

Chaotic maps on measure spaces and behavior of states

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Introduction. As well known, chaotic maps are considered as those φ 's which have the following property (cf.[1]).

- (1) The set of all periodic points for φ are dense.
- (2) φ is transitive.
- (3) φ depends on sensitive initial condition.

Those properties are concerned with the orbit of a given initial point. In this note, we consider how probability density functions changed by iteration of chaotic maps. More generally, we study behavior of states by $*$ -endomorphisms of von Neumann algebras associated with chaotic maps. In particular, we show some theorems concerning the limits of iterated states, which are stated as follows.

- (4) The sequence of iterated states by a chaotic map converges to a unique state in the norm topology.

In Section 1 and 2, we note some results related to $*$ -endomorphisms of von Neumann algebras and iterated states by chaotic maps respectively, which are stated without proof. Section 3 consists of examples only which give us the meaning of theorems in Section 2 and provide fruitful discussion on our theory. Moreover we can find deep relationship between our study and wavelets theory (cf.[4]). This note is a continuation of [5].

§1. A $*$ -endomorphism of von Neumann algebra associated with a family of isometries. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. In this note $\{V_i\}_{i=1}^n$ means a family of isometries on $\mathcal{H}$ satisfying the following property and is said to be a FIC on $\mathcal{H}$ for short.

$$(C.1) \quad \{V_i V_i^*\}_{i=1}^n \text{ is a set of mutually orthogonal projections and } \sum_{i=1}^n V_i V_i^* = I.$$

Of course, this family $\{V_i\}_{i=1}^n$ on $\mathcal{H}$ is the generators of the image of a representation of Cuntz-algebra $\mathcal{O}_n$ [3]. Moreover we can define a $*$ -endomorphism α_V of the full operator algebra $B(\mathcal{H})$ as follows.

$$(C.2) \quad \alpha_V(T) = \sum_{i=1}^n V_i T V_i^*, (T \in B(\mathcal{H}))$$

If a von Neumann algebra M on $\mathcal{H}$ is invariant for α_V , then α_V becomes a $*$ -endomorphism of M . For n and a positive integer k , we denote by $I(n)$ the set $\{1, 2, \dots, n\}$ and $I(n)^k$ the set of all k -tuples $\mu = (j_1, \dots, j_k)$ with j_i in $\{1, 2, \dots, n\}$. For μ in $I(n)^k$ we denote by $V(\mu)$ the isometry $V_{j_1} V_{j_2} \cdots V_{j_k}$ on $(\mathcal{H})$. Then $\{V(\mu) | \mu \in I(n)^k\}$ is a family of isometries whose final projections are mutually orthogonal. When α_V is a $*$ -endomorphism of M , α_V^n is of the form:

$$\alpha_V^k(T) = \sum_{\mu \in I(n)^k} V(\mu) T V(\mu)^*, \quad (T \in M).$$

Proposition 1.1. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$ and e a unit vector in $\mathcal{H}$ such that $V_1 e = e$. We put*

$$ONS(e, V) = \bigcup_{k=1}^{\infty} \{V(\mu)e | \mu \in I(n)^k\}.$$

Then $ONS(e, V)$ is an orthonormal system.

Remark. An orthonormal system $ONS(e, V)$ in the proposition above is regarded as the sequence $\{e_k\}_{k=1}^{\infty}$ which is inductively defined as follows: $e_1 = e$ and

$$e_{i+n(\ell-1)} = V_i e_{\ell} \quad (i \in I(n), \ell \in \mathbb{N}).$$

(c.f. 2 of [2])

For a von Neumann algebra M on $\mathcal{H}$, M_* denotes the predual of M . We denote by α_V^* the transpose map of α_V with respect to the duality of M and M_* . The vector state in M_* associated with unit vector ξ in $\mathcal{H}$ is denoted by ω_{ξ} , that is, for T in M , $\omega_{\xi}(T) = \langle T\xi, \xi \rangle$ and

$$\omega_{\xi}(\alpha_V(T)) = \langle \alpha_V(T)\xi, \xi \rangle = \alpha_V^*(\omega_{\xi})(T).$$

Moreover we have

$$\alpha_V^*(\omega_{\xi}) = \sum_{i=1}^n \omega_{V_i^* \xi}.$$

When e is a unit vector such that $V_1 e = e$, namely, it is an eigenvector for eigenvalue 1 of V_1 , we denote by $\mathcal{H}_e$ the subspace of $\mathcal{H}$ spanned by $ONS(e, V)$.

Proposition 1.2. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$. If there exists a unit vector e such that $V_1 e = e$, then for any unit vector ξ in the subspace $\mathcal{H}_e$ it follows that*

$$\lim_{n \rightarrow \infty} (\alpha_V^*)^n(\omega_{\xi}) = \omega_e \quad (\text{norm topology}).$$

Proposition 1.3. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$. If there exists a unit vector e such that $V_1 e = e$, then for any state ω of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi_k}$ where ξ_k 's are in $\mathcal{H}_e$, it follows*

that

$$\lim_{n \rightarrow \infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\text{norm topology}).$$

Proposition 1.4. Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$ and e a unit vector such that $V_1 e = e$, If $ONS(e, V)$ is complete, then for any state ω in the predual of $B(\mathcal{H})$ it follows that

$$\lim_{n \rightarrow \infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\text{norm topology}).$$

Proposition 1.5. Let M be a Neumann algebra on $\mathcal{H}$ and $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ be a couple of families of isometries on $\mathcal{H}$ satisfying (1.1). Suppose that M is invariant for α_V and α_W . Then following conditions are equivalent.

(1) $\alpha_V(T) = \alpha_W(T)$ for all T in M .

$$(2) (W_1, \dots, W_n) = (V_1, \dots, V_n) \begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix},$$

that is, $W_i = \sum_{j=1}^n V_j h_{ji}$, ($1 \leq i \leq n$), where each h_{ij} is a unitary element in the commutant M' of M on the Hilbert space $\mathcal{H}$.

§2. Chaotic maps and behavior of states. Let X be a measure space with measure m and φ a measurable map on X , Here we note some notations concerning X and φ .

- (1) $m \circ \varphi$ denotes the measure on X defined by $m \circ \varphi(E) = m(\varphi(E))$ and if the map φ is absolutely continuous with respect to m , the Radon-Nikodym derivative for $m \circ \varphi$ and m is denoted by $\frac{dm \circ \varphi}{dm}$
- (2) α_φ denotes the $*$ -endomorphism of $L^\infty(X) = L^\infty(X, m)$ defined by $\alpha_\varphi(f) = f(\varphi(x))$ for f in $L^\infty(X)$.
- (3) T_φ denotes the linear operator on the Hilbert space $\mathcal{H} = L^2(X) = L^2(X, m)$ defined by $(T_\varphi \xi)(x) = \xi(\varphi(x))$ for ξ in $\mathcal{H}$.
- (4) For a subset Y of X , χ_Y means the characteristic function of Y .
- (5) For a measurable function f on X , M_f denotes the multiplication operator on $L^2(X)$ defined by $M_f \xi = f \xi$ for ξ in $L^2(X)$.

- (6) For f in $L^\infty(X)$, $\pi(f)$ denotes the bounded multiplication operator on $L^2(X)$ defined by $\pi(f)\xi = f\xi$ for ξ in $L^2(X)$.

Definition 2.1. Let X is a measure space with measure m . A measurable map φ of X onto X is said to be a map with n -laps, MWnL for short, if there exists n measurable subsets $\{X_i\}_{i=1}^n$ of X such that

- (1) $\cup_{i=1}^n X_i = X$ and $X_i \cap X_j = \emptyset$ for $i \neq j$.
- (2) Each restriction φ_i of φ to X_i is a bimeasurable map of X_i onto X in the sense that φ_i is an surjective map of X_i onto $\varphi_i(X_i)$ with $m(X \setminus \varphi_i(X_i)) = 0$ and φ_i^{-1} is measurable, too.
- (3) For each i , φ_i and φ_i^{-1} are absolutely continuous with respect to m and non-singular in the sense that

$$\frac{dm \circ \varphi}{dm}(x) \neq 0, \text{ a.e. } x \quad \text{and} \quad \frac{dm \circ \varphi^{-1}}{dm}(x) \neq 0, \text{ a.e. } x.$$

For a measure space (X, m) and a measurable map φ of X into itself, M_f and T_φ is not necessarily defined on the full space $\mathcal{H}$. Then each isometry V_i in the following definition, if necessary, is considered as a uniquely extended bounded linear operator on the full Hilbert space $\mathcal{H}$.

Definition 2.2. Let φ be a MWnL on a measure space (X, m) . We define a family isometries $\{V_i(\varphi)\}_{i=1}^n$ associated with φ as follows.

$$V_i(\varphi) = M_{\sqrt{dm \circ \varphi / dm}} M_{\chi_{X_i}} T_\varphi \quad (i = 1, \dots, n),$$

By the definition we can see that

- (1) $V_i(\varphi)^* = M_{\sqrt{dm \circ \varphi_i^{-1} / dm}} T_{\varphi_i^{-1}} \quad (i = 1, \dots, n).$
- (2) $V_i(\varphi)V_i(\varphi)^* = M_{\chi_{X_i}} \quad (i = 1, \dots, n).$
- (3) $\int_X f(\varphi(x))\eta(x)dm(x) = \sum_{i=1}^n \int_X \frac{dm \circ \varphi_i^{-1}}{dm} \eta(\varphi_i^{-1}(x))dm$ for η in $L^1(X, m)$.

Proposition 2.3. Let φ be a MWnL on a measure space (X, m) and $\{V_i = V_i(\varphi)\}_{i=1}^n$ a family isometries associated with φ defined in Definition 2.2. Then it follows that

- (1) $\{V_i\}_{i=1}^n$ satisfies condition (C.1) in §1, that is, $\{V_i\}_{i=1}^n$ is a FIC on $L^2(X, m)$.
- (2) $\pi(\alpha_\varphi(f)) = \alpha_V(\pi(f))$ for all f in $L^\infty(X)$.

Proposition 2.3 (2) implies that α_V is a $*$ -endomorphism of the von Neumann algebra $M_{L^\infty(X)}$ and we denote by A_φ the transpose of the restriction of α_V to $M_{L^\infty(X)}$. Then we have

$$(A_\varphi \eta)(x) = \sum_{i=1}^n \frac{dm \circ \varphi_i^{-1}}{dm} \eta(\varphi_i^{-1}(x)).$$

The transformation A_φ is known as Perron-Frobenius operator on $L^1(X, m)$.

Theorem 2.4. *Let φ be a MWnL on a measure space (X, m) . Suppose that there exists a FIC $\{W_i\}_{i=1}^n$ such that W_1 has eigenvalue 1 with eigenvector e and*

$$\alpha_V(T) = \alpha_W(T) \quad \text{for } T \text{ in } M,$$

where M is a von Neumann algebra on $\mathcal{H}$. Then for any state ω of the form $\omega = \sum_{k=1}^\infty \omega_{\xi_k}$ where ξ_k 's are in $\mathcal{H}_e$, it follows that

$$\lim_{n \rightarrow \infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\text{norm topology on } M_*).$$

Moreover, this implies that

$$\lim_{n \rightarrow \infty} \|A_\varphi^n(\eta) - |e|^2\|_1 = 0.$$

where $\eta = |\xi|^2$ for ξ in $\mathcal{H}_e$.

Proposition 2.5. *Let φ be a MW2L on the interval $[0, 1]$ with Lebesgue measure m . Then the following conditions are equivalent.*

- (1) $V_1(\varphi)$ has eigenvalue 1 with eigenvector e .
- (2) $m(\{x \in [0, 1] \mid \frac{d\varphi_1}{dm}(x) = 1\}) > 0$.

Theorem 2.6. *Let φ be a MWnL on a measure space (X, m) and $e(x) = 1$ for a.e. x in X . Then following conditions are equivalent.*

- (1) *There exists a FIC $\{W_i\}_{i=1}^n$ such that $\alpha_V(T) = \alpha_W(T)$ for T in $M_{L^\infty(X)}$ and $W_1 e = e$.*
- (2) *T_φ is an isometry.*
- (3) $\sum_{i=1}^n \frac{dm \circ \varphi_i^{-1}}{dm}(x) = 1$ for a.e. x in X .

Definition 2.7. Let φ and ψ be two MWnL's on (X, m) . Two maps are said to be AC-topologically conjugate if there exists a bijective map h of X onto itself satisfying following conditions.

- (1) $\varphi = h \circ \psi \circ h^{-1}$.
- (2) Both $m \circ h$ and $m \circ h^{-1}$ are absolutely continuous and non-singular with respect to m .

Remark. Let h be a absolutely continuous map satisfying (2) of the definition above. We put

$$U(h) = M_{\sqrt{dm \circ h / dm}} T_h.$$

Then $U(h)$ is a unitary operator on $\mathcal{H}$.

Theorem 2.8. Let φ and ψ be two $MWnL$'s on (X, m) . Suppose that ψ is AC-conjugate to φ and there exists a FIC $\{W_i\}_{i=1}^n$ satisfying following conditions.

(1) W_1 has eigenvalue 1 with unit eigenvector e .

(2) $\alpha_{V(\varphi)}(T) = \alpha_W(T)$ for T in M ,

where M is a von Neumann algebra on $\mathcal{H}$. Let $f = U(h^{-1})e$. Then for any state ω of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi_k}$ where ξ_k 's are in $\mathcal{H}_f$, it follows that

$$\lim_{n \rightarrow \infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\text{norm topology on } (U(h)MU(h)^*)_*).$$

§3. Examples of $MWnL$. We give typical and interesting examples of map with n laps. Each number in each example indicates the following.

(1) Measure space (X, m) on which a map is given.

(2) Map φ with n laps on X .

(3) Number n and partition $\{X_i\}_{i=1}^n$ of X .

(4) $\{V_i\}_{i=1}^n = \{V_i(\varphi)\}_{i=1}^n$ defined in Definition 2.2.

(4-1) An eigenvector e for eigenvalue 1 of W_1 and $ONS(e, V) = \{e_k\}_{k=1}^{\infty}$.

(4-2) $ONS(e, V)$ is complete or not.

(5) $\{W_i\}_{i=1}^n$ such that $\alpha_V(T) = \alpha_W(T)$ for T in a von Neumann algebra M on $L^2(X, m)$.

(6) The von Neumann algebra M on which $\alpha_V = \alpha_W$.

(6-1) An eigenvector e for eigenvalue 1 of W_1 and $ONS(e, W) = \{e_k\}_{k=1}^{\infty}$.

(6-2) $ONS(e, W)$ is complete or not.

(7) Perron-Frobenius operator A_φ .

Example 3.1. (Tent map)

(1) $X = [0, 1]$, and $m = \text{Lebesgue measure}$.

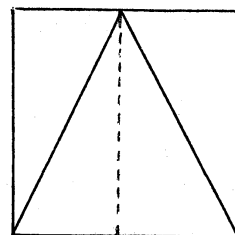
(2) φ is the map τ defined by

$$\tau(x) = 1 - |1 - 2x|.$$

(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$.

(4) $V_1 = \sqrt{2}M_{[0, 1/2)}T_\tau$, $V_2 = \sqrt{2}M_{[1/2, 1]}T_\tau$.

(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

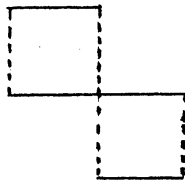
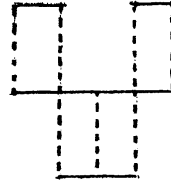
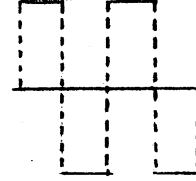


(6) $M = B(L^2[0, 1])$

(6-1) $e(x) = 1$ ($x \in [0, 1]$) and $e_1 = e$, $e_2 = M_{[0, 1/2]}e_1 - M_{[1/2, 1]}e_1$.

(6-2) $ONS(e, W)$ is complete.

(7) $A_r(\eta)(x) = \frac{1}{2} \left(\eta\left(\frac{x}{2}\right) + \eta\left(1 - \frac{x}{2}\right) \right)$.

 e_1  e_2  e_3  e_4

Example 3.2. (Generalized tent map)

(1) $X = [0, 1]$, and $m = \text{Lebesgue measure}$.

(2) $\varphi = \tau_c$, ($0 < c < 1$) defined by

$$\varphi_c(x) = \begin{cases} \frac{1}{c}x & \text{for } 0 \leq x \leq c, \\ \frac{1}{c-1}(x-1) & \text{for } c < x \leq 1. \end{cases}$$

(3) $n = 2$ and $X_1 = [0, 1/2]$, $X_2 = [1/2, 1]$.

(4) $V_1 = M_{\sqrt{1/c}} M_{\chi_{[0, c]}} T_{\tau_c}$, $V_2 = M_{\sqrt{1/(1-c)}} M_{\chi_{[c, 1]}} T_{\tau_c}$.

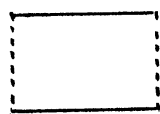
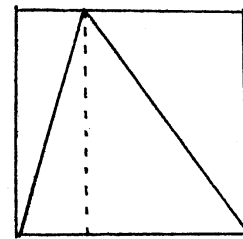
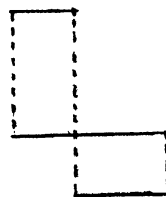
(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} \sqrt{c} & \sqrt{1-c} \\ \sqrt{1-c} & -\sqrt{c} \end{pmatrix}$.

(6) $M = B(L^2[0, 1])$

(6-1) $e(x) = 1$ ($x \in [0, 1]$) and $e_1 = e$, $e_2(x) = \begin{cases} \frac{1}{\sqrt{c}} & \text{for } 0 \leq x \leq c, \\ -\frac{1}{\sqrt{c-1}} & \text{for } c < x \leq 1. \end{cases}$

(6-2) $ONS(e, W)$ is complete.

(7) $A_{\tau_c}(\eta)(x) = c(\eta(cx)) + (1-c)\eta((c-1)x+1)$.

 e_1  e_2

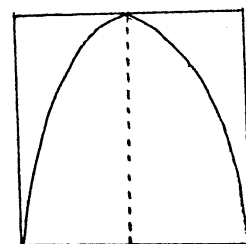
Remark. τ_c and τ_d are topologically conjugate (cf.[6],[8]) but they are AC-conjugate only if $c = d$.

Example 3.3. (Logistic map) (cf.[9])

(1) $X = [0, 1]$, and $m = \text{Lebesgue measure}$.

(2) φ is the map λ defined by

$\lambda(x) = 4x(1-x)$ (3) $n = 2$ and $X_1 = [0, 1/2]$, $X_2 = [1/2, 1]$.



$$(4) V_1 = \frac{1}{2\sqrt{1-2x}} M_{\chi_{[0,1/2]}} T_\lambda, \quad V_2 = \frac{1}{2\sqrt{2x-1}} M_{\chi_{[1/2,1]}} T_\lambda.$$

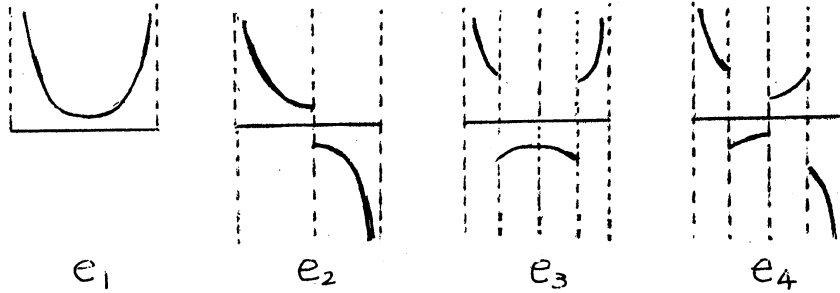
$$(5) (W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

$$(6) M = B(L^2[0, 1])$$

$$(6-1) e_1(x) = e(x) = 1/\sqrt{\pi\sqrt{x(1-x)}} \text{ and } e_2(x) = M_{\chi_{[0,1/2]}} e_1 - M_{\chi_{[1/2,1]}} e_1.$$

$$(6-2) ONS(e, W) \text{ is complete.}$$

$$(7) A_{\tau_c}(\eta)(x) = \frac{1}{4\sqrt{1-x}} \left(\eta \left(\frac{1-\sqrt{1-x}}{2} \right) - \eta \left(\frac{1+\sqrt{1-x}}{2} \right) \right).$$



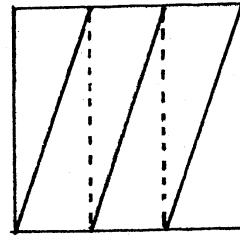
(The logistic map is topologically conjugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$ (cf.[7])).

Example 3.4. (Typical map with 3 laps)

$$(1) X = [0, 1], \text{ and } m = \text{Lebesgue measure.}$$

$$(2) \varphi \text{ is the map defined by}$$

$$\varphi(x) = \begin{cases} 3x & \text{for } 0 \leq x < 1/3, \\ 3x - 1 & \text{for } 1/3 \leq x < 2/3, \\ 3x - 2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$



$$(3) n = 3 \text{ and } X_1 = [0, 1/3], X_2 = [1/3, 2/3], X_3 = [2/3, 1].$$

$$(4) V_1 = \sqrt{3} M_{\chi_{[0,1/3]}} T_\varphi, \quad V_2 = \sqrt{3} M_{\chi_{[1/3,2/3]}} T_\varphi, \quad V_3 = \sqrt{3} M_{\chi_{[2/3,1]}} T_\varphi.$$

$$(5) (W_1, W_2, W_3) = (V_1, V_2, V_3) \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & (3-\sqrt{3})/6 & (-3-\sqrt{3})/6 \\ 1/\sqrt{3} & (-3-\sqrt{3})/6 & (3-\sqrt{3})/6 \end{pmatrix}.$$

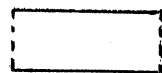
$$(6) M = B(L^2[0, 1])$$

$$(6-1) e(x) = 1 \text{ (} x \in [0, 1] \text{)} \text{ and } e_1 = e, \quad e_2(x) = \chi_{[0,1/3]} + \frac{\sqrt{3}-1}{2} \chi_{[1/3,2/3]} + \frac{-\sqrt{3}-1}{2} \chi_{[2/3,1]},$$

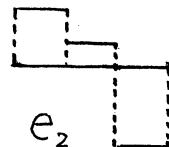
$$e_3(x) = \chi_{[0,1/3]} + \frac{-\sqrt{3}-1}{2} \chi_{[1/3,2/3]} + \frac{\sqrt{3}-1}{2} \chi_{[2/3,1]}.$$

$$(6-2) ONS(e, W) \text{ is complete.}$$

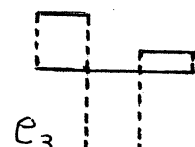
$$(7) A_\varphi(\eta)(x) = \frac{1}{3} \left(\eta \left(\frac{x}{3} \right) + \eta \left(\frac{x}{3} + \frac{1}{3} \right) + \eta \left(\frac{x}{3} + \frac{2}{3} \right) \right).$$



e_1



e_2



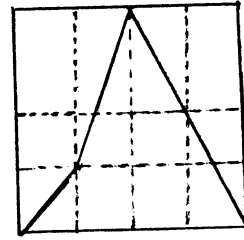
e_3

Example 3.5. (MW2L on $[0, 1]$ such that V_1 has an eigenvector for eigenvalue 1:a)

(1) $X = [0, 1]$ and m is the Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x) = \begin{cases} x & \text{for } 0 \leq x < 1/4, \\ (6x - 1)/2 & \text{for } 1/4 \leq x < 1/2, \\ -2x + 2 & \text{for } 1/2 \leq x \leq 1 \end{cases}$$



(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$

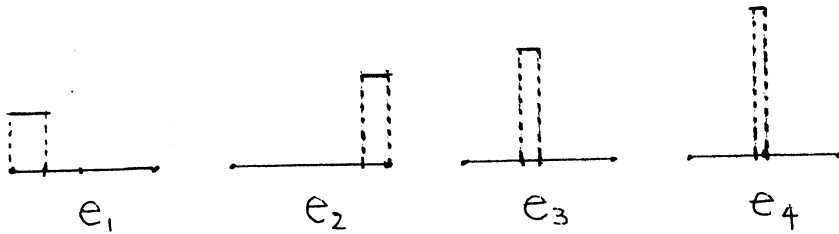
(4) $V_1 = M_{\chi_{[0,1/4)}} + \sqrt{3}M_{\chi_{[1/4,1/2)}}$, $V_2 = \sqrt{2}M_{\chi_{[1/2,1]}}$.

(4-1) $e_1 = e = 2\chi_{[0,1/4)}$, $e_2 = 2\sqrt{2}\chi_{(7/8,1]}$, $e_3 = 2\sqrt{6}\chi_{(11/24,1/2]}$, $e_4 = 4\chi_{[1/2,9/16)}$.

(4-2) $ONS(e, V)$ is not complete.

(6) $M = B(L^2[0, 1])$

(7) $A_\varphi(\eta)(x) = \eta(x)\chi_{[0,1/4]}(x) + \frac{1}{\sqrt{3}}\eta\left(\frac{2x+1}{6}\right)\chi_{[1/4,1]} + \frac{1}{\sqrt{2}}\eta\left(\frac{-x+2}{2}\right)$.

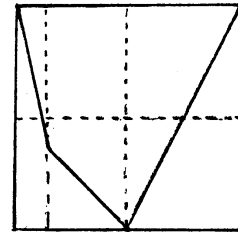


Example 3.6. (MW2L on $[0, 1]$ such that V_1 has an eigenvector for eigenvalue 1:b)

(1) $X = [0, 1]$ and m is the Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x) = \begin{cases} -5x + 1 & \text{for } 0 \leq x < 1/8, \\ -x + (1/2) & \text{for } 1/8 \leq x < 1/2, \\ 2x - 1 & \text{for } 1/2 \leq x \leq 1 \end{cases}$$



(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$

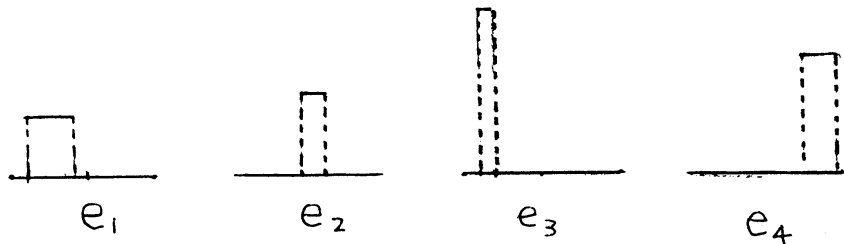
(4) $V_1 = \sqrt{5}M_{\chi_{[0,1/8)}} + M_{\chi_{[1/8,1/2)}}$, $V_2 = \sqrt{2}M_{\chi_{[1/2,1]}}$.

(4-1) $e_1 = e = 2\chi_{[1/8,3/8]}$, $e_2 = 2\sqrt{2}\chi_{[9/16,11/16]}$, $e_3 = 2\sqrt{10}\chi_{[5/80,7/80]}$, $e_4 = 4\chi_{[25/32,27/32]}$.

(4-2) $ONS(e, V)$ is not complete.

(6) $M = B(L^2[0, 1])$

(7) $A_\varphi(\eta)(x) = \eta\left(\frac{-2x+1}{2}\right)\chi_{[0,1/8)}(x) + \frac{1}{\sqrt{5}}\eta\left(\frac{-x+1}{5}\right)\chi_{[1/8,1]} + \frac{1}{\sqrt{2}}\eta\left(\frac{x+1}{2}\right)$.



Example 3.7. (Square root map)

(1) $X = [0, 1]$ and $m = \text{Lebesgue measure}$.

(2) φ is the map defined by

$$\varphi(x) = \begin{cases} \sqrt{2x} & \text{for } 0 \leq x < 1/2, \\ 1 - \sqrt{2x-1} & \text{for } 1/2 \leq x \leq 1. \end{cases}$$

(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$.

(4) $V_1 = (1/\sqrt{2x})M_{\chi_{[0,1/2)}}T_\varphi$, $V_2 = (1/\sqrt{2x-1})M_{\chi_{[1/2,1]}}T_\varphi$.

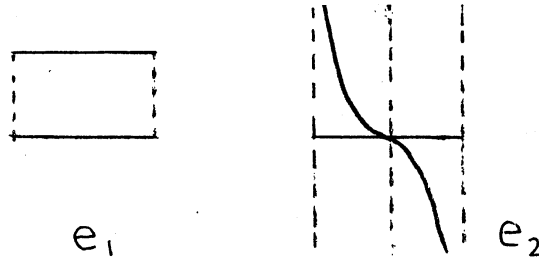
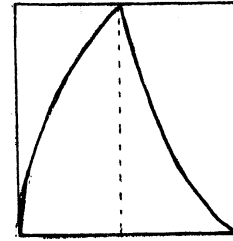
(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} M_{\sqrt{2cx}} & M_{\sqrt{1-2cx}} \\ M_{\sqrt{1-2cx}} & M_{\sqrt{2cx}} \end{pmatrix}$.

(6) $M = M_{L^\infty[0,1]}$

(6-1) $e_1(x) = e(x) = 1$, $e_2(x) = \sqrt{(1/\sqrt{2x}) - 1}\chi_{[0,1/2)}(x) - \sqrt{(1/\sqrt{2x-1}) - 1}\chi_{[1/2,1]}(x)$

(6-2) Now we cannot find whether $ONS(e, W)$ is complete or not.

(7) $A_\varphi(\eta)(x) = \frac{1}{x} \left(\eta\left(\frac{x^2}{2}\right) + \frac{1}{x-1} \eta\left(\frac{x^2-2x+2}{2}\right) \right)$.

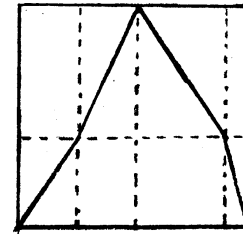


Example 3.8. (Map of broken line)

(1) $X = [0, 1]$ and $m = \text{Lebesgue measure}$.

(2) φ is the map defined by

$$\varphi(x) = \begin{cases} 8x/5 & \text{for } 0 \leq x < 1/4, \\ (12x-1)/5 & \text{for } 1/4 \leq x < 1/2, \\ (-12x+13)/7 & \text{for } 1/2 \leq x < 7/20, \\ (-8x+8)/3 & \text{for } 7/20 \leq x \leq 1, \end{cases}$$



(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$.

(4) $V_1 = (\sqrt{8/5}M_{\chi_{[0,1/4)}} + \sqrt{12/5}M_{\chi_{[1/4,1/2)}})T_\varphi$, $V_2 = (\sqrt{12/7}M_{\chi_{[1/2,7/20)}} + \sqrt{8/3}M_{\chi_{[7/20,1]}})T_\varphi$

(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} \sqrt{5/8}M_{\chi_{[0,2/5)}} + \sqrt{5/12}M_{\chi_{[2/5,1]}}, & \sqrt{3/8}M_{\chi_{[0,2/5)}} + \sqrt{7/12}M_{\chi_{[2/5,1]}} \\ \sqrt{3/8}M_{\chi_{[0,2/5)}} + \sqrt{7/12}M_{\chi_{[2/5,1]}}, & \sqrt{5/8}M_{\chi_{[0,2/5)}} - \sqrt{5/12}M_{\chi_{[2/5,1]}} \end{pmatrix}$.

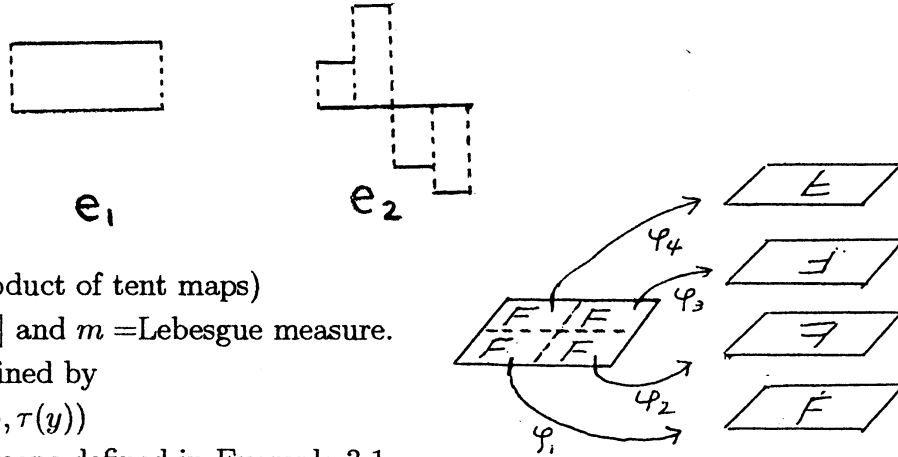
(6) $M = B(L^2[0, 2/5]) \oplus B(L^2[2/5, 1])$

(6-1) $e_1(x) = e(x) = 1$, $e_2(x) = \begin{cases} \sqrt{3/5} & \text{for } 0 \leq x < 1/4, \\ \sqrt{7/5} & \text{for } 1/4 \leq x < 1/2, \\ -\sqrt{5/7} & \text{for } 1/2 \leq x < 7/20, \\ -\sqrt{5/3} & \text{for } 7/20 \leq x \leq 1, \end{cases}$

(6-2) Now we cannot find whether $ONS(e, W)$ is complete or not.

$$(7) A_\varphi(\eta)(x) = \frac{5}{8}\eta\left(\frac{5x}{8}\right)\chi_{[0,2/5)}(x) + \frac{5}{12}\eta\left(\frac{5x+1}{12}\right)\chi_{[2/5,1)}(x)$$

$$+ \frac{3}{8}\eta\left(\frac{-3x+8}{8}\right)\chi_{[0,2/5)}(x) + \frac{7}{12}\eta\left(\frac{-7x+13}{12}\right)\chi_{[2/5,1)}(x).$$



Example 3.9. (Product of tent maps)

(1) $X = [0, 1] \times [0, 1]$ and $m = \text{Lebesgue measure}$.

(2) φ is the map defined by

$$\varphi(x, y) = (\tau(x), \tau(y))$$

where τ is the tent maps defined in Example 3.1.

(3) $n = 4$ and $X_1 = [0, 1/2] \times [0, 1/2]$, $X_2 = [1/2, 1] \times [0, 1/2]$, $X_3 = [1/2, 1] \times [1/2, 1]$, $X_4 = [0, 1/2] \times [1/2, 1]$.

$$(4) V_1 = 2M_{\chi_{[0,1/2] \times [0,1/2]}} T_\varphi, V_2 = 2M_{\chi_{[1/2,1] \times [0,1/2]}} T_\varphi, V_3 = 2M_{\chi_{[0,1/2] \times [1/2,1]}} T_\varphi, V_4 = 2M_{\chi_{[1/2,1] \times [1/2,1]}} T_\varphi.$$

$$(5) (W_1, W_2, W_3, W_4) = (V_1, V_2, V_3, V_4) \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix}$$

(6) $M = B(L^2([0, 1] \times [0, 1]))$

(6-1) $e_1(x, y) = e(x, y) = 1$ ($(x, y) \in [0, 1] \times [0, 1]$) and

$$e_2(x) = \chi_{[0,1/2] \times [0,1/2]} - \chi_{[1/2,1] \times [0,1/2]} + \chi_{[0,1/2] \times [1/2,1]} - \chi_{[1/2,1] \times [1/2,1]}.$$

(6-2) $ONS(e, W)$ is complete.

$$(7) A_\varphi(\eta)(x) = \frac{1}{4} \left(\eta\left(\frac{x}{2}, \frac{x}{2}\right) + \eta\left(1 - \frac{x}{2}, \frac{x}{2}\right) + \eta\left(\frac{x}{2}, 1 - \frac{x}{2}\right) + \eta\left(1 - \frac{x}{2}, 1 - \frac{x}{2}\right) \right).$$



Example 3.10. (Baker's transformation)

(1) $X = [0, 1] \times [0, 1]$ and $m = \text{Lebesgue measure}$.

(2) φ is the map β defined by

$$\beta(x, y) = \begin{cases} (2x, y/2) & \text{for } 0 \leq x < 1/2, \\ (2x - 1, (y + 1)/2) & \text{for } 1/2 \leq x \leq 1. \end{cases}$$

(3) $n = 1$ and $X_1 = X$

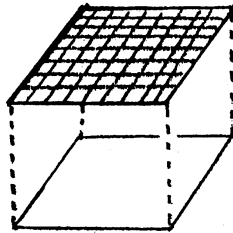
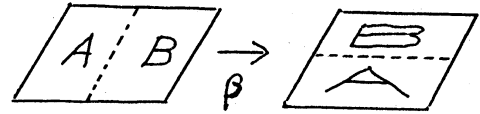
(4) $V_1 = T_\beta$

(4-1) $e_1(x, y) = e(x, y) = 1$

(4-2) $ONS(e, W) = \{e_1\}$ is not complete.

(6) $M = B(L^2([0, 1] \times [0, 1]))$

(7) $A_\beta(\eta)(x) = \eta(\beta(x))$



e_1

Remark. Baker's transformation is strong-mixing but $\{(\alpha_V^*)^n(\omega_\xi)\}_{n=1}^\infty$ does not converge to ω_e in the norm topology in M_* .

Example 3.11. (Unilateral shift map)

(1) $X = \prod_{n=1}^\infty \{1, 2\}$ and m = usual measure.

(2) φ is the map σ defined by

$$\sigma((x_1, x_2, x_3, \dots)) = (x_2, x_3, x_4, \dots),$$

(3) $n = 2$ and $X_1 = X(1) = \{(x_n)_{n=1}^\infty \in X | x_1 = 1\}$, $X_2 = X(2) = \{(x_n)_{n=1}^\infty \in X | x_1 = 2\}$

(4) $V_1 = \sqrt{2}M_{X(1)}T_\sigma$, $V_2 = \sqrt{2}M_{X(2)}T_\sigma$.

(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

(6) $M = B(L^2(X))$

(6-1) $e(x) = 1$ ($x \in X$) and $e_1 = e$, $e_2 = \chi_{X(1)}e_1 - \chi_{X(2)}e_1$.

(6-2) $ONS(e, W)$ is complete.

(7) $A_\sigma(\eta)(x) = \frac{1}{2}(\eta(\gamma_1) + \eta(\gamma_2))$,

where $\gamma_1((x_1, x_2, x_3, \dots)) = (1, x_1, x_2, \dots)$ and $\gamma_2((x_1, x_2, x_3, \dots)) = (2, x_1, x_2, \dots)$.

Example 3.12. (MW2L on the set N of all natural numbers)

(1) $X = N$ and m is the counting measure.

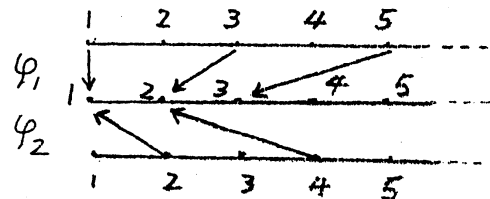
(2) φ is the map defined by

$$\varphi(2k - 1) = k \text{ and } \varphi(2k) = k \text{ } (k \in N)$$

(3) $n = 2$ and $X_1 = 2N - 1$, $X_2 = 2N$.

(4) $V_1 = M_{\chi_{(2N-1)}}T_\varphi$, $V_2 = M_{\chi_{(2N)}}T_\varphi$.

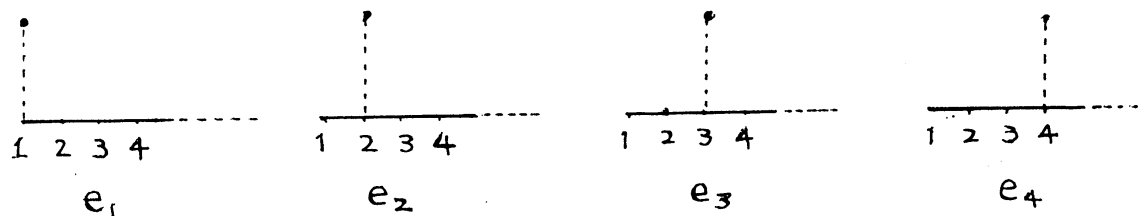
(4-1) $e = e_1 = \chi_{\{1\}}$ and the sequence $\{e_k\}_{k=1}^\infty$ is the canonical CONS of $\ell^2(N)$.



(4-2) $ONS(e, V)$ is complete.

(6) $M = B(\ell^2(N))$.

(7) $A_\varphi(\eta)(k) = \eta(2k - 1) + \eta(2k)$.



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