

Uniform convexity, uniform non-squareness and
von Neumann-Jordan constant for Banach spaces

Mikio KATO (加藤 幹雄) and Yasuji TAKAHASHI (高橋 泰嗣)

Kyushu Institute of Technology and
Okayama Prefectural University

Introduction

In connection with the famous work [7] of Jordan and von Neumann concerning inner products Clarkson [2] introduced the von Neumann-Jordan (NJ-) constant for Banach spaces X . Despite its fundamental nature very little is known on the NJ-constant by now. This note is a résumé of some recent results of the authors [12, 15] on the NJ-constant especially concerning some geometrical properties of Banach spaces such as uniform convexity, uniform non-squareness, and also super-reflexivity.

1. Definitions and preliminary results

The von Neumann-Jordan constant for a Banach space X ([2]), we denote it by $C_{NJ}(X)$, is defined to be the smallest constant C for which

$$(1.1) \quad \frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

hold for all $x, y \in X$ with $(x, y) \neq (0, 0)$. (Note that the left and right-hand side inequalities in (1.1) are equivalent; indeed, put $x+y=u$, $x-y=v$). The following facts are easily seen:

A. Proposition. (i) $C_{NJ}(X) = 2^{2/t-1}$, $1 \leq t \leq 2$, if and only if

$$\|A : l_2^2(X) \rightarrow l_2^2(X)\| = 2^{1/t},$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $l_2^2(X)$ denotes the X -valued l_2^2 -space; and hence

(ii) $C_{NJ}(X') = C_{NJ}(X)$, where X' is the dual space of X . (This was observed for L_p in Clarkson [2].)

Let us recall some classical and recent results in [7], [2], and [10], [9] (see also [11]), where the NJ-constant is calculated for some concrete Banach spaces:

B. Theorem (i) $1 \leq C_{NJ}(X) \leq 2$ for any Banach space X ; and $C_{NJ}(X) = 1$ if and only if X is a Hilbert space (Jordan and von Neumann [7]).

(ii) Let $1 \leq p \leq \infty$. Then, $C_{NJ}(L_p) = 2^{2/t-1}$, where $t = \min\{p, p'\}$, $1/p + 1/p' = 1$ (Clarkson [2]; see also [10]).

(iii) Let $1 \leq p, q \leq \infty$. Then, for $L_p(L_q)$ (L_q -valued L_p -space on arbitrary measure spaces), $C_{NJ}(L_p(L_q)) = 2^{2/t-1}$, where $t = \min\{p, q, p', q'\}$, $1/p + 1/p' = 1/q + 1/q' = 1$; and for the Sobolev space $W_p^k(\Omega)$, $C_{NJ}(W_p^k(\Omega)) = 2^{2/t-1}$, where $t = \min\{p, p'\}$ (Kato and Miyazaki [10]).

(iv) For $E = C_c(K)$ resp. $C_b(K)$ (the spaces of continuous functions on a locally compact Hausdorff space K which have compact support resp. are bounded), $C_{NJ}(E) = 2$ (Kato and Miyazaki [9]).

The above results (iii) (in particular, (ii)) can be obtained in a more simple way than [10] by using arguments in Hashimoto, Kato and Takahashi [8; Corollary 3.6; see also Theorem 3.2].

Let us recall some definitions. A Banach space X is called:

(i) *strictly convex* if $\|(x + y)/2\| < 1$ whenever $\|x\| = \|y\| = 1$, $x \neq y$,

(ii) *uniformly convex* provided for each ε ($0 < \varepsilon < 2$) there exists a $\delta > 0$ such that $\|(x + y)/2\| < 1 - \delta$ whenever $\|x - y\| \geq \varepsilon$, $\|x\| = \|y\| = 1$,

(iii) $(2, \varepsilon)$ -convex, $\varepsilon > 0$, (cf. [13]) provided $\min\{\|x + y\|, \|x - y\|\} \leq 2(1 - \varepsilon)$ whenever $\|x\| = \|y\| = 1$,

(iv) *uniformly non-square* ([5]; cf. [1], [3]) if there is a $\delta > 0$ such that there do not exist x and y in the closed unit ball of X for which $\|(x + y)/2\| > 1 - \delta$ and $\|(x - y)/2\| > 1 - \delta$. (Note that uniform non-squareness is equivalent to $(2, \varepsilon)$ -convexity.)

A Banach space Y is said to be *finitely representable* in X if for any $\lambda > 1$ and for each finite-dimensional subspace F of Y , there is an isomorphism T of F into X for which

$$\lambda^{-1} \|x\| \leq \|Tx\| \leq \lambda \|x\| \quad \text{for all } x \in F.$$

X is said to be *super-reflexive* ([6]; cf. [1], [3], [13], [14]) if no non-reflexive Banach space is finitely representable in X .

Super-reflexive spaces are characterized as those uniformly convexifiable:

C. Theorem (Enflo [4]; cf. [1], [3], [14]). A Banach space X is super-reflexive if and only if X admits an equivalent uniformly convex norm.

2. Uniform convexity, super-reflexivity and von Neumann-Jordan constant

We begin with the following proposition which will give effective examples later.

2.1 Proposition. Let $\lambda > 1$. Let $X_{2, \lambda}$ be the space l_2 equipped with the norm $\|x\|_{2, \lambda} := \max\{\|x\|_2, \lambda \|x\|_\infty\}$. Then:

(i) $X_{2, \lambda}$ is isomorphic to a Hilbert space and

$$C_{NJ}(X_{2, \lambda}) = \min\{\lambda^2, 2\}.$$

(ii) $X_{2, \lambda}$ is not strictly convex for any $\lambda > 1$.

Proof (sketch). (i) Since

$$\|x\|_2 \leq \|x\|_{2, \lambda} \leq \lambda \|x\|_2 \quad \text{for all } x \in X_{2, \lambda},$$

we have

$$\begin{aligned} \|x+y\|_{2, \lambda}^2 + \|x-y\|_{2, \lambda}^2 &\leq \lambda^2 (\|x+y\|_2^2 + \|x-y\|_2^2) \\ &= \lambda^2 2 (\|x\|_2^2 + \|y\|_2^2) \\ &\leq \lambda^2 2 (\|x\|_{2, \lambda}^2 + \|y\|_{2, \lambda}^2), \end{aligned}$$

or $C_{NJ}(X_{2, \lambda}) \leq \lambda^2$. By considering $x = (1/\lambda, 1/\lambda, 0, \dots)$ and $y = (1/\lambda, -1/\lambda, 0, \dots) \in X_{2, \lambda}$, we have $C_{NJ}(X_{2, \lambda}) = \min\{\lambda^2, 2\}$.

(ii) To see that $X_{2, \lambda}$ is not strictly convex, take an α satisfying $(1/\lambda)^2 + \alpha^2 \leq 1$ and $0 < \alpha \leq 1/\lambda$; then put $x = (1/\lambda, 0, 0, \dots)$ and $y = (1/\lambda, \alpha, 0, \dots)$.

Now, in the following two theorems we see that uniform convexity is nearly characterized by the condition $C_{NJ}(X) < 2$.

2.2 Theorem (i) If X is uniformly convex, then $C_{NJ}(X) < 2$; the converse is not true; indeed,

(ii) For any $\varepsilon > 0$ there exists a Banach space X (isomorphic to a Hilbert space) with $C_{NJ}(X) < 1 + \varepsilon$ which is not even strictly convex.

Proof (sketch). (i) Let X be uniformly convex. Let ε be any positive number with $0 < \varepsilon < 2^{1/2}$. Then, there exists a $\delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply

$$(2.1) \quad \|(x + y)/2\|^2 \leq (1 - \delta) \{(\|x\|^2 + \|y\|^2)/2\}$$

(cf. [1], p. 190). Now, let x and y be any elements in X with $\|x\|^2 + \|y\|^2 = 1$. We first assume that $\|x - y\| \geq \varepsilon$. Then, using (2.1), we have

$$(2.2) \quad \|x + y\|^2 + \|x - y\|^2 \leq 2(2 - \delta).$$

Next, if $\|x - y\| \leq \varepsilon$, we have

$$(2.3) \quad \begin{aligned} \|x + y\|^2 + \|x - y\|^2 &\leq 2(\|x\|^2 + \|y\|^2) + \varepsilon^2 \\ &\leq 2(1 + \varepsilon^2/2). \end{aligned}$$

Consequently, by (2.2) and (2.3) we obtain

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq 1 + \max\{1 - \delta, \varepsilon^2/2\},$$

or $C_{NJ}(E) < 2$.

(ii) By Proposition 2.1, for the spaces $X_{2,\lambda}$ ($\lambda > 1$) we have $C_{NJ}(X_{2,\lambda}) \rightarrow 1$ as $\lambda \rightarrow 1$, whereas $X_{2,\lambda}$ is not strictly convex.

Although the condition $C_{NJ}(X) < 2$ does not even imply strict convexity for X , it assures the existence of an equivalent norm on X for which X becomes uniformly convex (cf. Theorem C):

2.3 Theorem. Let $C_{NJ}(X) < 2$. Then, X is super-reflexive; the converse is not true.

Proof. Assume $C := C_{NJ}(X) < 2$. Let x and y be any elements in X with $\|x\| = \|y\| = 1$. Then,

$$\begin{aligned} \min_{\varepsilon_i = \pm 1} \|\varepsilon_1 x + \varepsilon_2 y\| &\leq \left\{ \frac{1}{2} (\|x + y\|^2 + \|x - y\|^2) \right\}^{1/2} \\ &\leq C^{1/2} (\|x\|^2 + \|y\|^2)^{1/2} \\ &= (2C)^{1/2}, \end{aligned}$$

that is, X is $(2, \varepsilon)$ -convex with some ε , or equivalently uniformly non-square, which implies that X is super-reflexive (James [6]; see also [1], [13]).

For the latter assertion, consider the space $X_{2,\sqrt{2}}$. Indeed, $X_{2,\sqrt{2}}$ is isomorphic to a Hilbert space and hence super-reflexive, whereas $C_{NJ}(X_{2,\sqrt{2}}) = 2$ by Proposition 2.1. (l_1^n and l_∞^n are also

such examples.)

2.4 Definition. Let $\tilde{C}_{NJ}(X)$ be the infimum of all NJ-constants for equivalent norms of X .

Theorems 2.2 and 2.3 assert that super-reflexivity is characterized by the condition $\tilde{C}_{NJ}(X) < 2$.

2.5 Theorem. The following are equivalent:

- (i) $\tilde{C}_{NJ}(X) < 2$.
- (ii) X is super-reflexive.
- (iii) X admits an equivalent uniformly convex norm.
- (iv) X admits an equivalent uniformly non-square norm.
- (v) X admits an equivalent uniformly smooth norm (cf. [1]).
- (vi) X is J-convex (cf. [1]).

For some further conditions equivalent to super-reflexivity, we refer the reader to [1], [3] and [14].

2.6 Corollary. $\tilde{C}_{NJ}(X) = 2$ if and only if X is not super-reflexive.

3. Uniform non-squareness and von Neumann-Jordan constant

Very recently the authors [15] proved some homogeneous characterizations of uniformly non-square spaces, one of which is similar to a well-known characterization of uniformly convex spaces (we have used it in the proof of Theorem 2.2!):

3.1 Theorem. Let $1 < p < \infty$. For a Banach space X the following are equivalent:

- (i) X is uniformly non-square.
- (ii) There exist some ε and δ ($0 < \varepsilon, \delta < 1$) such that $\|x - y\| \geq 2(1 - \varepsilon)$, $\|x\| \leq 1$, $\|y\| \leq 1$ implies

$$\left\| \frac{x + y}{2} \right\|^p \leq (1 - \delta) \frac{\|x\|^p + \|y\|^p}{2}.$$

- (iii) There exists some δ ($0 < \delta < 2$) such that for any x, y in X ,

$$\left\| \frac{x + y}{2} \right\|^p + \left\| \frac{x - y}{2} \right\|^p \leq (2 - \delta) \frac{\|x\|^p + \|y\|^p}{2}.$$

- (iv) $\|A : l_p^2(X) \rightarrow l_p^2(X)\| < 2$.

We omit the proof, which will appear elsewhere.

Owing to Theorem 3.1 a precise characterization of Banach spaces with NJ-constant less than two is obtained (cf. Theorems 2.2 and 2.3):

3.2 Theorem. The following are equivalent:

- (i) $C_{NJ}(X) < 2$.
- (ii) X is uniformly non-square.
- (iii) X is $(2, \varepsilon)$ -convex for some $\varepsilon > 0$.

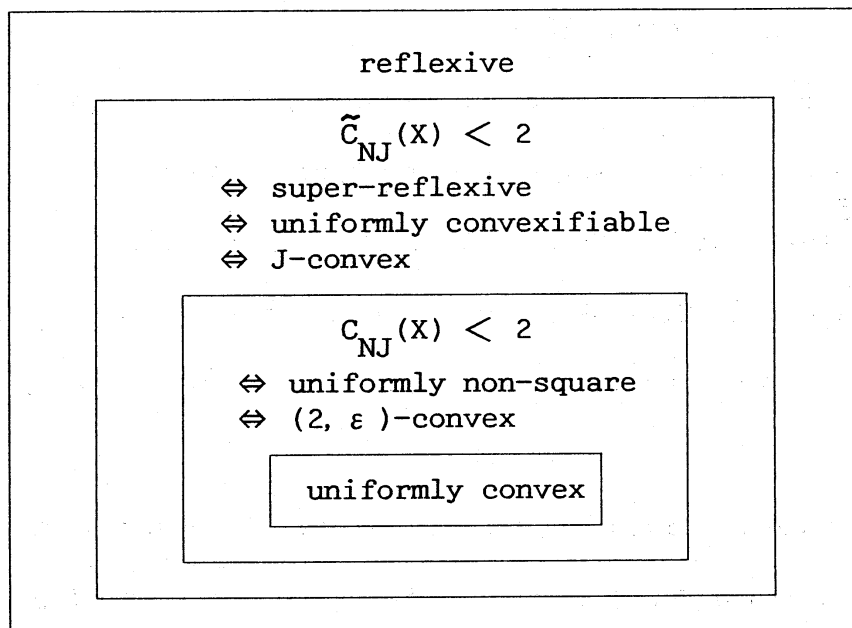
To see this, merely recall Proposition A.

3.3 Corollary. $C_{NJ}(X) = 2$ if and only if X is uniformly square.

3.4 Note. Further investigation on the NJ-constant is made in [12] especially for the spaces having NJ-constant $2^{2/p-1}$, $1 \leq p \leq 2$

(the same value of that of L_p -spaces; see Proposition B).

Our results stated in this note are summarized as follows:



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