

IMAGINARIES, STATIONARITY OF TYPES AND STABILITY

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ABSTRACT. Let T be a rosy theory having weak canonical bases with respect to a strict independence relation $\downarrow$. Suppose that any type over algebraically closed sets in the real sort is $\downarrow$ -stationary. THEN T is stable, non-forking relation coincides with $\downarrow$, and geometric elimination of imaginaries implies weak elimination of imaginaries.

1. NOTATIONS AND WELL-KNOWN FACTS ON STATIONARY TYPES

Let T be an L -theory, and $\mathcal{M}$ be a sufficiently saturated model of T . We work in $\mathcal{M}^{\text{eq}} := \{a_E : a_E \text{ is the } E\text{-class of } a, \text{ where } a \text{ is a finite tuple of } M \text{ and } E(x, y) \text{ is a } \phi\text{-definable equivalence relation with } \text{lh}(x) = \text{lh}(y) = \text{lh}(a)\}$. $A, B, C, \dots$ denote small subsets of $\mathcal{M}^{\text{eq}}$. $a, a', \alpha, \dots$ denote finite tuples in $\mathcal{M}$ (i.e. real finite tuples). We write $a \subset_\omega \mathcal{M}$. $A \equiv_B A'$ denotes $\text{tp}(A/B) = \text{tp}(A'/B)$ for any $A, A', B \subset \mathcal{M}^{\text{eq}}$. For $i \in \mathcal{M}^{\text{eq}}$, we write $i \in \text{acl}^{\text{eq}}(A)$ if the orbit of i by automorphisms fixing A pointwise is finite, and $i \in \text{dcl}^{\text{eq}}(A)$ if i is fixed by automorphisms fixing A pointwise. Adler [A] introduces a strict independence relation $* \downarrow_* *$ for triplet of small subsets of $\mathcal{M}^{\text{eq}}$ satisfying (1)-(9):

- (1) invariance: If $A \downarrow_B C$ and $ABC \equiv A'B'C'$, then $A' \downarrow_{B'} C'$.
- (2) monotonicity: If $A \downarrow_B C$, $A' \subseteq A$ and $C' \subseteq C$, then $A' \downarrow_B C'$.
- (3) right base monotonicity: If $A \downarrow_B D$ and $B \subseteq C \subseteq D$, then $A \downarrow_C D$.
- (4) left transitivity: If $B \subseteq C \subseteq D$, $D \downarrow_C A$ and $C \downarrow_B A$, then $D \downarrow_B A$.
- (5) left normality: $A \downarrow_B C$ implies $AB \downarrow_B C$.
- (6) extension: If $A \downarrow_B C$ and $C \subseteq D$, then there exists $A' (\equiv_{BC} A)$ such that $A' \downarrow_B D$.
- (7) left finite character: If $a \downarrow_B C$ for any $a \subset_\omega A$, then $A \downarrow_B C$.
- (8) local character: For any A there exists a cardinal $\kappa(A)$ such that, for any B there exists $B_0 \subseteq B$ with $|B_0| < \kappa(A)$ and $A \downarrow_{B_0} B$.
- (9) anti-reflexivity: $A \downarrow_B A$ implies $A \subseteq \text{acl}^{\text{eq}}(B)$.

T is said to be *rosy* if there exists a strict independence relation on $\mathcal{M}^{\text{eq}}$. Theorem 1.14 in [A] shows that (1)-(8) imply symmetry: $A \downarrow_B C \Leftrightarrow C \downarrow_B A$.

Fact 1.1. Let T be rosy with a strict independence relation $\downarrow$. Then we have $A \downarrow_B \text{acl}^{\text{eq}}(B)$ for any $A, B \subset \mathcal{M}^{\text{eq}}$.

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Proof. By local character and right base monotonicity, we have $A \downarrow_B B$. By extension, there exists $A' \equiv_B A$ such that $A' \downarrow_B \text{acl}^{\text{eq}}(B)$. By invariance, we have $A \downarrow_B \text{acl}^{\text{eq}}(B)$. $\square$

An algebraically closed set $C \subset \mathcal{M}^{\text{eq}}$ is said to be the $\downarrow$ -weak canonical base of $\text{tp}(a/B)$ if C is the smallest algebraically closed subset of $\text{acl}^{\text{eq}}(B)$ with $a \downarrow_C B$. $\text{wcb}_{\downarrow}(a/B)$ denotes the weak $\downarrow$ -weak cononical base of $\text{tp}(a/B)$. We say that a rosy T has $\downarrow$ -weak canonical bases if there exists $\text{wcb}_{\downarrow}(a/B)$ for any $a \subset_{\omega} \mathcal{M}$ and $B \subset \mathcal{M}^{\text{eq}}$. If T has $\downarrow$ -weak canonical bases, then $\downarrow = \downarrow^{\text{p}}$ by Theorem 3.3 in [A]. We say that T has geometric elimination of imaginaries (GEI) if there exists $\alpha \subset_{\omega} \mathcal{M}$ such that $a_E \in \text{acl}^{\text{eq}}(\alpha)$ and $\alpha \in \text{acl}^{\text{eq}}(a_E)$ for any $a \subset_{\omega} \mathcal{M}$ and any ϕ -definable equivalence relation $E(x, y)$ in T . We say that T has weak elimination of imaginaries (WEI) if T has GEI with $a_E \in \text{dcl}^{\text{eq}}(\alpha)$ in the above definition of GEI.

Definition 1.2. Let T be rosy with a strict independence relation $\downarrow$, and $a \subset_{\omega} \mathcal{M}$, $A \subset \mathcal{M}^{\text{eq}}$. We say $\text{tp}(a/A)$ is $\downarrow$ -stationary if for any $a' \subset_{\omega} \mathcal{M}$, $B = \text{acl}^{\text{eq}}(B) \subset \mathcal{M}^{\text{eq}}$ such that $A \subset B$, $a \equiv_A a'$, $a \downarrow_A B$ and $a' \downarrow_A B$, we have $a \equiv_B a'$.

The following is also well-known, which appeared in foot note on pp.20 in [Y2].

Fact 1.3. Let T be any rosy theory with a strict independence relation $\downarrow$.

- (1) If $\text{tp}(a/A)$ is $\downarrow$ -stationary and $i \in \text{dcl}^{\text{eq}}(a)$, then $\text{tp}(i/A)$ is also $\downarrow$ -stationary.
- (2) If $\text{tp}(i/A)$ is $\downarrow$ -stationary, $i \downarrow_A B$ and $i \in \text{dcl}^{\text{eq}}(B)$, then $i \in \text{dcl}^{\text{eq}}(A)$.

Proof. For (1): Suppose that $i \equiv_A i'$, $i \downarrow_A B$ and $i' \downarrow_A B$. We need to show $i \equiv_B i'$. By $i \in \text{dcl}^{\text{eq}}(a)$ and compactness, there exists a ϕ -definable function f such that $f(a) = i$. We may assume that $a \downarrow_{A_i} B$ by extension, so as $i \downarrow_A B$, we have $a \downarrow_A B$ by left transitivity, left normality and monotonicity. Take a' with $ia \equiv_A i'a'$. Again we may assume $a' \downarrow_{A_{i'}}$ by extension, so $a' \downarrow_A B$ follows. As $a \equiv_A a'$, $a \downarrow_A B$ and $a' \downarrow_A B$, by $\downarrow$ -stationarity of $\text{tp}(a/A)$, we have $a \equiv_B a'$. Since $i = f(a)$, $i' = f(a')$, we see $i \equiv_B i'$, as desired.

For(2): Note that $i \in \text{acl}^{\text{eq}}(A)$. Suppose that $i' \equiv_A i$. As $i' \in \text{acl}^{\text{eq}}(A)$ and Remark 1.1 and symmetry, $i' \downarrow_A B$ follows. By $\downarrow$ -stationarity of $\text{tp}(i/A)$, we see $i \equiv_B i'$, so $i = i'$ by $i \in \text{dcl}^{\text{eq}}(B)$. $\square$

2. STABILITY

Remark 2.1. If T has GEI, stationarity of types over algebraically closed sets in the real sort implies stationarity of types over algebraically closed sets in imaginary sorts.

Proof. Take $a \subset_{\omega} \mathcal{M}$, $A = \text{acl}^{\text{eq}}(A) \subset \mathcal{M}^{\text{eq}}$ arbitrarily. Suppose that $A \subset B = \text{acl}^{\text{eq}}(B) \subset \mathcal{M}^{\text{eq}}$, $a \equiv_A a'$, $a \downarrow_A B$ and $a' \downarrow_A B$. We need to show $a \equiv_B a'$. By GEI, there exists $A_0 = \text{acl}(A_0)$, $B_0 = \text{acl}(B_0) \subset \mathcal{M}$ such that $\text{acl}^{\text{eq}}(A_0) = A$, $\text{acl}^{\text{eq}}(B_0) = B$. As $A_0 \subseteq A$, we have $a \equiv_{A_0} a'$. By Remark 1.1, we have $a \downarrow_{A_0} \text{acl}^{\text{eq}}(B_0)$, $a' \downarrow_{A_0} \text{acl}^{\text{eq}}(B_0)$. By stationarity over algebraically closed sets in the real sorts, we see $a \equiv_{\text{acl}^{\text{eq}}(B_0)=B} a'$. $\square$

The following proposition is pointed out by Byunghan Kim at RIMS model theory workshop 2020.

Proposition 2.2. *Let T be rosy.*

- (1) *Suppose that $\text{tp}(a/\text{acl}^{\text{eq}}(A))$ is $\downarrow^{\text{p}}$ -stationary for any finite real tuple $a \subset_{\omega} \mathcal{M}$ and small $A \subset \mathcal{M}^{\text{eq}}$. Then T is stable and thorn-forking coincides with forking.*
- (2) *Suppose that $\text{tp}(a/\text{acl}(A))$ is $\downarrow^{\text{p}}$ -stationary for any finite real tuple $a \subset_{\omega} \mathcal{M}$ and small $A \subset \mathcal{M}$. Then T is stable and thorn-forking coincides with forking.*
- (3) *Suppose that $\text{tp}(a/\text{bdd}^{\text{heq}}(A))$ is $\downarrow^{\text{p}}$ -stationary for any finite real tuple $a \subset_{\omega} \mathcal{M}$ and small $A \subset \mathcal{M}^{\text{heq}}$. (i.e. for any $B = \text{bdd}^{\text{heq}}(B) \supset \text{bdd}^{\text{heq}}(A)$, if $a \downarrow_{\text{bdd}^{\text{heq}}(A)}^{\text{p}} B, a' \downarrow_{\text{bdd}^{\text{heq}}(A)}^{\text{p}} B$ and $a \equiv_{\text{bdd}^{\text{heq}}(A)} a'$ imply $a \equiv_B a'$.) Then T is stable and thorn-forking coincides with forking.*

Proof. (1): If $M \models T$, then we have $\text{acl}^{\text{eq}}(M) = \text{dcl}^{\text{eq}}(M)$. In particular $\text{tp}(a/M) = \text{tp}(a/\text{dcl}^{\text{eq}}(M)) = \text{tp}(a/\text{acl}^{\text{eq}}(M))$ and $a \downarrow_M^{\text{p}} \text{acl}^{\text{eq}}(M)$ by Fact 1.1. As $\text{tp}(a/\text{acl}^{\text{eq}}(M)) = \text{tp}(a/M)$ is $\downarrow^{\text{p}}$ -stationary, T has the independence theorem over models with respect to thorn-forking. By Theorem 2.7 (2) and Fact 2.4 in [EO], for any finite real tuple a and any small $A \subset \mathcal{M}^{\text{eq}}$ there exists $A_0 \subseteq A$ with $|A_0| \leq |T|$ and $a \downarrow_{A_0}^{\text{p}} A$. (Otherwise there would exist a chain of thorn-forking extension of length $|T|^+$, then $\text{p}_{\psi, \theta, k}$ -rank drops infinitely many times for a fixed triplet (ψ, θ, k) , so T would not be rosy.) By Kim-Pillay criterion: Theorem 3.3.1 in [K], thorn-forking coincides with forking. As any type of a finite real tuple over a model M is forking-stationary, by Corollary 2.3.17 in [W], it is definable over M . By Theorem 2.15 in [P], we see that T is stable.

(2): A similar proof of (1) works for (2).

(3): By Remark and Definition 4.1.4 (2) in [K], note that $\text{bdd}^{\text{heq}}(M) = \text{dcl}^{\text{heq}}(M)$ for any model M of T . In particular $\text{tp}(a/M) = \text{tp}(a/\text{dcl}^{\text{heq}}(M)) = \text{tp}(a/\text{bdd}^{\text{heq}}(M))$ and $a \downarrow_M^{\text{p}} \text{bdd}^{\text{heq}}(M)$ by the proof of Fact 1.1. Now, a similar proof of (1) works for (3) in T^{heq} . $\square$

3. GEI AND WEI IN ROSY THEORIES

The author's proof to the following proposition is improved by Anand Pillay as follows.

Proposition 3.1. *Let T be a rosy theory having weak canonical bases with respect to a strict independence relation $\downarrow$. Suppose that any type over algebraically closed set in the real sort is $\downarrow$ -stationary. THEN $\text{GEI} \Leftrightarrow \text{WEI}$. ($\Leftarrow$ is clear.)*

Proof. Let $i = a_E$ and take b such that $b \equiv_i a$ and $b \downarrow_i a$. By GEI, there exist $B \subset \mathcal{M}$ such that $\text{wcb}_{\downarrow}(b/ai) = \text{acl}^{\text{eq}}(B)$. Put $C := \text{acl}(B)$.

Note that $C \subseteq \text{acl}^{\text{eq}}(B) \subseteq \text{acl}^{\text{eq}}(i)$, as $b \downarrow_i ai$. We also have $b \downarrow_C \text{dcl}^{\text{eq}}(ai)$ by Fact 1.1, symmetry, left transitivity and monotonicity. As $\text{tp}(b/C)$ is $\downarrow$ -stationary and $i \in \text{dcl}^{\text{eq}}(b) \cap \text{dcl}^{\text{eq}}(ai)$, $i \in \text{dcl}^{\text{eq}}(C)$ by Fact 1.3. So we have $C \subseteq \text{acl}^{\text{eq}}(i)$ and $i \in \text{dcl}^{\text{eq}}(C)$, as desired. $\square$

Theorem 3.2. *Let T be a rosy theory having weak canonical bases with respect to a strict independence relation $\downarrow$. Suppose that any type over algebraically closed*

set in the real sort is $\downarrow$ -stationary. THEN T is stable, non-forking coincides $\downarrow$, and GEI implies WEI.

Proof. By Theorem 3.3 in [A], we have $\downarrow = \downarrow^p$. By Proposition 2.2 (2), T is stable and non-forking coincides with $\downarrow = \downarrow^p$. By Proposition 3.1, GEI implies WEI. $\square$

Corollary 3.3. *Suppose T has a strict independence relation $\downarrow$ having the intersection property. (i.e. $a \downarrow_A B$ and $a \downarrow_B A$ imply $a \downarrow_{A \cap B} AB$ for any $a \subset_\omega \mathcal{M}$ and any small subsets of $\mathcal{M}$ such that $A = \text{acl}(A), B = \text{acl}(B)$.) If any type over algebraically closed set in the real sort is $\downarrow$ -stationary, then T is stable with WEI and non-forking coincides with $\downarrow$ in T .*

Proof. By Proposition 3.4 in [Y3] and Theorem 3.2. $\square$

- Remark 3.4.** (1) Let T be stable. Suppose that $\text{tp}(a/A) \vdash \text{tp}(a/\text{acl}^{\text{eq}}(A))$ for any $a, A = \text{acl}(A) \subset \mathcal{M}$. As $\text{tp}(a/\text{acl}^{\text{eq}}(A))$ is stationary by stability of T , we have $\text{GEI}=\text{WEI}$ in T by Proposition 3.1. (Note that $a \downarrow_A B$ implies $a \downarrow_A \text{acl}^{\text{eq}}(B)$ by symmetry, Fact 1.1 and left transitivity.) Now we show the stationarity of $\text{tp}(a/A)$ i.e. if $\text{acl}(A) = A \subseteq B = \text{acl}^{\text{eq}}(B), a \equiv_A a', a \downarrow_A B$ and $a' \downarrow_A B$, then $a \equiv_B a'$: Note that $A \subseteq \text{acl}^{\text{eq}}(A) \subseteq B$. By $\text{tp}(a/A) \vdash \text{tp}(a/\text{acl}^{\text{eq}}(A))$ and $a \equiv_A a'$, we have $a \equiv_{\text{acl}^{\text{eq}}(A)} a'$. By the above we have $a \downarrow_A B, a' \downarrow_A B$ and $a \equiv_{\text{acl}^{\text{eq}}(A)} a'$, by stationarity of strong types in stability theory, we have $a \equiv_B a'$.
- (2) Let T be a rosy theory with CM-triviality in the real sort. Then T is CM-trivial with GEI and weak canonical bases. (See Theorem 4.2 and Theorem 2.4 in [Y3].) If any type over algebraically closed sets in the real sort is $\downarrow$ -stationary in T , then T is stable with WEI, non-forking coincides with $\downarrow$ by Theorem 3.2.
- (3) Any relational structure M such that $\text{Th}(M)$ is amalgamation over closed sets ARE stable CM-trivial in the real sort and having non-forking stationarity of types over algebraically closed sets in the real sort (see [Y1] and [VY]). So, $\text{Th}(M)$ has WEI by (2).
- (4) D.M.Evans gives an example of $\aleph_0$ -categorical CM-trivial SU-rank one graph which does not have GEI. See [E] and [Y3]. This example is NOT CM-trivial in the real sort.
- (5) If T is a theory having free amalgamation ternary relation, then T has NSOP₄, elimination of hyperimaginaries and WEI. See Theorem 4.4 and Theorem 5.6 in [C].
- (6) We said that a definable set $D \subset \mathcal{M}$ is stably embedded if for any $n < \omega$ and $\mathcal{M}$ -definable subset X of $\mathcal{M}^n$, there exists D -definable subset Y of $\mathcal{M}^n$ such that $X \cap D^n = Y \cap D^n$. The following are equivalent by Lemma 2.3.2 in [CH].
- (a) $D \subset \mathcal{M}$ is stably embedded and D has WEI.
- (b) $\text{tp}(i/\text{acl}^{\text{eq}}(i) \cap D) \vdash \text{tp}(i/D)$ for any $i \in \mathcal{M}^{\text{eq}}$.
- (7) Let D be a strongly minimal set with infinite $D \cap \text{acl}(\phi)$. Then D has WEI. See Lemma 1.6 in [P1].
- (8) D.M.Hoffmann points out a simple theory CCMA(=Compact Complex Manifolds with an Automorphism) having GEI and finite coding in Theorem 4.3.6 and Lemma 4.3.7 in [Ho]. But CCMA does not having EI by

Corollary 3.6 in [BHM]. As WEI+finite coding=EI by Proposition 1.6 in [CF], CCMA does not have WEI.

- (9) The N -dimensional free pseudospace is ω -stable and N -ample but not $(N+1)$ -ample with WEI. See [T] and [BMZ].
- (10) The theory of the free group is stable, n -ample for all $n < \omega$ and has GEI up to adding some reasonable sorts. See Theorem 4.6 in [S].
- (11) $\text{ACF}, \text{DCF}, \text{SCF}_e^* (e < \omega)$ eliminate imaginaries. SCF_e for $e \in \omega \cup \{\infty\}$ in the language of fields does not eliminate imaginaries and has finite coding, so it does not have WEI. See [MMP].

Question 3.5. Does SCF_e for $e \in \omega \cup \{\infty\}$ in the language of fields geometrically eliminate imaginaries? Find a stable theory (moreover, strongly minimal set) having GEI not having WEI.

REFERENCES

- [A] H.Adler, Explanation of independence, Ph.D thesis, Freiburg, 2005, arXiv:math.LO/0511616 v1, 24 Nov 2005.
- [BHM] M.Bays, M.Hils and R.Moosa, Model theory of compact complex manifolds with an automorphism, Transactions of the AMS, volume 369 (2017), number 6, 4485–4516.
- [BMZ] A.Baudisch, A.Martin-Pizarro, M. Ziegler, Ample Hierarchy, Fund. Math. 224 (2014), 97–153.
- [C] G.Conant, An axiomatic approach to free amalgamation, J. Symbolic Logic, 82, No.2 (2017), 648–671.
- [CF] E.Casanovas and R.Farré, Weak forms of elimination of imaginaries, Math.Logic Quarterly, 50, No.2 (2004), 126–140.
- [CH] G.Cherlin, E.Hrushovski, Finite structures with few types, Annals of Mathematics Studies 152 (2003).
- [EO] C.Ealy and A.Onshuus, Characterizing rosy theories, J. Symbolic Logic, 72, No.3, (2007) 919–940.
- [E] D.M.Evans, $\aleph_0$ -categorical structures with a predimension, Ann. Pure Appl. Logic 116 (2002), 157–186
- [Ho] D.M.Hoffmann, Model theoretic dynamics in Galois fashion, Ann. Pure Appl. Logic 170 (2019), 755–804.
- [Hr] E.Hrushovski, A new strongly minimal set, Ann. Pure Appl. Logic 62 (1993), 147–166.
- [K] B.Kim, Simplicity theory, Oxford Logic Guides 53, Oxford Science Publications (2014).
- [MMP] D.Marker, M.Messmer, A.Pillay, Model theory of fields, Lecture note in logic 5 (1996).
- [P] A.Pillay, An introduction to stability theory, Oxford Logic Guides 8, Oxford University Press (1983).
- [P1] A.Pillay, Model theory of algebraically closed fields, Model Theory and Algebraic Geometry, edit by E. Bouscaren, Lecture Note in Mathematics 1696 (1998), 61–84.
- [S] Z.Sela, Diophantine geometry over groups IX: Envelopes and imaginaries, arXiv:0909.0774 (2009).
- [T] K.Tent, The free pseudospace is N -ample. but not $(N+1)$ -ample, J. Symbolic Logic 79, No. 2 (2014), 410–428
- [VY] V.Verbovskiy and I.Yoneda, CM-triviality and relational structures, Ann. Pure Appl. Logic 122(2003), 175–194.
- [W] F.O.Wagner, Simple theories, Mathematics and Its Applications, Kluwer Academic Publishers (2000).
- [Y1] I.Yoneda, CM-triviality and generic structures, Arch. Math. Logic 42 (2003), 423–433.
- [Y2] I.Yoneda, On CM-triviality, RIMS kokyuroku 1390 (2004), pp18–22.
- [Y3] I.Yoneda, Some remarks on CM-triviality, J.Math.Soc.Japan 61 (2009), 423–433

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