phi_1, \phi_2 \in \mathcal{H}$, the polarization identity leads to $(\phi_1, PP_pP\phi_2) = a(\phi_1, P\phi_2)$. The lemma follows. $\square$

Lemma 3.6 Suppose $|e| < e_*$ and $|e| < 1/\sqrt{\theta(p)}$. Then $\text{Tr} P_p \geq 2$.

Proof: Suppose $\text{Tr} P_p = 1$. Let $P = |\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|$ and $P_p = |\psi_p\rangle\langle\psi_p|$. Lemma 3.5 yields that

$$\begin{aligned} PP_pP &= (|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|)|\psi_p\rangle\langle\psi_p|(|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|) \\ &= |(\varphi_+, \psi_p)|^2 |\varphi_+\rangle\langle\varphi_+| + |(\varphi_-, \psi_p)|^2 |\varphi_-\rangle\langle\varphi_-| \\ &\quad + (\varphi_+, \psi_p)(\psi_p, \varphi_-) |\varphi_+\rangle\langle\varphi_-| + (\varphi_-, \psi_p)(\psi_p, \varphi_+) |\varphi_-\rangle\langle\varphi_+| \\ &= a(|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|). \end{aligned} \tag{3.2}$$

It follows that $(\varphi_+, \psi_p)(\psi_p, \varphi_-) = 0$. Let us assume $(\psi_p, \varphi_-) = 0$. It implies in terms of (3.2) that $|(\varphi_+, \psi_p)|^2 |\varphi_+\rangle\langle\varphi_+| = a(|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|)$. This contradicts $(\varphi_+, \psi_p) \neq 0$ and $a \neq 0$. Thus the lemma follows. $\square$

We define

$$e_0 = \inf \left\{ |e| \mid |e| < 1/\sqrt{3\theta(p)}, |e| < e_* \right\}. \quad (3.3)$$

A proof of Theorem 2.1

By Lemma 3.6, $\text{Tr}P_p \geq 2$, and by Lemma 3.3, $\text{Tr}P_p \leq 2$. Hence $\text{Tr}P_p = 2$ follows. Without loss of generalization we may assume that $p = (0, 0, 1)$. Then $\varphi_{\pm} \in \mathcal{H}(\pm 1/2)$. Let $\psi_{\pm}$ be ground states of H_p such that $\psi_+ \in \mathcal{H}(z)$ and $\psi_- \in \mathcal{H}(z')$ with some $z, z' \in \mathbb{Z} + 1/2$. Since $PP_pP = aP$ we have $(\varphi_{\pm}, P_p\varphi_{\pm}) = a > 0$. Let $Q_{\pm}$ be the projections to $\mathcal{H}(\pm 1/2)$. Then $Q_+P_p\varphi_+ \neq 0$ and $Q_-P_p\varphi_- \neq 0$. The alternative $Q_+\psi_+ \neq 0$ or $Q_+\psi_- \neq 0$ holds, or the alternative $Q_-\psi_+ \neq 0$ or $Q_-\psi_- \neq 0$ holds. We may set $Q_+\psi_+ \neq 0$. Then $\psi_+ \in \mathcal{H}(+1/2)$ and $\psi_- \in \mathcal{H}(-1/2)$. The theorem follows. $\square$

4 Confining potentials

In this section we set $\omega(k) = |k|$ and

$$H = \frac{1}{2}(-i\nabla_x - eA_{\varphi}(x))^2 + H_f - \frac{e}{2}\sigma B_{\varphi}(x) + V.$$

Let V be relatively bounded with respect to $-\Delta/2$ with a relative bound strictly smaller than one. It has been established in [10, 11] that H is self-adjoint on $D(-\Delta) \cap D(H_f)$ and bounded from below, for *arbitrary* e . A confining potential V breaks the total momentum invariance,

$$[P_{\text{total}}, H] \neq 0. \quad (4.1)$$

Existence of ground states of H is expected by (4.1). Actually by many authors it has been established that H has ground states, e.g., [1, 6, 7, 8, 14, 13], and in a spinless case, the ground state is unique [9].

Let $H_0 = H_{\text{el}} + H_f$ and $H_{\text{el}} = \frac{1}{2}p^2 + V$. We set $E = \inf \sigma(H)$, $E_{\text{el}} = \inf \sigma(H_{\text{el}})$ and $\Sigma_{\text{el}} = \inf \sigma_{\text{ess}}(H_{\text{el}})$.

We define a class of external potentials.

Definition 4.1 (1) We say $V = Z + W \in V_{\text{exp}}$ if the following (i)–(iv) hold, (i) $Z \in L^1_{\text{loc}}(\mathbb{R}^3)$, (ii) $Z > -\infty$, (iii) $W < 0$, (iv) $W \in L^p(\mathbb{R}^3)$ for some $p > 3/2$.

(2) We say $V \in V(m)$, $m \geq 1$, if (i) $V \in V_{\text{exp}}$, (ii) $Z(x) \geq \gamma|x|^{2m}$, outside a compact set for some positive constant γ .

(3) We say $V \in V(0)$, $m \geq 1$, if (i) $V \in V_{\text{exp}}$, (ii) $\liminf_{|x| \rightarrow \infty} Z(x) > \inf \sigma(H)$.

We assume that V satisfies that (1) $\|Vf\| \leq a\|(p^2/2)f\| + b\|f\|$ with some $a < 1$ and some $b \geq 0$, (2) $V \in V(m)$ with some $m \geq 0$, (3) $V(x) = V(-x)$, (4) $\Sigma_{\text{el}} - E_{\text{el}} > 0$ and the ground state ϕ_0 of H_{el} is unique and real.

(1) guarantees self-adjointness of H , (2) derives a boundedness of $\|x|\psi_0\|$ for ground states ψ_0 of H , and (3) will be needed to estimate a lower bound of the multiplicity of ground states of H . (4) ensures that H has ground states and H_0 has twofold ground states. Actually H_0 has the two ground states, $\phi_+ = \begin{pmatrix} \phi_0 \otimes \Omega \\ 0 \end{pmatrix}$ and $\phi_- = \begin{pmatrix} 0 \\ \phi_0 \otimes \Omega \end{pmatrix}$.

Let P_{ϕ_0} denote the projection onto $\{\mathbb{C}\phi_0\}$. Define

$$P = P_{\phi_0} \otimes P_{\Omega}, \quad Q = P_{\phi_0}^{\perp} \otimes P_{\Omega}.$$

Furthermore P_e denotes the projection onto the space spanned by ground states of H . Let ψ be arbitrary ground state of H . It is proven in [1] that

$$\|N_f^{1/2}\psi\|^2 \leq \theta_1(e)\|x|\psi\|^2, \quad (4.2)$$

and in [2, 12] that

$$\|x|^k\psi\|^2 \leq \theta_2(e)\|\psi\|^2. \quad (4.3)$$

Then together with (4.2) and (4.3), we have

$$\|N_f^{1/2}\psi\|^2 \leq \theta_1(e)\theta_2(e)\|\psi\|^2. \quad (4.4)$$

Suppose $\Sigma_{\text{el}} - E > 0$. Then there exists $\theta_3(e)$ such that

$$\|Q\psi\|^2 \leq \theta_3(e)\|\psi\|^2. \quad (4.5)$$

Note that $\lim_{|e| \rightarrow 0} \theta_j(e) = 0$.

Lemma 4.2 Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1$. Then $(\psi_0, P\psi_0) > 0$.

Proof: It follows from (4.4), (4.5) and $P \geq 1 - N_f - Q$. □

Lemma 4.3 Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr}P_e \leq 2$.

Proof: It can be proven in the similar way as Lemma 3.3. □

Next we estimate $\text{Tr}P_e$ from below using the realness argument used in the previous section. Let F denote the Fourier transformation on $L^2(\mathbb{R}^3)$. We define the unitary operator $\mathcal{O}$ on $\mathcal{H}$ by $\mathcal{O} = (F \otimes 1)e^{ix \otimes P_f}$. Then $\mathcal{O}$ maps $D(-\Delta) \cap D(H_f)$ onto $D(|x|^2) \cap D(H_f)$ with

$$\widetilde{H} = \mathcal{O}H\mathcal{O}^{-1} = \frac{1}{2}(x - P_f - eA(0))^2 + \widetilde{V} + H_f - \frac{e}{2}\sigma \cdot B(0).$$

Here $\widetilde{V}$ is defined by

$$\widetilde{V}f = FVF^{-1}f = \widehat{V} * f$$

where $*$ denotes the convolution. By the assumption $V(x) = V(-x)$ we see that $\widetilde{V}$ is a reality preserving operator. Let

$$\widetilde{H}_0 = \frac{1}{2}(x - P_f)^2 + H_f + \widetilde{V}.$$

Lemma 4.4 *We have $(\widetilde{H}_0 - z)^{-n} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$ for all $z \in \mathbb{R}$ with $z \notin \sigma(\widetilde{H}_0)$ and $n \in \mathbb{R}$.*

Proof: We have

$$(\widetilde{H}_0 - z)^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{-1+n} e^{-t\widetilde{H}_0} e^{tz} dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is enough to prove $e^{-t\widetilde{H}_0} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. Since by the Trotter product formula,

$$e^{-t\widetilde{H}_0} = s\text{-}\lim_{n \rightarrow \infty} \left(e^{-(t/n)(P_f - x)^2/2} F^{-1} e^{-(t/n)V} F \right)^n,$$

$$F^{-1} e^{-sV} F \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F})),$$

and

$$e^{-s(P_f - x)^2} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F})),$$

it follows that $e^{-t\widetilde{H}_0} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. The lemma follows. $\square$

From this lemma it follows that $(\widetilde{H}_0 - z)^{-1}, (\widetilde{H}_0 - z)^{-1/2} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. We decompose $\widetilde{H} = \widetilde{H} - E$ as $\widetilde{H} = \widetilde{H}_0 + \widetilde{H}_1$, where

$$\widetilde{H}_1 = -\frac{e}{2}(x - P_f)A_\varphi(0) - \frac{e}{2}A_\varphi(0)(x - P_f) + \frac{e^2}{2}A_\varphi^2(0) - \frac{e}{2}\sigma B_\varphi(0) - E.$$

Lemma 4.5 *There exists $e_c > 0$ such that for all $|e| < e_c$, $\text{Tr}P_e \geq 2$.*

Proof: First we prove $PP_eP = aP$ with some $a > 0$ in the similar way as Lemma 3.4 with H_p and H_{Ip} replaced by $\widetilde{H}$ and $\widetilde{H}_I$, respectively. Then the lemma follows from the proof of Lemma 3.6. $\square$

Theorem 4.6 *Suppose $\Sigma_{el} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr}P_e = 2$.*

Proof: It follows from Lemmas 4.3 and 4.5. $\square$

Suppose that V is rotation invariant. Let

$$\mathcal{J}_{\text{total}} = x \times (-i\nabla_x) + J_f + S_f + \frac{1}{2}\sigma.$$

Then we have for $\theta \in \mathbb{R}$, $\vec{n} \in \mathbb{R}^3$ with $|\vec{n}| = 1$,

$$e^{i\theta\vec{n} \cdot \mathcal{J}_{\text{total}}} H e^{-i\theta\vec{n} \cdot \mathcal{J}_{\text{total}}} = H.$$

Since $\sigma(\vec{n} \cdot \mathcal{J}_{\text{total}}) = \mathbb{Z} + 1/2$ for each $\vec{n}$, $\mathcal{H}$ and H are decomposable as $\mathcal{H} = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} \mathcal{H}(z)$, and $H = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} H(z)$. In the same way as the proof of Theorem 2.1 one can prove the following corollary.

Corollary 4.7 *Suppose that V is translation invariant, and $\Sigma_{el} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then H has two orthogonal ground states, $\psi_{\pm}$, with $\psi_{\pm} \in \mathcal{H}(\pm 1/2)$.*

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