

A SOLUTION TO A PROBLEM OF TEODOR PRZYMUSIŃSKI

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A subset A of a space X is C^* -embedded in X if every bounded real-valued continuous function on A is continuously extendable to the whole of X . If this holds for all real-valued continuous functions on A , then A is C -embedded in X .

The present note provides detailed suggestions to the solution of the following problem. For a *non-discrete metric* space M and a subset A of a space X , does the C^* -embedding of $A \times M$ in $X \times M$ imply that it is also C -embedded in $X \times M$, i.e.

$$A \times M \overset{C^*}{\hookrightarrow} X \times M \implies A \times M \overset{C}{\hookrightarrow} X \times M ?$$

The problem was stated as Problem 3 of [T. Przymusiński, *Notes on extendability of continuous functions from products with a metric factor*, unpublished note, May 1983], later on as Problem 4.14 of [T. Hoshina, *Extensions of mappings II*, Topics in General Topology (K. Morita and J. Nagata, eds.), North-Holland, Amsterdam, 1989, pp. 41–80] and Problem 3.1 of [T. Hoshina, *Extensions of mappings*, Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), North-Holland, Amsterdam, 1992, pp. 405–416].

THE SOLUTION

To state the main result we call in use also the following imbedding-like properties. Let λ be an infinite cardinal number.

P^λ -embedding: A subset A of a space X is P^λ -embedded in X , or briefly $A \overset{P^\lambda}{\hookrightarrow} X$, if every continuous $f : A \rightarrow Y$ in a Banach space Y of $w(Y) \leq \lambda$ is continuously extendable to the whole of X .

U^ω -embedding: A subset A of a space X is U^ω -embedded in X , or briefly $A \overset{U^\omega}{\hookrightarrow} X$, if for every continuous $f : A \rightarrow \mathbb{R}$ there exists a continuous $g : X \rightarrow \mathbb{R}$ with $f(x) \leq g(x)$ whenever $x \in A$.

It should be mentioned that A is C -embedded in X if and only if it is P^ω -embedded in X , while A is P^ω -embedded in X if and only if it is both U^ω - and C^* -embedded in X . That is, always

$$C = P^\omega = U^\omega + C^*.$$

The following recent result was obtained together with Haruto Ohta.

Theorem. *For a P^λ -embedded subset A of a space X and a metric space M , the following conditions are equivalent*

- (a) $A \times M \xrightarrow{P^\lambda} X \times M$
- (b) $A \times M \xrightarrow{C^*} X \times M$
- (c) $A \times M \xrightarrow{U^\omega} X \times M$

Note that $A \times M \xrightarrow{C^*} X \times M$ implies $A \xrightarrow{C} X$ provided M is non-discrete because, in this case, M contains an infinite compact subset. Hence, the above result provides a complete positive solution to the problem of interest. For the proper understanding of this theorem, a word should be said also about the last condition (c). The statement that it is equivalent to the previous ones should be compared with Rudin-Starbird's result that, for a non-discrete metric space M , the normality of $X \times M$ implies the countable paracompactness of $X \times M$. Namely, the U^ω -embedding has a quite nice and useful reading just in terms of Ishikawa's characterization of countable paracompactness.

ON THE WAY TO THE PROOF

Special cases of (a) $\Leftrightarrow$ (b): $X \times M$ an M -independent product and $\lambda = \omega$ (Przymusiński, 1983); $M = \mathbb{P}$ the space of irrational numbers and $\lambda = \omega$ (Ohta, 1993); M σ -locally compact (Yamazaki, 1997); M^2 homeomorphic to M (Hoshina and Yamazaki, 199?).

FIRST STEP: A reduction to "nice" metric factors

For a space Y , let $\mathcal{P}(Y)$ be the set of all closed subsets of Y . Let A , X and M be as in our theorem. To M we associate the family of *all solutions*, or the *Przymusiński* family for M , by

$$\mathfrak{P} = \{S \subset M : A \times S \xrightarrow{P^\lambda} X \times S\}.$$

The following important fact will play a central role in this part of the proof.

Fact 1 (Michael). $S \in \mathfrak{P} \implies \mathcal{P}(S) \subset \mathfrak{P}$.

It will be useful to illustrate the idea first on a partial case. For the purpose, let $M^{(\mathcal{K},0)} = M$, and, for every ordinal $\alpha > 0$, let

$$M^{(\mathcal{K},\alpha)} = X \setminus \bigcup \{K \subset M \text{ compact} : K \subset M^{(\mathcal{K},\beta)} \text{ is open for some } \beta < \alpha\}.$$

Take an ordinal γ with $M^{(\mathcal{K},\gamma)} = M^{(\mathcal{K},\gamma+1)}$. Then,

1. $M^{(\mathcal{K},\gamma)} \in \mathcal{P}(M)$ is **nowhere locally compact**;
2. $M \setminus M^{(\mathcal{K},\gamma)}$ is σ -**locally compact**.

Now, suppose that M is a Polish space with $\dim(M) = 0$. Then, relaying on the known partial solution and Fact 1, we get the following series of implications.

$$M^{(\mathcal{K},\gamma)} = \emptyset \implies M \text{ is } \sigma\text{-locally compact} \implies M \in \mathfrak{P}.$$

On the other hand,

$$\begin{aligned} M^{(\mathcal{K},\gamma)} \neq \emptyset &\implies M^{(\mathcal{K},\gamma)} = \mathbb{P} \\ &\Downarrow \\ &M^{(\mathcal{K},\gamma)} \in \mathfrak{P} \\ &\Downarrow \\ M \in \mathcal{P}(\mathbb{P}) = \mathcal{P}(M^{(\mathcal{K},\gamma)}) &\subset \mathfrak{P}. \end{aligned}$$

That is, always $M \in \mathfrak{P}$.

Let $\mathcal{K} = \{S \in \mathcal{P}(M) : S \text{ is compact}\}$. Then, by the known results, $\mathcal{K} \subset \mathfrak{P}$. On the other hand, $M^{(\mathcal{K},\gamma)}$ is a resulting set by a $\mathcal{K}$ -*scattered* procedure and, hence, a procedure that is *scattered* also with respect to a part of the members of $\mathfrak{P}$. This arguments suggest that, for a better result, we need to call in use all members of $\mathfrak{P}$, i.e. to arrange a $\mathfrak{P}$ -*scattered* procedure on M .

Turning to this case, we change our definition as follows. Let $S \subset M$, and let $S^{(\mathfrak{P},0)} = S$. Next, for any ordinal $\alpha > 0$, we consider the set

$$S^{(\mathfrak{P},\alpha)} = S \setminus \bigcup \{U \subset S : U \text{ is open and } \text{cls}_S(U) \cap S^{(\mathfrak{P},\beta)} \in \mathfrak{P} \text{ for some } \beta < \alpha\}.$$

Suppose that $M \notin \mathfrak{P}$, and let $S \in \mathcal{P}(M) \setminus \mathfrak{P}$ be such that

$$w(S) = \min\{w(F) : F \in \mathcal{P}(M) \setminus \mathfrak{P}\}.$$

Then, as before, take an ordinal γ with $S^{(\mathfrak{P},\gamma)} = S^{(\mathfrak{P},\gamma+1)}$. As a result, we get that

1. $S^{(\mathfrak{P},\gamma)} \in \mathcal{P}(S)$ is **weight-homogeneous**, that is, $w(U) = w(S)$ for every non-empty open $U \subset S$;
2. $S \setminus S^{(\mathfrak{P},\gamma)}$ has a σ -**discrete closed cover** $\Sigma \subset \mathfrak{P}$.

On the other hand, for the members of $\mathfrak{P}$, we have that

Fact 2. $\mathcal{D} \subset \mathfrak{P}$ discrete in $\bigcup \mathcal{D} \implies \bigcup \mathcal{D} \in \mathfrak{P}$.

In view of our next arguments, let us make the following

Assumption. $S(\mathfrak{P}, \gamma) \in \mathfrak{P}$.

As a result, we now get that

Conclusion 3. There exists a countable cover $\mathcal{F}$ of S with $\mathcal{F} \subset \mathcal{P}(S) \cap \mathfrak{P}$.

Conclusion 4. $A \times S \xrightarrow{\text{well}} X \times S$.

Here, $A \times S \xrightarrow{\text{well}} X \times S$ if $A \times S$ is completely separated from any zero-set of $X \times S$ which doesn't meet $A \times S$. To involve Conclusion 4, we also need the following *weak embedding* properties:

C_1 -embedding: A subset B of Y is C_1 -embedded in Y , or briefly $B \xrightarrow{C_1} Y$, if $F \xrightarrow{\text{well}} Y$ for every zero-set F of B . That is, for any zero-set F of B and any zero-set Z of Y , with $Z \cap F = \emptyset$, there exists a zero-set Z_F of Y such that $F \subset Z_F$ and $Z_F \cap Z = \emptyset$.

CU -embedding: A subset B of Y is CU -embedded in Y , or briefly $B \xrightarrow{CU} Y$, if for any zero-set F of B and any zero-set Z of Y , with $Z \cap F = \emptyset$, there exists a zero-set Z_F of Y such that $F \subset Z_F$ and $Z_F \cap Z \cap B = \emptyset$.

The relations between our weak-embedding properties could be now summarized into the following diagram.

Observation 5.

$$\begin{array}{ccc} C^* & & U^\omega \\ & \searrow \quad \swarrow & \\ C_1 & = CU + \text{well} & \end{array}$$

Then, by Conclusion 4, we have

Conclusion 6. $A \times S \xrightarrow{C_1} X \times S$.

According to Conclusion 3, this implies

Final Conclusion. $S \in \mathfrak{P}$.

The so obtained contradiction provides the following result which accomplishes the first step of the proof of our theorem.

Theorem A. $M \in \mathfrak{P}$ provided $S \in \mathfrak{P}$ for any weight-homogeneous and nowhere locally compact $S \in \mathcal{P}(M)$.

SECOND STEP: Separating the factors

NOTATIONS: For sets D and R , let R^D denote all maps from D to R , and 2^R — all subsets of R . For cardinals κ and μ , let $\kappa^{<\mu} = \bigcup \{\kappa^\delta : \delta < \mu\}$. For reasons of convenience, we regard κ^0 as the singleton $\{\emptyset\}$. To every $\sigma \in \kappa^\delta$ and $\alpha < \kappa$ we associate another map $\sigma \hat{\alpha} \in \kappa^{\delta+1}$ defined by $\sigma \hat{\alpha}(\delta) = \sigma$ and $\sigma \hat{\alpha}(\delta) = \alpha$. Also, to every $\mathcal{H} : T \rightarrow (2^R)^D$ we associate another map $\langle \mathcal{H}, D \rangle : T \rightarrow 2^R$ defined by

$$\langle \mathcal{H}, D \rangle(t) = \bigcup \mathcal{H}[t](D) \quad \forall t \in T.$$

Finally, for a space Y , we shall use $\text{coz}(Y)$ to denote the collection of all *cozero-sets* of Y and $\text{zero}(Y)$ for that of all *zero-sets* of Y .

CONCEPTS:

Monotone decreasing map: $\mathcal{H} : \kappa^{<\omega} \rightarrow (2^R)^D$ if $\mathcal{H}[\sigma \hat{\alpha}](D)$ refines $\mathcal{H}[\sigma](D)$ for every $\sigma \in \kappa^{<\omega}$ and $\alpha < \kappa$.

Sieve: $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(Y)$ if $\mathcal{S}(\emptyset) = Y$ and $\mathcal{S}(\sigma) = \bigcup \{\mathcal{S}(\sigma \hat{\alpha}) : \alpha < \kappa\}$ for every $\sigma \in \kappa^{<\omega}$.

Strong Sieve: $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(Y)$ if $\mathcal{S}$ is a sieve such that $\emptyset \notin \mathcal{S}(\kappa^{<\omega})$, each family $\mathcal{S}(\kappa^n)$, $n < \omega$, is a locally finite in Y and, whenever $y \in \bigcap \{\mathcal{S}(t|n) : n < \omega\}$ for some $t \in \kappa^\omega$, the collection $\mathcal{S}(t|n)$, $n < \omega$, stands for a local base at y in Y .

$\mathcal{S}$ -free map: $\mathcal{G} : \kappa^{<\omega} \rightarrow (2^Y)^\kappa$, where $\mathcal{S}$ is a map $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(M)$, if for every $t \in \kappa^\omega$ we have that $\bigcap \{\text{cl}_Y((\langle \mathcal{G}, \kappa \rangle(t|n)) \times \mathcal{S}(t|n)) : n < \omega\} = \emptyset$.

Expansion: $\mathcal{H} : \kappa^{<\omega} \rightarrow (2^X)^\kappa$ of $\mathcal{G} : \kappa^{<\omega} \rightarrow (2^Y)^\kappa$, where $Y \subset X$, if $\mathcal{G}[\sigma](\alpha) = \mathcal{H}[\sigma](\alpha) \cap Y$ whenever $\sigma \in \kappa^{<\omega}$ and $\alpha < \kappa$.

The second step of the proof of our theorem reads now as follows.

Theorem B. *Under the conditions of the main theorem, let, in addition, M be weight homogeneous and nowhere locally compact. Also, let $w(M) = \kappa$. Then, the following conditions are equivalent.*

- (a) $A \times M \xrightarrow{C^*} X \times M$
- (b) Whenever $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(M)$ is a strong sieve, every monotone decreasing and $\mathcal{S}$ -free map $\mathcal{G} : \kappa^{<\omega} \rightarrow \text{coz}(A)^\kappa$ has a monotone decreasing and $\mathcal{S}$ -free expansion $\mathcal{G} : \kappa^{<\omega} \rightarrow \text{coz}(X)^\kappa$.
- (c) $A \times M \xrightarrow{P^\lambda} X \times M$

Here is a brief scheme of (a) $\implies$ (b). Suppose that $\mathcal{G} : \kappa^{<\omega} \rightarrow \text{coz}(A)^\kappa$ and $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(M)$ are as in (b). Then, the statement that $\mathcal{G}$ is an $\mathcal{S}$ -free map becomes equivalent to the statement that the family $\{(\mathcal{G}, \kappa)(\sigma) \times \mathcal{S}(\sigma) : \sigma \in \kappa^{<\omega}\}$ is locally finite in $A \times M$. The last becomes “almost” equivalent to the existence of $F_{(\mathcal{G}, \mathcal{S})}^0, F_{(\mathcal{G}, \mathcal{S})}^1 \in \text{zero}(A \times M)$ such that $F_{(\mathcal{G}, \mathcal{S})}^0 \cap F_{(\mathcal{G}, \mathcal{S})}^1 = \emptyset$. However, by (a), $A \times M \xrightarrow{C^*} X \times M$. Hence, there are $Z_{(\mathcal{H}, \mathcal{S})}^0, Z_{(\mathcal{H}, \mathcal{S})}^1 \in \text{zero}(X \times M)$ such that

$$F_{(\mathcal{G}, \mathcal{S})}^i \subset Z_{(\mathcal{H}, \mathcal{S})}^i, \quad i < 2, \quad \text{and} \quad Z_{(\mathcal{H}, \mathcal{S})}^0 \cap Z_{(\mathcal{H}, \mathcal{S})}^1 = \emptyset.$$

Relying on the “almost” equivalence mentioned above, these two zero-sets of $X \times M$ yield a monotone decreasing and $\mathcal{S}$ -free expansion $\mathcal{H} : \kappa^{<\omega} \rightarrow \text{coz}(X)^\kappa$ of $\mathcal{G}$.

Here is also a brief scheme of (b) $\implies$ (c). This implication is based on the following chain of arguments.

Fact 1. There exists a strong sieve $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(M)$ on M such that

$$\mathcal{S}_n(z) = \bigcup \{ \mathcal{S}(\sigma) : \sigma \in \kappa^n \text{ \& } z \in \text{cl}_M(\mathcal{S}(\sigma)) \}, \quad n < \omega,$$

constitute a local base at z for every $z \in M$.

A CONCEPT MORE: Let $\mathbb{I} = [0, 1]$.

Sieve partition of unity: $\xi : \kappa^{<\omega} \rightarrow C(M, \mathbb{I})$, or a function version of strong sieve, if $\xi[\emptyset]$ is the constant function on M with the value of 1, and $\xi[\sigma] = \sum \{ \xi[\sigma \hat{\ } \alpha] : \alpha < \kappa \}$ for every $\sigma \in \kappa^{<\omega}$.

Fact 2. For every strong sieve $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(M)$ there exists a sieve-partition of unity $\xi : \kappa^{<\omega} \rightarrow C(M, \mathbb{I})$ such that $\text{supp}(\xi[\sigma]) \subset \mathcal{S}(\sigma)$ for every $\sigma \in \kappa^{<\omega}$.

Let $(Y, \|\cdot\|)$ be a Banach space, and let $f : A \times M \rightarrow Y$ be a continuous map. The statement of (c) becomes now equivalent to the existence of a continuous map $g : X \times M \rightarrow Y$ with $g|_{A \times M} = f$. Towards this end, for every space T we shall associate a map Δ_T

$$T \longrightarrow \Delta_T : C(T \times M, Y) \rightarrow C(T, Y)^{\kappa^{<\omega}}$$

that defines into the following manner. Let $\mathcal{S} : \kappa^{<\omega} \rightarrow \text{coz}(M)$ be a strong sieve on M as in Fact 1. Take a dense $D \subset M$ with $|D| = \kappa$, and then define a map $\theta : \kappa^{<\omega} \rightarrow M$ by $\theta(\alpha) \in D \cap \mathcal{S}(\alpha)$ for every $\alpha \in \kappa^{<\omega}$. Finally, our Δ_T is defined by $\Delta_T(h)[\sigma](x) = h(x, \theta(\sigma))$ whenever $h \in C(T \times M, Y)$, $\sigma \in \kappa^{<\omega}$ and $x \in T$.

The correspondence Δ_T is “nice” invertible on the image of $C(T \times M, Y)$ under Δ_T . That is, one could restore in full $h \in C(T \times M, Y)$ relying only on $\Delta_T(h)$. Namely,

let $\xi : \kappa^{<\omega} \rightarrow C(M, \mathbb{I})$ be a sieve partition of unity on M as in Fact 2 applied to $\mathcal{S}$. Then,

$$(*) \quad h = \lim_{n \rightarrow \infty} \sum \{\xi[\sigma] \cdot \Delta_T(h)[\sigma] : \sigma \in \kappa^n\}.$$

The idea of (b) $\Rightarrow$ (c) could be now stated in the following abstract setting. To the map f we associate the corresponding one $\Phi = \Delta_A(f) : \kappa^{<\omega} \rightarrow C(A, Y)$. In this way, the correspondence Δ_T transforms our extension problem to an extension problem for Φ . Namely, it is now sufficient to find $\Gamma : \kappa^{<\omega} \rightarrow C(X, Y)$ subject to the following

Extension Condition:

$$(EC) \quad \Gamma[\sigma] \Big|_A = \Phi[\sigma], \quad \text{for every } \sigma \in \kappa^{<\omega};$$

Continuity Condition:

$$(CC) \quad \Gamma \in \Delta_X(C(X \times M, Y)).$$

If one could deal with this last problem, then merely $g = \Delta_X^-(\Gamma) \in C(X \times M, Y)$ will be the required extension of f . Turning to this, let us observe that

$$A \xrightarrow{P^\lambda} X \quad \Longrightarrow \quad \text{“many” solutions of (EC)}$$

$$???????? \quad \Longrightarrow \quad \text{at least one solution of (CC)}$$

To discover the nature of (CC) we call in use (*) and thus we get the following its more concrete setting:

$$(CC)^* \quad \lim_{n \rightarrow \infty} \sum \{\xi[\sigma] \cdot \Gamma[\sigma] : \sigma \in \kappa^n\} \in C(X \times M, Y).$$

We are now ready for the final realization of this implication. Namely, the hidden property “????????” becomes the **controlled** extending of monotone decreasing $\mathcal{S}$ -free maps. That is, just these maps will take care about the control on (CC). Briefly, to the map Φ we associate a sequence $\{\mathcal{F}_\ell : \ell < \omega\}$ of monotone decreasing and $\mathcal{S}$ -free maps $\mathcal{F}_\ell : \kappa^{<\omega} \rightarrow \text{coz}(A)^\kappa$. According to (b), each $\mathcal{F}_\ell$ admits a monotone decreasing and $\mathcal{S}$ -free expansion $\mathcal{G}_\ell : \kappa^{<\omega} \rightarrow \text{coz}(X)^\kappa$.

The fact that $\Phi = \Delta_A(f)$ could be now stated as

$$\begin{aligned} \ell < \omega, \quad m \leq n < \omega \quad \& \quad \sigma \in \kappa^n \\ \Downarrow \\ \|\Phi[\sigma](x) - \Phi[\sigma|m](x)\| &\leq \frac{1}{2^{\ell+1}} \quad \forall x \in A \setminus \langle \mathcal{F}_\ell, \kappa \rangle(\sigma|m) \end{aligned}$$

Relying on this, we finally construct Γ just satisfying the same condition, i.e. such that

$$\begin{aligned}
 (\text{CC})^{**} \quad & \ell \leq m \leq n < \omega \quad \& \quad \sigma \in \kappa^n \\
 & \Downarrow \\
 & \|\Gamma[\sigma](x) - \Gamma[\sigma|_m](x)\| \leq \frac{1}{2^{\ell+1}} \quad \forall x \in X \setminus \langle \mathcal{G}_\ell, \kappa \rangle(\sigma|_m)
 \end{aligned}$$