

Saturated fusion systems and Brauer indecomposability of Scott modules

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1. INTRODUCTION

Let p be a prime number, G a finite group, and k an algebraically closed field of characteristic p . For a finite dimensional kG -module M and a p -subgroup Q of G , we denote by $M(Q)$ the Brauer quotient of M with respect to Q . The Brauer quotient $M(Q)$ is naturally a $kN_G(Q)$ -module. A kG -module M is said to be Brauer indecomposable if $M(Q)$ is indecomposable or zero as a $kQC_G(Q)$ -module for any p -subgroup Q of G ([1]). Brauer indecomposability of p -permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

In [1], a relationship between Brauer indecomposability of p -permutation modules and saturated fusion systems was given. For a p -subgroup P of G , we denote by $\mathcal{F}_P(G)$ the fusion system of G over P . One of the main result in [1] is the following.

Theorem 1 ([1, Theorem 1.1]). *Let P be a p -subgroup of G and M an indecomposable p -permutation kG -module with vertex P . If M is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.*

In the special case that P is abelian and M is the Scott kG -module $S(G, P)$, the converse of the above theorem holds.

Theorem 2 ([1, Theorem 1.2]). *Let P be an abelian p -subgroup of G . If $\mathcal{F}_P(G)$ is saturated, then $S(G, P)$ is Brauer indecomposable.*

In general, the above theorem does not hold for non-abelian P . However, there are some cases in which the Scott kG -module $S(G, P)$ is Brauer indecomposable, even if P is not necessarily abelian.

We study the condition that $S(G, P)$ to be Brauer indecomposable where P is not necessarily abelian. The following result gives an equivalent condition for Scott kG -module with vertex P to be Brauer indecomposable.

Theorem 3. *Let G be a finite group and P a p -subgroup of G . Suppose that $M = S(G, P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.*

- (i) *M is Brauer indecomposable.*
- (ii) *For each fully normalized subgroup Q of P , the module $\text{Res}_{QC_G(Q)}^{N_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable.*

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The following theorem shows that $\text{Res}_{QC_G(Q)}^{N_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable if Q satisfies some conditions.

Theorem 4. *Let G be a finite group, P a p -subgroup of G and Q a fully normalized subgroup of P . Suppose that $\mathcal{F}_P(G)$ is saturated. Moreover, we assume that there is a subgroup H_Q of $N_G(Q)$ satisfying following two conditions:*

- (i) $N_P(Q) \in \text{Syl}_p(H_Q)$
- (ii) $|N_G(Q) : H_Q| = p^a$ ($a \geq 0$)

Then $\text{Res}_{QC_G(Q)}^{N_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable.

The following is a consequence of above two theorems.

Corollary 5. *Let G be a finite group and P a p -subgroup of G . Suppose that $\mathcal{F}_P(G)$ is saturated. If for every fully normalized subgroup Q of P there is a subgroup H_Q of $N_G(Q)$ satisfies the conditions of 4, then $S(G, P)$ is Brauer indecomposable.*

Throughout this article, we denote by $L \cap_G H$ the set $\{^g L \cap H \mid g \in G\}$ for subgroups L and K of G .

2. PRELIMINARIES

2.1. Scott modules. First, We recall the definition of Scott modules and some of its properties:

Definition 6. For a subgroup H of G , the Scott kG -module $S(G, H)$ with respect to H is the unique indecomposable summand of $\text{Ind}_H^G k_H$ that contains the trivial kG -module.

If P is a Sylow p -subgroup of H , then $S(G, H)$ is isomorphic to $S(G, P)$. By definition, the Scott kG -module $S(G, P)$ is a p -permutation kG -module.

By Green's indecomposability criterion, the following result holds.

Lemma 7. *Let H be a subgroup of G such that $|G : H| = p^a$ (for some $a \geq 0$). Then $\text{Ind}_H^G k_H$ is indecomposable. In particular, we have that*

$$S(G, H) \cong \text{Ind}_H^G k_H.$$

Hence, for p -subgroup P of G , if there is a subgroup H of G such that P is a Sylow p -subgroup of H and $|G : H| = p^a$, then we have that

$$S(G, P) \cong \text{Ind}_H^G k_H.$$

The following theorem gives us information of restrictions of Scott modules.

Theorem 8 ([3, Theorem 1.7]). *Let H be a subgroup of G and P a p -subgroup of G . If Q is a maximal element of $P \cap_G H$, then $S(H, Q)$ is a direct summand of $\text{Res}_H^G S(G, P)$.*

2.2. Brauer quotients. Let M be a kG -module and H a subgroup of G . Let M^H be the set of H -fixed elements in M . For subgroups L of H , we denote by Tr_H^G the trace map $\text{Tr}_L^H : M^L \rightarrow M^H$. Brauer quotients are defined as follows.

Definition 9. Let M be a kG -module. For a p -subgroup Q of G , the Brauer quotient of M with respect to Q is the k -vector space

$$M(Q) := M^Q / (\sum_{R < Q} \text{Tr}_R^Q(M^R)).$$

This k -vector space has a natural structure of $kN_G(Q)$ -module.

Proposition 10. Let P be a p -subgroup of G and $M = S(G, P)$. Then $M(P) \cong S(N_G(P), P)$.

Proposition 11. Let M be an indecomposable p -permutation kG -module with vertex P . Let Q be a p -subgroup of G . Then $Q \leq_G P$ if and only if $M(Q) \neq 0$.

2.3. Fusion systems. For a p -subgroup P of G , the fusion system $\mathcal{F}_P(G)$ of G over P is the category whose objects are the subgroups of P , and whose morphisms are the group homomorphisms induced by conjugation in G .

Definition 12. Let P be a p -subgroup of G

- (i) A subgroup Q of P is said to be fully normalized in $\mathcal{F}_P(G)$ if $|N_P({}^x Q)| \leq |N_P(Q)|$ for all $x \in G$ such that ${}^x Q \leq P$.
- (ii) A subgroup Q of P is said to be fully automized in $\mathcal{F}_P(G)$ if $p \nmid |N_G(Q) : N_P(Q)C_G(Q)|$.
- (iii) A subgroup Q of P is said to be receptive in $\mathcal{F}_P(G)$ if it has the following property: for each $R \leq P$ and $\varphi \in \text{Iso}_{\mathcal{F}_P(G)}(R, Q)$, if we set

$$N_\varphi := \{g \in N_P(Q) \mid \exists h \in N_P(R), c_g \circ \varphi = \varphi \circ c_h\},$$

then there is $\bar{\varphi} \in \text{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\bar{\varphi}|_R = \varphi$.

Saturated fusion systems are defined as follows.

Definition 13. Let P be a p -subgroup of G . The fusion system $\mathcal{F}_P(G)$ is saturated if the following two conditions are satisfied:

- (i) P is fully normalized in $\mathcal{F}_P(G)$.
- (ii) For each subgroup Q of P , if Q is fully normalized in $\mathcal{F}_P(G)$, then Q is receptive in $\mathcal{F}_P(G)$.

For example, if P is a Sylow p -subgroup of G , then $\mathcal{F}_P(G)$ is saturated.

3. SKETCH OF PROOF

In this section, let P be a p -subgroup of G and M the Scott module $S(G, P)$.

Lemma 14. If $Q \leq P$ is fully normalized in $\mathcal{F}_P(G)$, then $N_P(Q)$ is a maximal element of $P \cap_G N_G(Q)$.

By above lemma, we can show that $S(N_G(Q), N_P(Q))$ is a direct summand of $M(Q)$ for each fully normalized subgroup Q of P . Therefore, we have that Theorem 3 (i) implies 3 (ii).

Assume that Theorem 3 (ii) holds. We prove that $\text{Res}_{QC_G(Q)}^{N_G(Q)}(M(Q))$ is indecomposable for each $Q \leq P$ by induction on $|P : Q|$. Without loss of generality, we can assume that

Q is fully normalized. If $M(Q)$ is decomposable, then by the following lemma, we can show that there is a subgroup R such that $Q < R \leq P$ and $\text{Res}_{RC_G(R)}^{N_G(R)}$ is decomposable, this contradicts the induction hypothesis.

Lemma 15. *Suppose that a subgroup Q of P is fully automized and receptive. Then for any $g \in G$ such that $Q \leq {}^gP$, we have that $N_{{}^gP}(Q) \leq_{N_G(Q)} N_P(Q)$.*

Hence, $M(Q)$ is indecomposable, and isomorphic to $S(N_G(Q), N_P(Q))$. Consequently, Theorem 3 (ii) implies 3 (i).

Theorem 4 is proved by using properties of Scott modules and the following lemma.

Lemma 16. *If Q is fully automized subgroup of P , and there is a subgroup $H_Q \leq N_G(Q)$ containing $N_P(Q)$ such that $|N_G(Q) : H_Q| = p^a$, then $C_G(Q)H_Q = N_G(Q)$.*

4. EXAMPLE

We set $p = 2$ and

$$\begin{aligned} G &:= \langle a, x, y \mid a^4 = x^2 = e, a^2 = y^2, \\ &\quad xax = a^{-1}, ay = ya, xy = yx \rangle, \\ P &:= \langle a, xy \rangle. \end{aligned}$$

Then G is a finite group of order 16, and P is isomorphic to the quaternion group of order 8. Hence, P is a non-abelian p -subgroup of G . One can easily show that G and P satisfy the hypothesis of the Corollary 5. Therefore, $S(G, P)$ is Brauer indecomposable.

In particular, if G is a p -group and $\mathcal{F}_P(G)$ is saturated for a p -subgroup P of G , then G and P satisfy the hypothesis of the Corollary 5, and hence $S(G, P)$ is Brauer indecomposable.

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