

Scattering problem for nonlinear Schrödinger and Hartree equations

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We consider the existence and asymptotic completeness of the wave operators for nonlinear Schrödinger equations of the form

$$\text{NLS} \quad i\partial_t u + \frac{1}{2}\Delta u = f(u),$$

where u is a complex-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbf{R}^n$, and f describes the nonlinear interaction. Typical form of f is given by

- (power-type nonlinearity) $f(u) = \lambda|u|^{p-1}u$, where $\lambda \in \mathbf{R}$ and $p > 1$,

and by

- (Hartree-type nonlinearity) $f(u) = (V * |u|^2)u$, where $V(x) = \lambda|x|^{-\gamma}$, $\lambda \in \mathbf{R}$, $\gamma > 0$, and $*$ denotes the convolution in $\mathbf{R}^n$.

There is a large literature on the scattering theory for NLS (see for instance [1-39] and references therein), where the existence and asymptotic completeness of the wave operators is the most basic problem. The problem is usually formulated as follows. Let $U(t) = \exp(i(t/2)\Delta) = \mathcal{F}^{-1} \exp(-i(t/2)|\xi|^2)\mathcal{F}$ be the unitary group associated with the free Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = 0.$$

Let $v_+(t) = U(t)\phi_+$ be a free solution with Cauchy data ϕ_+ in a suitable space X . The first half of the problem is the existence and uniqueness of solutions of NLS behaving as v_+ in X as $t \rightarrow +\infty$. If that is the case, the map $W_+ : \phi_+ \mapsto u(0)$ is well defined in X and is called the wave operator for positive time. The wave operator for negative time $W_- : \phi_- \mapsto u(0)$ is similarly introduced on the basis of the existence and uniqueness of solutions of NLS behaving as $v_-(t) = U(t)\phi_-$ in X as $t \rightarrow -\infty$. We call $\phi_\pm$ asymptotic states at $\pm\infty$ and $\phi = u(0)$ an interacting state. The second half of the problem is the existence of asymptotic states $\phi_\pm$ at $\pm\infty$ for a given interacting state ϕ in X in the sense that the solution u of NLS with $u(0) = \phi$ behaves as $v_\pm(t) = U(t)\phi_\pm$ in X

as $t \rightarrow \pm\infty$. If that is the case, the ranges of wave operators $W_{\pm}$ are characterized as $\text{Ran}(W_+) = \text{Ran}(W_-) = X$ and we say that asymptotic completeness holds.

Asymptotic completeness is a much more difficult problem than the existence of wave operators except in the small data setting. Regarding asymptotic completeness for large data in the sense as above, we need sharp a priori estimates for solutions of NLS to impose strong assumptions on the nonlinear interaction, such as admissible ranges of p and γ and some repulsivity condition.

In the case of NLS with power-type nonlinearity as above, the available results on the existence of the wave operators are summarized as follows, where we assume that $p < 1 + 4/(n - 2)$ if $n \geq 3$ throughout.

- (A1) In the energy space $X = H^1$ the wave operators exist if $p > 1 + 4/n$ [11].
- (A2) In the weighted energy space $X = H^1 \cap \mathcal{F}(H^1)$ the wave operators exist if $p > \text{Max}(1 + 2/n, 1 + 4/(n + 2))$ [4].
- (A3) In the space $X = H^{s'} \cap \mathcal{F}(H^s)$ with $0 < s, s' < 2$ the wave operators exist if $\text{Max}(1 + 2/n, 1 + 4/(n + 2s), s, s') < p < \begin{cases} 1 + 4/n & \text{if } s' < 1, \text{ or if } s' \geq 1 \text{ and } \lambda < 0, \\ 1 + 4/(n - 2) & \text{if } s' \geq 1 \text{ and } \lambda > 0 \end{cases}$ [8].
- (A4) In the space $X = H^s \cap \mathcal{F}(H^s)$ with $0 < s < 2$ the wave operators exist if $p = 1 + 4/(n + 2s)$ and $p > \text{Max}(1 + 2/n, s)$ [29].
- (A5) The wave operators do not exist if $p \leq 1 + 2/n$ in the sense that the convergence $U(-t)u(t) \rightarrow \phi_+$ in L^2 implies $u(0) = \phi_+ = 0$ [1, 32, 37].

We note that (A2) is a special case of (A3). The number $p = 1 + 2/n$ is the borderline where the usual framework of scattering as above breaks down and a special treatment is required by taking long-range effect of nonlinearity into account [7, 32]. Therefore, in the present setting the optimal result is provided by (A2) for $n \leq 2$ and by (A3) for $n \leq 3$.

As regards the asymptotic completeness, the available results are summarized as follows, where we assume that $\lambda > 0$ and that $p < 1 + 4/(n - 2)$ if $n \geq 3$ throughout.

- (B1) In the space $X = H^1$ the asymptotic completeness holds if $p > 1 + 4/n$ [11, 27].
- (B2) In the space $X = H^1 \cap \mathcal{F}(H^1)$ the asymptotic completeness holds if $p > \gamma(n) \equiv (n + 2 + \sqrt{n^2 + 12n + 4})/(2n)$ [18, 37, 38], or if $p = \gamma(n)$ for $n \neq 2$ [2, 4].

In the case of Hartree-type nonlinearity, the corresponding results available so far is summarized as follows.

- (C1) In the space $X = H^1$ the wave operators exist and are asymptotically complete if $2 < \gamma < \text{Min}(4, n)$ and $\lambda > 0$ [13, 28].
- (C2) In the space $X = H^1 \cap \mathcal{F}(H^1)$ the wave operators exist and are asymptotically complete if $4/3 < \gamma < \text{Min}(4, n)$ and $\lambda > 0$ [16, 17].
- (C3) In $X = H^1 \cap \mathcal{F}(H^1)$ the wave operators exist if $1 < \gamma < \text{Min}(2, n)$ [31].
- (C4) The wave operators do not exist in the sense of (A5) if $\gamma \leq 1$ [14, 16, 17].

The purpose of this talk is to present a sharp framework of function spaces where the corresponding Strichartz estimates work and contribution of decay factors of the form $|t|^{-\nu}$ keeps a scaling invariance up to the limiting case that has been excluded.

Theorem 1. *Let s, s' satisfy $0 \leq s', s < 2$. Let p satisfy*

$$\text{Max}(1 + 2/n, s, s') < p < \begin{cases} 1 + 4/n \text{ if } s' < 1, \text{ or if } s' \geq 1 \text{ and } \lambda < 0, \\ 1 + 4/(n - 2) \text{ if } s' \geq 1 \text{ and } \lambda > 0, \end{cases}$$

$$p \geq 1 + 4/(n + 2s).$$

Then for NLS with power-type nonlinearity the wave operators exist in the space $X = H^{s'} \cap \mathcal{F}(H^s)$.

Theorem 2. *Let $p = \gamma(n)$ and $\lambda > 0$. Then for NLS with power-type nonlinearity asymptotic completeness holds in the space $X = H^1 \cap \mathcal{F}(H^1)$.*

Theorem 3. *Let $\gamma = 4/3, n \geq 2$, and $\lambda > 0$. Then for NLS with Hartree type nonlinearity asymptotic completeness holds in the space $X = H^1 \cap \mathcal{F}(H^1)$.*

Theorem 1 improves (A2)(A3)(A4). Note that NLS with $p = 1 + 4/(n + 2s)$ has a scaling invariance in the homogeneous space $\mathcal{F}(\dot{H}^s) = |x|^{-s}L^2$. Theorem 2 closes a gap in (B2). Our method of the proof is independent of the pseudo-inversion (pseudo-conformal transformation)

$$u(t, x) \mapsto (it)^{-n/2} \exp(i|x|^2/2t) \overline{u(1/t, x/t)}$$

that has been used in [4, 17, 29, 37, 38, 39] and provides a unified treatment regardless of the spatial dimension n . Theorem 3 closes a gap in (C2).

Our method of the proof depends on:

- (1) The treatment of regularity of functions in space directions in terms of the operators $U(t)\psi U(-t)$ instead of $\mathcal{F}^{-1}\psi\mathcal{F} = \psi(-i\nabla)$. In particular, we adopt it in the Besov style by way of the Littlewood-Paley decomposition, which enables us to make the most of the power behavior of nonlinearity at the origin compatible with regularity of fractional order.
- (2) The treatment of integrability in time in terms of the Lorentz space instead of the usual Lebesgue space. This enables us to make the most of decay factors such as $|t|^{-\nu}$ up to the limiting case keeping up a scaling invariance.
- (3) The treatment of the pseudo-conformal charge that is compatible with the framework given by (1) and (2). This provides us with sharp a priori estimates for solutions of NLS with repulsive interaction.

To be more specific, for the existence of the wave operators we consider the integral equations of the form

$$\begin{aligned} u(t) &= U(t)\phi_+ + i \int_t^\infty U(t-t')f(u(t'))dt' \\ &\equiv U(t)\phi_+ + i(Gf(u))(t). \end{aligned}$$

We solve the integral equation by a contraction argument on a closed ball of a suitable function space over the time interval $[T, \infty)$ with $T > 0$ sufficiently large. For that purpose we use the following Strichartz estimates, where the integrability and regularity in space are measured respectively by the Lebesgue and homogeneous Besov spaces and the integrability in time is measured by the Lorentz spaces.

Proposition. *Let $q, r, q_j, r_j, j = 1, 2$, satisfy*

$$\begin{aligned} 0 &< 2/q = n/2 - n/r < 1, \\ 0 &< 2/q_j = n/2 - n/r_j < 1. \end{aligned}$$

Let $\rho \in \mathbf{R}$. Then

$$\begin{aligned} \| U(\cdot)\phi; L^{q,2}(\mathbf{R}; L^r(\mathbf{R}^n)) \| &\leq C \| \phi; L^2(\mathbf{R}^n) \|, \\ \| |t|^\rho M^{-1}U\phi; L^{q,2}(\mathbf{R}; \dot{B}_{r,2}^\rho(\mathbf{R}^n)) \| &\leq C \| |x|^\rho \phi; L^2(\mathbf{R}^n) \|, \\ \| Gf(u); L^{q_1,2}(\mathbf{R}; L^{r_1}(\mathbf{R}^n)) \| &\leq C \| f(u); L^{q'_2,2}(\mathbf{R}; L^{r'_2}(\mathbf{R}^n)) \|, \\ \| |t|^\rho M^{-1}Gf(u); L^{q_1,2}(\mathbf{R}; \dot{B}_{r_1,2}^\rho(\mathbf{R}^n)) \| &\leq C \| |t|^\rho M^{-1}f(u); L^{q'_2,2}(\mathbf{R}; \dot{B}_{r'_2,2}^\rho(\mathbf{R}^n)) \|, \end{aligned}$$

where $M = M(t) = \exp(i|x|^2/2t)$ and p' is the dual exponent of p defined by $1/p + 1/p' = 1$.

The above integral equation is treated in a similar way as in [8] on the basis of the proposition and the generalized Hölder inequality [23] so that the resulting time integral in [8] that just diverges in the critical case $p = 1 + 4/(n + 2s)$ is replaced by the finite Lorentz (weak-Lebesgue) norm $\| |t|^{-s(p-1)}; L^{\theta, \infty} \|$ with $1/\theta = s(p-1)$. To make the contraction factor sufficiently small, it suffices to notice that $\|U(\cdot)\phi; L^{q,2}(T, \infty; L^r)\| \rightarrow 0$ as $T \rightarrow \infty$. That is the essential idea for the proof of Theorem 1. For the proof of Theorem 2, we regard the conformal identity as

$$\frac{d}{dt}(t^{-\nu}P(t)) = -\nu t^{-\nu-1} \|(x + it \nabla)u(t); L^2\|^2,$$

where $\nu = n(1 + 4/n - p)/2$ and

$$P(t) = \|(x + it \nabla)u(t); L^2\|^2 + \frac{2\lambda t^2}{p+1} \|u(t); L^{p+1}\|^{p+1}.$$

This yields the a priori estimate

$$\|t^{-\nu} \|(x + it \nabla)u; L^2\|; L^\infty \cap L^2(1, \infty; \frac{dt}{t})\| \leq CP(1),$$

from which together with the Sobolev embedding and the generalized Hölder inequality provides the required a priori estimate in a space which constitutes admissible spaces for the Strichartz estimates, provided that $\nu \leq 1 - s$, which is equivalent to $p \geq \gamma(n)$. For details, see [30].

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