

$\mathcal{A}$ -SCHEMES AND THEIR APPLICATIONS

Satoshi Takagi / 高木 聡
Department of Math., Kyoto Univ.

1 Motivation (I)

Let $(\mathbf{Top})$ (resp. $(\mathbf{CHaus})$) be the category of topological spaces (resp. compact Hausdorff spaces).

Stone-Čech compactification

$$\beta : (\mathbf{Top}) \rightleftarrows (\mathbf{CHaus}) : U$$

The compact Hausdorff space βX is called the Stone-Čech compactification of the given space X .

To construct the left adjoint, we need the following lemma:

Tychonoff's Theorem

$(\mathbf{CHaus})$ is small complete.

We may hope that similar things hold for schemes: schemes are extended concepts of commutative rings. However,

The category $(\mathbf{Ring})$ of rings is (small) complete and co-complete.

$\Updownarrow$ (compare)

The category $(\mathbf{Sch})$ of schemes is NOT complete NOR co-complete.

This becomes an obstruction, when we try to construct a universal compactification of schemes using the above method.

Why is the category of schemes not complete?

2 Motivation (II)

There are several attempts to construct 'scheme-like topological objects' from other algebraic type, such as:

- (1) Monoids ("schemes over $\mathbb{F}_1$ ")
- (2) Semirings (Tropical geometry).

What are the common aspects between them?

3 Results

Properties of $\mathcal{A}$ -schemes (I)

Let $(\mathbf{CohSch})$ (resp. $(\mathbf{CohLRS})$, $(\mathcal{A}\text{-}\mathbf{Sch})$) be the category of coherent schemes (resp. coherent locally ringed spaces, $\mathcal{A}$ -schemes).

- (1) Then, we have an adjunction

$$\mathbf{Spec} : (\mathbf{Ring}) \rightleftarrows (\mathcal{A}\text{-}\mathbf{Sch})^{\mathrm{op}} : \Gamma$$

- (2) We have a fully faithful functors

$$(\mathbf{CohSch}) \xrightarrow{j} (\mathcal{A}\text{-}\mathbf{Sch}) \rightarrow (\mathbf{CohLRS}),$$

and j preserves fibre products and finite quasi-compact open patchings.

Properties of $\mathcal{A}$ -schemes (II)

- (1) The category $(\mathcal{A}\text{-}\mathbf{Sch})$ is **small complete and co-complete**.
- (2) We have the notions of **separatedness and properness** (we exclude the 'of finite type' condition). Valuable criteria exist.
- (3) Fix a base $\mathcal{A}$ -scheme S , and $(\mathbf{p}\text{-}\mathcal{A}\text{-}\mathbf{Sch}/S)$ be the category of proper $\mathcal{A}$ -schemes over S . Then, we have the adjunction

$$\mathbf{ZR}_S : (\mathcal{A}\text{-}\mathbf{Sch}/S) \rightleftarrows (\mathbf{p}\text{-}\mathcal{A}\text{-}\mathbf{Sch}/S) : U$$

The $\mathcal{A}$ -scheme $\mathbf{ZR}_S(X)$ is the **Zariski-Riemann space** associated to X .

As a corollary, we obtain the well known application for usual schemes:

Cor (Nagata embedding)

Let $X \rightarrow S$ be a separated, of finite type morphism of coherent schemes. Then, there exists an open immersion $X \rightarrow \overline{X}$ of S -schemes, where $\overline{X}$ is proper, of finite type over S .

Unfortunately, we don't have enough space to show here the precise definition of $\mathcal{A}$ -schemes. The key point is to

Abandon the feature of schemes, that it is locally a spectrum of a ring.

References

- [1] Takagi, S.: $\mathcal{A}$ -schemes and Zariski-Riemann spaces, to appear in *Rendiconti del Seminario Matematico della Univ. Padova*, arXiv: mathAG/1101.2796
- [2] Takagi, S.: *Nagata embeddings and $\mathcal{A}$ -schemes*, preprint, arXiv: mathAG/1107.3414