

Monodromy and bifurcations of the Hénon map

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Dedicated to Professor Ushiki on his 60th birthday

1 Monodromy of the *complex* Hénon Map

We discuss the structure of the parameter space of the *complex* Hénon map

$$H_{a,c} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 : (x, y) \mapsto (x^2 + c - ay, x), \quad (a, c) \in \mathbb{C}^2$$

and the pruning front of the *real* Hénon map $H_{a,c}|_{\mathbb{R}^2}$ for $(a, c) \in \mathbb{R}^2$. Let $\mathcal{H}$ be the subset of $\mathbb{C}^2$ which consists of the parameter values (a, c) such that $K_{a,c}^{\mathbb{C}} := \{p \in \mathbb{C}^2 : \{H_{a,c}^n(p)\}_{n \in \mathbb{Z}} \text{ is bounded}\}$ is uniformly hyperbolic and conjugate to the full shift $\sigma : \Sigma_2 \rightarrow \Sigma_2$ of two symbols. We denote $K_{a,c}^{\mathbb{C}} \cap \mathbb{R}^2$ by $K_{a,c}^{\mathbb{R}}$.

Let us fix a basepoint $(a_0, c_0) \in \mathcal{H}$ and a topological conjugacy $h_0 : K_{a_0, c_0}^{\mathbb{C}} \rightarrow \Sigma_2$. Given a loop $\gamma : [0, 1] \rightarrow \mathcal{H}$ based at (a_0, c_0) , we construct a continuous family of conjugacies $h_t : K_{\gamma(t)}^{\mathbb{C}} \rightarrow \Sigma_2$ along γ . Then we define $\rho(\gamma) := h_1 \circ (h_0)^{-1} : \Sigma_2 \rightarrow \Sigma_2$. It is easy to see that ρ defines a group homomorphism $\rho : \pi_1(\mathcal{H}, (a_0, c_0)) \rightarrow \text{Aut}(\Sigma_2)$ where $\text{Aut}(\Sigma_2)$ is the group of the automorphisms of Σ_2 . We call ρ the monodromy homomorphism. Let us denote the image of ρ by Γ .

In analogy with the one dimensional complex dynamics, John Hubbard raised the following conjecture, which implies that the topological structure of $\mathcal{H}$ should be extremely rich.

Hubbard's Conjecture. $\Gamma \cup \{\sigma\}$ generates $\text{Aut}(\Sigma_2)$.

Theorem 1. *The image Γ satisfies the following properties:*

- (1) Γ contains non-trivial elements. In particular, it contains elements of infinite order [1].
- (2) Γ does NOT contain any odd-time iteration of σ . Moreover, with respect to the decomposition $\text{Aut}(\Sigma_2) = \mathbb{Z}\langle\sigma\rangle \oplus \text{Inert}(\Sigma_2)$ where $\text{Inert}(\Sigma_2)$ is the subgroup of inert automorphisms, we have $\Gamma \subset \mathbb{Z}\langle\sigma^2\rangle \oplus \text{Inert}(\Sigma_2)$.

The proof for the statement (1) is computer-assisted (see [1, 2]). To prove (2), we make use of the following algebraic condition on the automorphism of Σ_2 .

Let $\phi \in \text{Aut}(\Sigma_2)$ be an automorphism. The n -th sign number $s_n(\phi)$ of ϕ is the sign ± 1 , of the permutation induced by ϕ on the set of periodic orbits of least period n . For each periodic orbit U of least period n , we choose an arbitrary element $x_U \in U$. Since $\phi(x_U)$ and $x_{\phi(U)}$ are in the same periodic orbit, we can find an integer $k(U)$ such that $\phi(x_U) = \sigma^{k(U)}(x_{\phi(U)})$. Then we define the n -th gyration number $g_n(\phi) \in \mathbb{Z}_n$ by

$$g_n(\phi) := \sum_U k(U) \pmod n$$

where the sum is taken over all the periodic orbit of least period n .

We say that a map $\phi : \Sigma_2 \rightarrow \Sigma_2$ satisfies the sign-gyration compatibility condition (SGCC, see [3]) if

$$g_{2^m q}(\phi) = \begin{cases} 0 & \text{if } \prod_{j=0}^{m-1} s_{2^j q}(\phi) = 1, \\ 2^{m-1} q & \text{if } \prod_{j=0}^{m-1} s_{2^j q}(\phi) = -1. \end{cases}$$

for every odd positive integer q and every non-negative integer m . It is known that every inert automorphism satisfies SGCC.

Note that if ϕ satisfies SGCC, it follows from the condition for $q = 1$ and $m = 1$ that ϕ interchanges the two fixed points if and only if it rotates the period 2 orbit. By investigating the configuration of bifurcation curves of periodic orbit of period 1 and 2, we can prove that this property also holds for all automorphisms in Γ .

Lemma 2. *Let $\phi \in \Gamma$. Then ϕ interchanges the two fixed points if and only if it rotates the period 2 orbit.*

Proof of Theorem 1 (2). Let $\phi \in \Gamma$ and assume $\phi = (\sigma^k, \phi')$ where k is odd. Then $\sigma^{-k} \circ \phi \in \text{Inert}(\Sigma_2)$ and therefore satisfies SGCC. However, by Lemma 2 and the assumption k is odd, $\sigma^{-k} \circ \phi$ does not satisfy SGCC. This is a contradiction. $\square$

2 Application to pruning fronts of the real Hénon Map

The key to relate the monodromy of the complex Hénon map to the pruning front of the real Hénon map is the following theorem.

Theorem 3 (ZA [1]). *For $(a, c) \in \mathcal{H} \cap \mathbb{R}^2$ and a path α connecting (a_0, b_0) to (a, c) , define $\gamma := \alpha \cdot (\bar{\alpha})^{-1}$. Then $\rho(\gamma)$ is an involution and $H_{a,c} : K_{a,c}^{\mathbb{R}} \rightarrow K_{a,c}^{\mathbb{R}}$ is topologically conjugate to $\sigma|_{\text{Fix}(\rho(\gamma))} : \text{Fix}(\rho(\gamma)) \rightarrow \text{Fix}(\rho(\gamma))$.*

By virtue of the theorem, we can define the pruning front for these real Hénon map by

$$P := ([0] \cap (\rho(\gamma))^{-1}[1]) \cup ([1] \cap (\rho(\gamma))^{-1}[0]).$$

The pruning front P completely determines the dynamics of the real Hénon map acting on $K_{a,c}^{\mathbb{R}}$ (see [1]).

It is known that SGCC holds for any automorphisms of a full shift which is a composition of finite-order automorphisms. It follows that

Proposition 4. *If γ is symmetric (i.e. $\bar{\gamma} = \gamma$) then $\rho(\gamma)$ must satisfy SGCC.*

This implies there is a restriction on the shape of pruning fronts. For example, although $0_1^0 10$ is possible (and in fact, is the pruning front for $a = -1, c = -5$), $0_1^0 100$ is not allowed.

Recently, Nicholas Long proved the following related result posing an algebraic restriction on subshifts that can be the fixed point set of an involution.

Theorem 5 (Long [5]). *If a SFT Y is the fixed point set of an inert involution of a mixing shift of finite type X , then $\text{Per}(X) \setminus \text{Per}(Y)$ is the disjoint union of 2-cascades.*

This theorem suggests that in a hyperbolic SFT that appears via pruning, if a periodic orbit O is missing, then all periodic orbits on the period-doubling cascade beginning at O should also be missing.

References

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