

Introduction to Jet Schemes and Arc Spaces

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0 Introduction

1968 Nash, preprint (Nash problem)

1995 published in Duke Math [25]

1995 Kontsevich Lecture at Orsay [19]

Motivic integration $\implies$ application

birational Calabi-Yau have the same Hodge number

Denef-Loeser [6][7] research on motivic integration

Mustaŭă, Ein, Lazarsfeld, Yasuda

characterization of singularities by jet schemes [9][10][11][23][24], ([11] without motivic integration)

1 Jet Scheme and Arc Space

1.1 Existence

Notation. $k = \bar{k}$, $\text{char } k \geq 0$. X : variety/ k , $\dim X = n$ (fixed).

Definition 1.1. Let $K \supset k$: field extension, $m \in \mathbb{Z}_{\geq 0}$.

A k -morphism $\alpha : \text{Spec } K[t]/(t^{m+1}) \rightarrow X$ is called an m -jet of X .

More precisely, “ K -valued m -jet of X ”.

A k -morphism $\alpha : \text{Spec } K[[t]] \rightarrow X$ is called an arc of X .

$\text{Spec } K[[t]] = \{0 (= \text{closed point}), \eta (= \text{generic point})\}$.

More precisely, “ K -valued arc of X ”.

Theorem 1.2. X : variety/ k . For $\forall m$, $\exists X_m$: k -scheme of finite type/ k s.t.

$$\text{Hom}_k(Z \times \text{Spec } k[t]/(t^{m+1}), X) \cong \text{Hom}_k(Z, X_m)$$

for $\forall Z$: k -scheme. Here $\text{Hom}_k(Z \times \text{Spec } k[t]/(t^{m+1}), X)$: space of m -jets parametrized by Z . “ $\times$ ” means “ $\times_{\text{Spec } k}$ ”.

In particular, in case $Z = \text{Spec } K$ ($K \supset k$, field extension).

$$\begin{array}{ccc} \text{Hom}(\text{Spec } K[t]/(t^{m+1}), X) & \cong & \text{Hom}(\text{Spec } K, X_m) \\ \Downarrow & & \Downarrow \\ \alpha & \longleftrightarrow & \alpha \text{ (use the same notation)} \\ K\text{-valued } m\text{-jet of } X & & K\text{-valued point of } X \end{array}$$

Proof. There are three proofs: (1) See BLR [4]. (2) Concrete construction. (3) See Vojta [33], more general construction, by “Hasse-Schmidt derivation”. For $X : S$ -scheme not necessary of finite type, $\forall S : \text{scheme}$.

(2) Here we show the second proof. $X = \text{Spec } R$, $R = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$. $Z = \text{Spec } A$.

$$\begin{aligned}
& \text{Hom}_k(Z \times \text{Spec } k[t]/(t^{m+1}), X) \\
&= \text{Hom}_k(k[x_1, \dots, x_N]/(f_1, \dots, f_r), A[t]/(t^{m+1})) \\
&= \{ \varphi : k[x_1, \dots, x_N] \longrightarrow A[t]/(t^{m+1}) \mid \varphi(f_i) = 0 \text{ for } \forall i \} \\
&\quad \varphi : x_j \longmapsto a_j^{(0)} + a_j^{(1)}t + \dots + a_j^{(m)}t^m, \quad a_j^{(\ell)} \in A \\
&\quad \varphi(f_i) = F_i^{(0)}(a_j^{(\ell)}) + F_i^{(1)}(a_j^{(\ell)})t + \dots + F_i^{(m)}(a_j^{(\ell)})t^m, \\
&\quad \quad F_i^{(m)}(a_j^{(\ell)}) : \text{polynomial in } a_j^{(\ell)} \\
&\quad \varphi(f_i) = 0 \implies F_i^{(0)}(a_j^{(\ell)}), F_i^{(1)}(a_j^{(\ell)}), \dots, F_i^{(m)}(a_j^{(\ell)}) = 0 \\
&= \{ \varphi : k[x_1, \dots, x_N, x_1^{(1)}, \dots, x_N^{(1)}, \dots, x_N^{(m)}] \longrightarrow A \mid F_i^{\ell'}(a_j^{(\ell)}) = 0 \} \\
&\quad \quad (x_j^{(\ell)} \longmapsto a_j^{(\ell)}) \\
&= \text{Hom}(k[x_j, x_j^{(1)}, \dots, x_j^{(m)}]/F_i^{\ell'}(x_j^{(\ell)}), A) \\
&= \text{Hom}(\text{Spec } A, \text{Spec } R_m), \quad X_m = \text{Spec } R_m
\end{aligned}$$

□

How to get equations of $X_m \subset \mathbb{A}_k^M$ ($\text{char } k = 0$). Derivation D on $k[x_j, x_j^{(1)}, \dots, x_j^{(m)}]$ is defined as follows:

$$D(x_j^{(\ell)}) = x_j^{(\ell+1)} \quad (\ell < m), \quad D(x_j^{(m)}) = 0 \text{ (otherwise)}.$$

By the embedding

$$\begin{array}{ccc}
X_m & \longrightarrow & \mathbb{A}^M \\
\psi & & \psi \\
\alpha & \longmapsto & (j! a_j^{(\ell)}),
\end{array}$$

we have equations $\{D^j(f_i)\}$ of $X_m \subset \mathbb{A}^M$.

Note 1.3.

- $X : \text{affine} \implies X_m : \text{affine}$
- $X : \text{finite type over } k \implies X_m : \text{finite type over } k$.

Example 1.4.

- $X : \text{reduced variety, } \dim X = 0 \implies X_m \cong X$.
- $X = \mathbb{A}^N \implies X_m = \mathbb{A}_k^{(m+1)N}$.

Definition 1.5. We define $\psi_{m,m-1} : X_m \longrightarrow X_{m-1}$ as follows. Let $\alpha \in X_m$, $f \in A$. $\alpha(f) := a_0 + a_1 t + \dots + a_{m-1} t^{m-1} + a_m t^m$. $\alpha' = \psi_{m,m-1}(\alpha)$ is defined by $\alpha'(f) := a_0 + \dots + a_{m-1} t^{m-1}$.

More formally,

$$k[t]/(t^{m+1}) \longrightarrow k[t]/(t^m). \quad \dots\dots\dots (*)$$

(**) $\forall Z : k$ -scheme, we have

$$\begin{aligned} Z \times \text{Spec } k[t]/(t^{m+1}) &\longleftarrow Z \times \text{Spec } k[t]/(t^m) \\ \text{Hom}(Z \times \text{Spec } k[t]/(t^{m+1}), X) &\longrightarrow \text{Hom}(Z \times \text{Spec } k[t]/(t^m), X) \\ &= \text{Hom}(Z, X_m) \quad \quad \quad = \text{Hom}(Z, X_{m-1}). \end{aligned}$$

Put $Z := X_m$. We get

$$\begin{aligned} \text{Hom}(X_m, X_m) &\longrightarrow \text{Hom}(X_m, X_{m-1}) \\ \downarrow &\quad \quad \downarrow \\ \text{id} &\longmapsto \psi_{m,m+1}. \end{aligned}$$

We say this argument (**) is the argument “induced from (*)”.

Let $m' > m$. Define $\psi_{m',m} := \psi_{m+1,m} \circ \dots \circ \psi_{m',m'-1}$.

Example 1.6. X : non-singular variety, $\psi : X_{m'} \longrightarrow X_m$ locally trivial fibration with the fiber $\mathbb{A}^{(m'-m)n}$.

In case $X = \mathbb{A}_k^n$, we have

$$\begin{aligned} X_{m'} = \mathbb{A}_k^{(m'+1)n} &\longrightarrow X_m = \mathbb{A}_k^{(m+1)n}, \\ \downarrow & \\ \alpha & \\ \alpha(x_i) = \sum_{j=0}^{m'} a_{ij} t^j &\longmapsto \sum_{j=0}^m a_{ij} x_j^j \\ k[x_j, x_j^{(1)}, \dots, x_j^{(m')}] &\longleftrightarrow k[x_j, x_j^{(1)}, \dots, x_j^{(m)}] \\ \mathbb{A}^{(m'+1)n} &\longrightarrow \mathbb{A}^{(m+1)n} \text{ canonical projection.} \end{aligned}$$

Definition 1.7. We define

$$\begin{aligned} \pi_m : X_m &\longrightarrow X \\ \downarrow &\quad \quad \downarrow \\ \alpha &\longmapsto \alpha(0) \end{aligned} \quad \left(\begin{array}{ccc} \text{Spec } K[t]/(t^{m+1}) &\longrightarrow & X \\ \downarrow &\quad \quad \downarrow \\ \{0\} &\longmapsto & \alpha(0) \end{array} \right),$$

induced from (*), $k[t]/(t^{m+1}) \longrightarrow k$ (discussion as before).

Example 1.8. Even if X is irreducible, X_m is not necessarily irreducible.

For example, $X = \{x^2 - y^2 + x^3 = 0\} \subset \mathbb{C}^3 \implies X_1 = Z_1 \cup Z_2$ irreducible decomposition. Here $\pi_1^{-1}(0) = \mathbb{A}^2 = Z_1$, $\pi_1^{-1}(X_{\text{reg}}) = \mathbb{A}^1$ -bundle. Z_2 : closure of $\pi_1^{-1}(X_{\text{reg}})$.

Definition 1.9. $m' > m \Rightarrow \psi_{m',m} : X_{m'} \longrightarrow X_m$ projective system. From

$$X_{m'} \xrightarrow{\psi_{m',m}} X_m \xrightarrow{\psi_{m,m''}} X_{m''},$$

we can define $X_\infty := \varprojlim_{m \rightarrow \infty} X_m$. Note that $X = \text{Spec } R \Rightarrow X_m = \text{Spec } R_m$. Put $\varprojlim R_m =: R_\infty$, $X_\infty := \text{Spec } R_\infty$.

Theorem 1.10. For any k -scheme Z ,

$$\text{Hom}_k(Z \hat{\times} \text{Spec } k[[t]], X) \cong \text{Hom}_k(Z, X_\infty).$$

$\therefore$ $\text{Hom}_k(Z \times \text{Spec } k[[t]]/(t^{m+1}), X) \cong \text{Hom}_k(Z, X_m)$. Taking projective limit $m \rightarrow \infty$, we have $\text{Hom}_k(Z \hat{\times} \text{Spec } k[[t]], X) = \text{Hom}_k(Z, X_\infty)$.

Put $Z = \text{Spec } A$. We obtain $\text{Hom}_k(\text{Spec } A[[t]], X) \cong \text{Hom}_k(\text{Spec } A, X_\infty)$. NB. $A[[t]] \neq A \otimes k[[t]]$. For example, $A = k[x]$.

Example 1.11. $X = \mathbb{A}_k^n$. $X_\infty = \text{Spec } k[x_j, \dots, x_j^{(1)}, \dots, x_j^{(2)}, \dots] =: \mathbb{A}_k^\infty$.

$$m \in \mathbb{N} : \{\text{closed point of } \mathbb{A}_k^m\} = k^m,$$

$$m = \infty : \{\text{closed point of } \mathbb{A}_k^\infty\} \neq k^\infty (\text{if } \#k = \aleph_0).$$

Definition 1.12. Define a k -morphism as follows:

$$\begin{array}{ccc} \psi_m : X_\infty & \longrightarrow & X_m \\ \downarrow \psi & & \downarrow \psi \\ \alpha & \longmapsto & \alpha_m \end{array}$$

$$\alpha(f) = \sum_{j=0}^{\infty} a_j t^j, \quad \alpha_m(f) = \sum_{j=0}^m a_j t^j.$$

Formally this morphism is “induced from” $k[[t]] \longrightarrow k[[t]]/(t^{m+1})$. Here α is

$$\text{arc } \alpha : \text{Spec } K[[t]] \longrightarrow \text{Spec } A \subset X \longleftrightarrow \text{ring hom } K[[t]] \xleftarrow{\alpha} A.$$

Define a k -morphism

$$\begin{array}{ccc} \pi : X_\infty & \longrightarrow & X \\ \downarrow \psi & & \downarrow \psi \\ \alpha & \longmapsto & \alpha(0) \end{array}$$

where 0 is the closed point of $\text{Spec } K[[t]]$. Formally this morphism π is “induced from” $k[[t]] \rightarrow k$.

We have $X_\infty \xrightarrow{\psi_{m'}} X_{m'} \xrightarrow{\psi_{m',m}} X_m \xrightarrow{\pi_m} X$, $X_\infty \xrightarrow{\pi} X$. If X : smooth then $\psi_{m'}$: surjective.

Proposition 1.13. $\psi_m(X_\infty)$ is constructible set (finite union of locally closed subsets).

1.2 Functoriality

Proposition 1.14. Let $m \in \mathbb{N} \cup \{\infty\}$, $f : X \rightarrow Y$ k -morphism.

$$\begin{array}{ccc} \mathrm{Spec} K[t]/(t^{m+1}) & \xrightarrow{\alpha} & X \\ & \searrow & \downarrow f \\ & & Y \end{array}$$

Then

$$\begin{array}{ccc} f_m : X_m & \longrightarrow & Y_m \quad k\text{-morphism} \\ \downarrow \Psi & & \downarrow \Psi \\ \alpha & \longmapsto & f \circ \alpha \\ \pi_m^X \downarrow & & \downarrow \pi_m^Y \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative.

Proposition 1.15. Let $m \in \mathbb{N} \cup \{\infty\}$. Then $X \xrightarrow{f} Y$ is étale $\implies X_m \cong Y_m \times_Y X$.

Corollary 1.16. Let m be as above. Then $U \hookrightarrow X$: open immersion $\implies U_m \cong (\pi_m^X)^{-1}(U)$. In particular, $U_m \hookrightarrow X_m$ is an open immersion.

Definition 1.17. Main part of $X_m \stackrel{\text{def}}{\iff} \overline{\pi_m^{-1}(X_{\text{reg}})}$

Note 1.18. $Z \subset X$ closed immersion $\implies Z_m \subset X_m$ closed immersion.

$\therefore Z \subset X \subset \mathbb{A}^N$. $\mathcal{O}_{\mathbb{A}^N} \supset \mathcal{I}_Z \supset \mathcal{I}_X$, $\mathcal{O}_{\mathbb{A}^{N(m+1)}} \supset \mathcal{I}_{Z_m} \supset \mathcal{I}_{X_m}$.

NB. $Z \subset X$ closed, $Z_m \subsetneq \pi_m^{-1}(Z)$, $Z_\infty \subsetneq \pi_\infty^{-1}(Z)$. ($\text{codim } Z_\infty = \infty$.)

Proposition 1.19. Let $m \in \mathbb{N} \cup \{\infty\}$. Then $(X \times_{\mathrm{Spec} k} Y)_m = X_m \times_{\mathrm{Spec} k} Y_m$.

Theorem 1.20 (Kolchin [18], Ishii-Kollár [14, Lemma 2.12]). Let $\text{char } k = 0$. Then X : irreducible $\implies X_\infty$: irreducible.

Example 1.21. $\text{char } k = p > 0$. X_∞ is not necessarily irreducible. $X = \{x^p - y^p z = 0\} \subset \mathbb{A}_k^3 \implies X_\infty$ is not irreducible. (See IK [14].)

Note 1.22.

- $f : X \rightarrow Y$ surjective $\nRightarrow f_m : X_m \rightarrow Y_m$ surjective,
- $f : X \rightarrow Y$ proper $\nRightarrow f_m : X_m \rightarrow Y_m$ proper.

Assume that X has A_n singularity $\subset \mathbb{A}^3$. Then f is proper surjective:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{f} & \mathbb{C}^2/G = X \\ \uparrow & & \uparrow \\ \mathbb{C}_m^2 & \xrightarrow{f_m} & X_m \quad f_m : \text{dominant.} \end{array}$$

X_m is irreducible (Mustață) and $\mathbb{C}^2 \ni 0 \xrightarrow{f} P \in X$, $(\pi_m^{\mathbb{C}^2})^{-1}$ is irreducible. But if $m \gg 0$, $(\pi_m^X)^{-1}(P)$ has n irreducible components (Nash). Therefore f_m is not surjective, in particular it is not proper.

1.3 Cylinders and valuation

Definition 1.23 (ELM[11]). Let $C \subset X_\infty$ be an irreducible constructible set.

$$\begin{aligned} C : \text{thin} &\stackrel{\text{def}}{\iff} \exists Z \subset X \text{ proper closed } (C \subset Z_\infty) \\ C : \text{fat} &\stackrel{\text{def}}{\iff} \text{not thin.} \end{aligned}$$

Note 1.24. Let $\alpha \in C$ irreducible constructible set $\subset X$. Then

$$\begin{aligned} C : \text{fat} &\iff \alpha(\eta) : \text{generic point of } X \\ &\iff \begin{array}{ccc} k[[t]] & \xleftarrow{\alpha} & A \quad (X = \text{Spec } A) \\ & \cap & \downarrow \\ K((t)) & \xleftarrow{\alpha} & K(X) \text{ extendable,} \end{array} \end{aligned}$$

where η is the generic point of $\text{Spec } K[[t]]$.

Let $v_C(f) := \text{ord } \alpha(f)$ ($f \in K(X) \setminus \{0\}$). Then v_C : discrete valuation.

Definition 1.25. $C \subset X_\infty$ is a cylinder $\stackrel{\text{def}}{\iff} \exists S \subset X_\infty$ constructible set, $C = \psi_m^{-1}(S)$.

Note 1.26. X : non-singular, $C = \psi_m^{-1}(S)$: cylinder.

- (0) $X_\infty \longrightarrow X_m$ locally trivial fibration with the fiber $\mathbb{A}^\infty$.
- (1) $S = S_1 \cup S_2 \cup \dots \cup S_r$ irreducible decomposition
 $\implies C = \psi_m^{-1}(S_1) \cup \dots \cup \psi_m^{-1}(S_r)$ irreducible decomposition.
- (1)' In particular, cylinder's irreducible components are finite.
- (2) C : cylinder $\implies \overline{C} = \psi_m^{-1}(\overline{S})$ cylinder.
- (3) $\forall$ irreducible components of cylinder is fat.
 Corresponding valuation is divisorial valuation, i.e., $\exists E$: divisor over X ,
 $v_C = q \text{ val}_E$ ($q \in \mathbb{N}$).

If X has singularity, (1)' also holds but (0), (1) and (3) are not affirmative. (2) is open problem.

Example 1.27 (Thin cylinder [De Fernex-Ein-Ishii, preprint]). Let $F = x^2 - y^2 z$, $X = \{F = 0\} \subset \mathbb{A}_{\mathbb{C}}^3$, $\alpha_m \in X_m$ closed point.

$$\mathbb{C}[[t]]/(t^{m+1}) \xleftarrow{\alpha_m} \mathbb{C}[x, y, z]/(F), \quad \begin{cases} \alpha_m(x) = t \\ \alpha_m(y) = 0 \\ \alpha_m(z) = 0 \end{cases}.$$

Then cylinder $\psi_m^{-1}(\alpha_m) \subset (\text{Sing } X)_\infty$ is thin!

Proposition 1.28 (De Fernex-Ein-Ishii). X : singular.

- (1) $\#(\text{components of cylinder}) < \infty$.
- (2) A thin component of a cylinder $\subset (\text{Sing } X)_\infty$.
- (3) C : fat component of a cylinder $\implies v_C$: divisorial valuation.

Proposition 1.29. C : cylinder $\implies \forall m \in \mathbb{N}$, $\psi_m(C)$ is a constructible set.

2 Motivic Integration

Exposition text Craw [5], Veys [32], Loeser [8].

On Nash problem, see [3],[12],[14]–[17],[20]–[22],[27],[28],[30],[31].

2.1 Grothendieck ring

$\mathcal{V}ar_{\mathbb{C}} := \{\text{variety } / \mathbb{C}\}$. $K_0(\mathcal{V}ar_{\mathbb{C}}) :=$ abelian group generated by $\{[V] \mid V \in \mathcal{V}ar_{\mathbb{C}}\}/(\text{equiv.})$, equiv. means as follows:

$[V] = [W]$ if $V \cong W$, $[V] = [V \setminus Z] + [Z]$, $Z \subset V$ closed.

This has a multiplication $[V][W] := [V \times W]$.

$K_0(\mathcal{V}ar_{\mathbb{C}}) =$ “Grothendieck ring.”

$\forall C$: constructible set in some variety V . $C = \coprod A_i \implies$ naturally $[C] := \sum [A_i] \in K_0(\mathcal{V}ar_{\mathbb{C}})$. Here A_i : locally closed.

Convention: $[\text{point}] = 1$, $[\mathbb{A}^1] =: \mathbb{L}$.

Example 2.1.

- (1) $X = \{y^2 - x^3 = 0\} \in \mathbb{A}^2 \implies [X] = [\mathbb{A}^1 \setminus \{0\}] + [\{0\}] = [\mathbb{A}^1] = \mathbb{L}$.
- (2) $\forall f : Y \longrightarrow X$ piecewise trivial fibration with fiber F , i.e., $X = \coprod X_i$ locally closed and

$$f|_{f^{-1}(X_i)} : f^{-1}(X_i) \longrightarrow X_i. \\ \cong X_i \times F$$

then $[Y] = [X][F]$.

$\therefore [Y] = \sum [f^{-1}(X_i)] = [F] \sum [X_i] = [F][X]$, since $[f^{-1}(X_i)] = [X_i] \times [F]$, $\sum [X_i] = [X]$.

Note 2.2. $K_0(\mathcal{V}ar_{\mathbb{C}})$ is not integral domain. See Poonen [29]. $\exists A, B$: Abelian varieties, $[A] \neq [B]$ and $A \times A \cong B \times B \implies ([A] - [B])([A] + [B]) = [A]^2 - [B]^2 = 0$.

Definition 2.3 (Hodge-Deligne polynomial). Let $V \in \mathcal{V}ar_{\mathbb{C}}$, $\dim V = n$.

$$H(V, u, v) := \sum_{p,q=0}^n \sum_{i=0}^{2n} (-1)^i h^{pq}(H_C^i(V, \mathbb{C})) u^p v^q \in \mathbb{Z}[u, v]$$

where h^{pq} is dimension of (p, q) -Hodge components. In particular, $H(V, 1, 1) = \chi(V)$ Euler characteristic.

Note 2.4.

$$\begin{array}{ccc} \mathcal{V}ar_{\mathbb{C}} & \xrightarrow{H} & \mathbb{Z}[u, v] \text{ factors.} \\ & \searrow & \nearrow H \\ & K_0(\mathcal{V}ar_{\mathbb{C}}) & \end{array}$$

Example 2.5.

$$(1) \quad H(\mathbb{P}^n, u, v) = 1 + uv + (uv)^2 + \cdots + (uv)^n.$$

$$\therefore h^{pq}(H^i(\mathbb{P}^n, \mathbb{C})) = \begin{cases} 1 & (p = q = \frac{i}{2}) \\ 0 & (\text{otherwise}) \end{cases}.$$

$$(2) \quad H(\mathbb{A}^n, u, v) = (uv)^n. \text{ In particular, } H(\mathbb{L}, u, v) = uv.$$

$$\therefore \begin{array}{ccc} [\mathbb{P}^n] & = & [\mathbb{A}^n] + [\mathbb{P}^{n-1}] \\ 1 + uv + \cdots + (uv)^n & & (uv)^n \quad 1 + uv + \cdots + (uv)^{n-1} \end{array}$$

Definition 2.6. $\mathcal{M} := K_0(\mathcal{V}ar_{\mathbb{C}})_{\mathbb{L}}$, localization by $\mathbb{L}$.

Definition 2.7. $F^m :=$ subgroup generated by $\frac{[S]}{\mathbb{L}^i}$. $\dim S - i \leq -m. \implies \{F^m\}$ descending filtration, $F^m \cdot F^n \subseteq F^{m+n}$. $\mathcal{M}_{\mathbb{C}}/F^m \rightarrow \mathcal{M}_{\mathbb{C}}/F^{m-1}$: projective system. $\widehat{\mathcal{M}}_{\mathbb{C}} := \varprojlim_m \mathcal{M}_{\mathbb{C}}/F^m$,

We say “ $\sum_{m \in \mathbb{Z}} a_m \mathbb{L}^{-m}$ converge” ($a_m \in \mathcal{V}ar_{\mathbb{C}}$) $\iff \sum_{m \in \mathbb{Z}} a_m \mathbb{L}^{-m} \in \widehat{\mathcal{M}}_{\mathbb{C}}$.

$$\left(\iff \sum_{\dim a_m - m > k} a_m \mathbb{L}^{-m} \in \widehat{\mathcal{M}}_{\mathbb{C}}/F^k \right)$$

For example, $\sum_{m \in \mathbb{Z}} \mathbb{L}^{-m}$ does not converge.

2.2 Motivic integration

There are two ways.

(1) in $\widehat{\mathcal{M}}_{\mathbb{C}}$, Denef, Loeser, Veys. (2) in $\mathbb{Z}[[u^{-1}v^{-1}]] [u, v]$, Mustață.

(1) C : cylinder $\subset X_{\infty}$, $n = \dim X$.

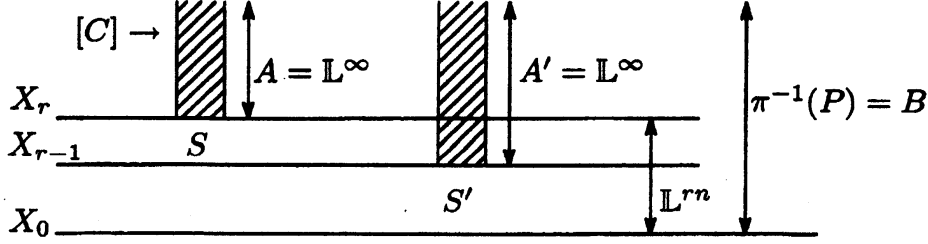
$$\text{motivic measure } \mu(C) = \lim_{n \rightarrow \infty} \frac{[\psi_m(C)]}{\mathbb{L}^{(m+1)n}} \in \widehat{\mathcal{M}}_{\mathbb{C}}. \quad (\text{DL}[6])$$

(Note that Veys defines $\mu(C) = \lim_{n \rightarrow \infty} \frac{[\psi_m(C)]}{\mathbb{L}^{mn}}$.)

In case X is non-singular, $C = \psi_r^{-1}(S)$, $S \subset X_r$.
 $m > r$, $\psi_m(C) = \psi_{m,r}^{-1}(S)$, $[\psi_m(C)] = [S]\mathbb{L}^{(m-r)n}$.

$$\frac{[\psi_m(C)]}{\mathbb{L}^{(m+1)n}} = \frac{[S]}{\mathbb{L}^{(r+1)n}} \quad \text{for } \forall m. \quad \mu(C) = \frac{[S]}{\mathbb{L}^{(r+1)n}}.$$

Picture



$$"[C]" = [S]A = [S]B/\mathbb{L}^{rn} \Rightarrow \frac{[S]}{\mathbb{L}^{rn}}$$

Here by putting $B = 1$, we can think that $\frac{[S]}{\mathbb{L}^{rn}}$ is a "volume" of C .

Example 2.8 (Singular case). $X = \{xy = 0\} \subset \mathbb{A}^2$. X_∞ : cylinder, $[\psi_m(X_\infty)] = 2\mathbb{L}^{m+1} - 1$. When $m = 1$, $\psi_1(X_\infty)$ = horizontal part. $\mathbb{A}^2 \cup \mathbb{A}^2$, $\mathbb{L}^2 + \mathbb{L}^2 - 1 = 2\mathbb{L}^2 - 1$.

$$\mu(X_\infty) = \lim_{m \rightarrow \infty} \frac{[\psi_m(X_\infty)]}{\mathbb{L}^{m+1}} = \lim_{m \rightarrow \infty} \frac{2\mathbb{L}^{m+1} - 1}{2\mathbb{L}^{m+1}} = 2.$$

Definition 2.9 (Motivic Integration). $F : X_\infty \rightarrow \mathbb{Z} \cup \{\infty\}$ function s.t. $F^{-n}(n)$ is a cylinder.

$$\int_{X_\infty} \mathbb{L}^{-F} d\mu := \sum_{m \in \mathbb{Z}} \mu(F^{-1}(m)) \mathbb{L}^{-m} \in \widehat{\mathcal{M}}_{\mathbb{C}}$$

Example 2.10 (Example of F). $Z \subset X = \text{Spec } A$ closed subscheme. $F_Z : X_\infty \rightarrow \mathbb{Z} \cup \{\infty\}$ ($\alpha \mapsto \text{ord}_\alpha(Z) = \text{ord}_t \alpha(I_Z)$) satisfies the condition for F .
 $\therefore F_Z(\mathbb{Z}_{\geq m}) = \psi_{m-1}^{-1}(Z_{m-1})$ (Exercise). $F_Z^{-1}(m) = \psi_{m-1}^{-1}(Z_{m-1}) \setminus \psi_m^{-1}(Z_\infty)$.

(2) by Mustařá. F : as above.

$$\begin{aligned} \int_{X_\infty} e^{-F} &:= \sum_{m \in \mathbb{Z}} H(\mu(F^{-1}(m)) \mathbb{L}^{-m}) \in \mathbb{Z}[[u^{-1}v^{-1}]] [u, v] \\ &= \sum_{m \in \mathbb{Z}} H(\mu(F^{-1}(m))) (uv)^{-m} \in \mathbb{Z}[[u^{-1}v^{-1}]] [u, v]. \end{aligned}$$

Theorem 2.11 (Change of variables formula, DL [6]). $\varphi : Y \rightarrow X$ proper birational morphism of non-singular varieties. Then

$$\begin{aligned} \int_{X_\infty} \mathbb{L}^{-F} d\mu &= \int_{Y_\infty} \mathbb{L}^{-F \circ \varphi_\infty - F_{K_{Y/X}}} d\mu. \\ \int_{X_\infty} e^{-F} &= \int_{Y_\infty} e^{-F \circ \varphi_\infty - F_{K_{Y/X}}}. \end{aligned}$$

Corollary 2.12. $X, X' : \text{smooth Calabi-Yau varieties. } X \sim X' \text{ birational} \Rightarrow [X] = [X']$.

Proof. $K_{Y/X} = K_{Y/X'} = K_Y$.

$$\begin{array}{ccc} & Y & \\ \varphi \swarrow & & \searrow \varphi' \\ X & & X' \end{array}$$

$X_\infty \xrightarrow{F} \mathbb{Z} \cup \{\infty\}$ (zero map), $X'_\infty \xrightarrow{F'} \mathbb{Z} \cup \{\infty\}$ (zero map) $\Rightarrow F \circ \varphi_\infty = F' \circ \varphi_\infty : \text{zero map}$.

$$\begin{aligned} \int_{X_\infty} \mathbb{L}^{-F} d\mu &= \mu(F^{-1}(0)) = \frac{[\pi(X_\infty)]}{\mathbb{L}^{(0+1)n}} = \frac{[X]}{\mathbb{L}^n}, \\ \int_{Y_\infty} \mathbb{L}^{-F \circ \varphi_\infty - F_{K_{Y/X}}} &= \int_{X'_\infty} \mathbb{L}^{-F'} d\mu = \frac{[X']}{\mathbb{L}^n} \Rightarrow [X] = [X'] \in \widehat{\mathcal{M}}_{\mathbb{C}}. \end{aligned}$$

□

Corollary 2.13. $X, X' : \text{birational Calabi-Yau varieties} \Rightarrow h^{pq}(X) = h^{pq}(X')$.

2.3 Application of motivic integration

Characterization of singularities via jets.

Theorem 2.14 (Mustaă, [23]). $X : \text{smooth, } Y \subset X \text{ closed subscheme. Then}$

$$\text{log canonical threshold } c(X, Y) = \dim X - \sup_{m \geq 1} \frac{\dim Y_m}{m+1}.$$

Theorem 2.15 (Ein-Mustaă-Yoshida, [9]). $X : \text{locally complete intersection variety (lci for short). Then}$

$X \text{ has canonical singularity} \iff X_m \text{ is irreducible for } \forall m \in \mathbb{N}.$
 $(\iff X : \text{rational in this situation})$

Theorem 2.16 (Ein-Mustaă, [10]). $X : \text{normal lci. } Y = \sum a_i Y_i \text{ } (a_i \in \mathbb{R}), Y_i \subset X \text{ irreducible closed subscheme. Then } x \mapsto \text{mld}(x; X, Y) \text{ (minimal log discrepancy) is upper semi-continuous.}$

We show Mustață's proof of Theorem 2.14.

Theorem 2.14'(M[23]). X : smooth, $Y \subset X$ closed. Then

$$(X, qY) \text{ is log canonical} \iff \dim Y = (m+1)(\dim X - q), \forall m \in \mathbb{N}.$$

Theorem 2.14' $\implies$ Theorem 2.14.

$$\therefore \text{ log canonical threshold} = \sup\{q \mid (X, qY) : \text{log canonical}\}.$$

□

Let $\varphi : X' \longrightarrow X$ be a log resolution of (X, Y) . $\varphi^{-1}(Y) = \sum_{i=1}^r a_i D_i$, $a_i \geq 1$, $D_i \in X'$ irreducible divisor. $K_{X'/X} = \sum_{i=1}^r b_i D_i$, $b_i \geq 0$. Then

$$\begin{aligned} (X, qY) : \text{log canonical} &\iff b_i - qa_i \geq -1 \quad (i = 1, \dots, r) \\ &\iff qa_i - b_i - 1 \leq 0. \end{aligned}$$

Here $(X, qY) : \text{log canonical} \stackrel{\text{def}}{\iff}$

$$K_{X'} = \varphi^* K_X + q\varphi^{-1}(Y) + \sum r_i D_i, \quad r_i \geq -1 \quad \text{for } \forall i.$$

From this,

$$\begin{aligned} K_{X'} - \varphi^* K_X &= q\varphi^{-1}(Y) + \sum r_i D_i \\ \parallel &\parallel \\ K_{X'/X} = \sum b_i D_i &= q \sum a_i D_i, \quad b_i = qa_i + r_i, \quad r_i \geq -1. \end{aligned}$$

Recall motivic integration.

$$X_\infty \xrightarrow{F_Y} \mathbb{Z} \cup \{\infty\} \xrightarrow{f} \mathbb{Z} \cup \{\infty\}.$$

where $F^{-1}(s) : \text{cylinder for } \forall s \in \mathbb{Z}$. We define f later.

We have two expressions

$$\int_{X_\infty} e^{-F} = \int_{X'_\infty} e^{-F \circ \varphi_\infty - F_{K_{X'/X}}}.$$

We compare the left and right of this equality.

NB. $F_Y^{-1}(m) = \psi_{m-1}^{-1}(Y_{m-1}) \setminus \psi_m^{-1}(Y_m)$.

$$\mu(F_Y^{-1}(m)) = \frac{[Y_{m-1}]}{\mathbb{L}^{mn}} - \frac{[Y_m]}{\mathbb{L}^{(m+1)n}}.$$

Put $f(m) = s$.

$$\begin{aligned} \int_{X_\infty} e^{-F} &= \sum_{s \in \mathbb{Z}_{\geq 0}} H(\mu(F^{-1}(s)))(uv)^{-s} = \sum_{m \in \mathbb{Z}_{\geq 0}} H(\mu(F_Y^{-1}(m)))(uv)^{-f(m)} \\ &\quad \uparrow F^{-1}(s) = F_Y(f^{-1}(s)) \\ &= \sum_{m \in \mathbb{Z}_{\geq 0}} (H(Y_{m-1})(uv)^{-mn} - H(Y_m)(uv)^{-(m+1)n})(uv)^{-f(m)} \\ &= \sum H(Y_{m-1})(uv)^{-mn-f(m)} - \sum H(Y_m)(uv)^{-(m+1)n-f(m)} \\ &=: S_1 - S_2. \end{aligned}$$

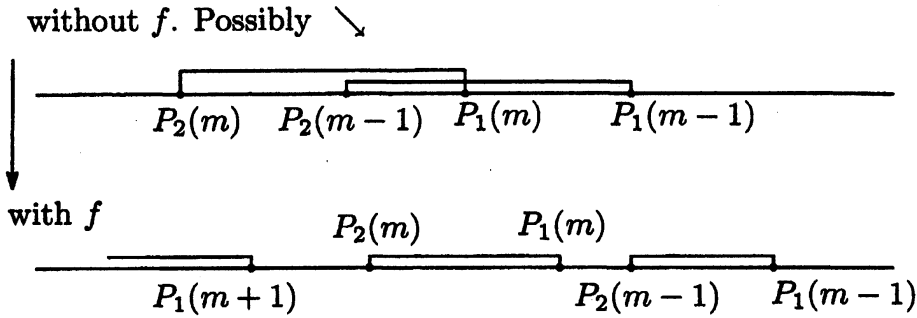
Definition 2.17 (Definition of f). We define $f : \mathbb{Z} \cup \{\infty\} \hookrightarrow \mathbb{Z} \cup \{\infty\}$ as follows:

$$\begin{cases} m \leq 0 : & f(m) = m \\ m > 0 : & \text{inductively, } f(m+1) > f(m) + \dim Y_m + c(m+1), \end{cases}$$

where $c > \left| n - \frac{b_i - 1}{a_i} \right|$ for $\forall i$.

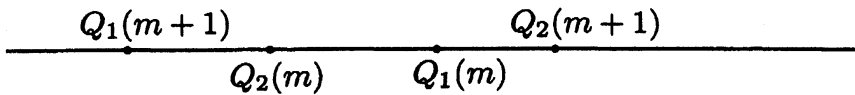
Now, looking at S_1 , we have

$$\begin{aligned} 2(-mn - f(m)) &< \left\{ \begin{array}{c} \text{deg of monomials in} \\ H(Y_{m-1})(uv)^{-mn-f(m)} \end{array} \right\} < 2(-mn - f(m) + \dim Y_{m-1}). \\ &=: 2P_2(m) \qquad \qquad \qquad =: 2P_1(m) \end{aligned}$$



Next, looking at S_2 , we have

$$\begin{aligned} 2(-(m+1)n - f(m)) &< \left\{ \begin{array}{c} \text{deg of monomials in} \\ H(Y_m)(uv)^{-(m+1)n-f(m)} \end{array} \right\} < 2(-(m+1)n - f(m) + \dim Y_m). \\ &=: 2Q_2(m) \qquad \qquad \qquad =: 2Q_1(m) \end{aligned}$$



Compare $Q_1(m)$, $P_1(m)$. We have

$$P_1(m) \geq Q_1(m).$$

Here, “=” holds $\iff \dim Y_m = \dim Y_{m-1} + n$.

$\therefore$) By $\dim Y_m \leq \dim Y_{m-1} + n$ the inequality of P_1 and Q_1 follows. $\square$

Put $\ell(Y_{m-1}) = \#(\text{maximal dim components})$. Then

- The term of $\deg = -2P_1(m)$ in $S_1 = \ell(Y_{m-1})(uv)^{-P_1(m)}$ (Exercise).
- The term of $\deg = -2Q_1(m)$ in $S_2 = \ell(Y_m)(uv)^{-Q_1(m)}$.

Change of variables formula,

$$\int_{X'_\infty} e^{-F \circ \varphi_\infty - F_{K_{X'/X}}} = \sum_{J \subset \{1, \dots, r\}} S_J, \quad (\text{For "=" , see Batyrev[1], Theorem 36})$$

$$S_J = \sum H(D_J^0)(uv - 1)^{|J|} (uv)^{-n - \sum \alpha_i(b_i + 1) - f(\sum \alpha_i a_i)} =: \sum (\star),$$

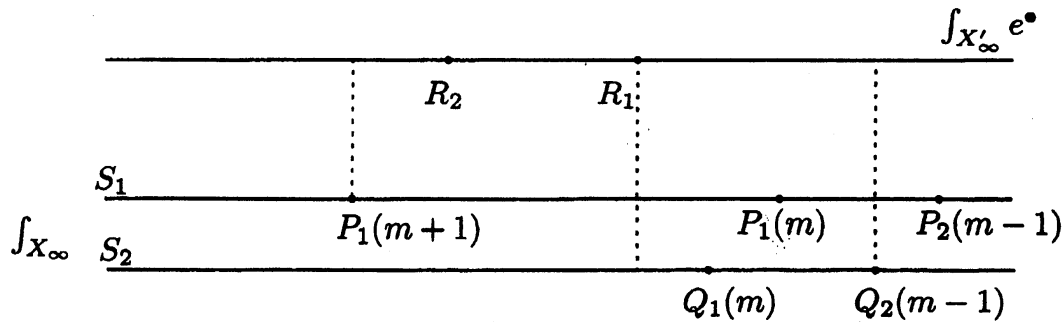
$$D_J = \left(\bigcap_{i \in J} D_i \right) \setminus \bigcup_{i \notin J} D_i.$$

We have

$$\begin{aligned} 2R_1(\alpha_i | i \in J) - 2n &< \deg \text{ of } (\star) < 2 \sum_{i \in J} (\alpha_i(b_i + 1) - f(\sum \alpha_i a_i)) \\ &=: 2R_2(\alpha_i | i \in J) &=: 2R_1(\alpha_i | i \in J) \end{aligned}$$

Put $\tau(m) = \dim Y_m - (m + 1)(n - q)$.

$$\begin{aligned} R_1(\alpha_i | i \in J) &= \\ P_1(\sum \alpha_i a_i) - \tau(\sum \alpha_i a_i - 1) &=: (*) + \sum \alpha_i(qa_i - b_i - 1) &=: (**) \dots \dots (1) \\ P_1(\sum \alpha_i a_i + 1) &< R_2(\alpha_i | i \in J) \\ &< R_1(\alpha_i | i \in J) < \min\{Q_2(\sum \alpha_i a_i - 1), P_2(\sum \alpha_i a_i - 1)\} \end{aligned}$$



Assume $(X, qY) : \log$ canonical. Then $(**) \leq 0$.

If $\exists m$ s.t. $\tau(m - 1) > 0 \iff (*) > 0$ for $m = \sum \alpha_i a_i$

$\implies R_1(\alpha_i | i \in J) < P_1(\sum \alpha_i a_i)$.

$\implies (uv)^{P_1(m)}$ does not appear in $\int_{X_\infty} e^{-F}$. Therefore, the monomial $(uv)^{P_1(m)}$ in S_1 should be cancelled by a term of S_2 .

$\implies P_1(m) = Q_1(m) \implies \dim Y_m = \dim Y_{m-1} + n$.

$\tau(m) = \tau(m + 1) - q > 0 \xrightarrow{\text{same argument}} P_1(m + 1) = Q_1(m + 1)$

$$\Rightarrow \dim Y_{m+1} = \dim Y_m + n \Rightarrow \tau(m+1) = \tau(m+2) - q > 0 \Rightarrow \dots$$

Therefore, $\exists m_0$ s.t. $\forall m \geq m_0 \Rightarrow \dim Y_m = \dim Y_{m-1} + n$

$$\Rightarrow Y_\infty \supset \psi_m^{-1}(Y_m^0), Y_m^0 : \text{maximal dim component.}$$

Y_∞ : thin, $\psi_m^{-1}(Y_m^0)$: cylinder, fat. “thin $\supset$ fat” : contradiction.

(The proof of the converse was not shown in the talk because of the shortness of time. One who are interested in can see it in Mustață's paper [23]. Here we see just the sketch of it.)

Assume $\tau(m) \leq 0$ for all $m \in \mathbb{N}$. Fix m such that $a_i \mid m+1$ for every i . Here, if $qa_j - b_j - 1 > 0$ for some j , define

$$J = \{j\} \text{ and } \alpha_i = \begin{cases} \frac{m+1}{a_j} & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Then,

$$\begin{aligned} \Rightarrow R_1(\alpha_i \mid i \in J) &= P_1(m+1) - \frac{\tau(m)}{\leq 0} + \frac{m+1}{a_j}(qa_j - b_j - 1)_{>0} \\ \Rightarrow P_1(m+1) &< R_1(\alpha_i \mid i \in J). \end{aligned}$$

By this and some other discussions, it follows that there is d in the interval $(P_1(m+1), \min\{P_2(m), Q_2(m)\})$ such that the term $(uv)^{-d}$ appears in $\int_{X'_\infty} e^{-F \circ \varphi_\infty - F_{X'/X}}$. But for any d in this interval, the term $(uv)^{-d}$ does not appear in $\int_{X_\infty} e^{-F}$, a contradiction. Therefore the inequality $qa_i - b_i - 1 \geq 0$ should hold for every i , i.e., (X, qY) is log-canonical. $\square$

References

1. V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. **1** (1999) 5-33.
2. C. Bouvier, *Diviseurs essentiels, composantes essentielles des variétés toriques singulières*, Duke Math. J. **91** (1998) 609-620
3. C. Bouvier and G. Gonzalez-Sprinberg, *Système générateur minimal, diviseurs essentiels et G-désingularisations de variétés toriques*, Tohoku Math. J. **47**, (1995) 125-149.
4. S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron Models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **21** (1990) Springer-Verlag.
5. A. Craw, *An introduction to motivic integration*, math.AG/9911179
6. J. Denef and F. Loeser, *Germes of arcs on singular varieties and motivic integration*, Invent. Math. **135**, (1999) 201-232.
7. J. Denef and F. Loeser, *Motivic integration, quotient singularities and the McKay correspondence*, Compositio Math. **131**, (2002) 267-290.

8. F. Loeser, *Seattle lectures on motivic integration*. preprint 2006 in his web page.
9. L. Ein, M. Mustață and T. Yasuda, *Jet schemes, log discrepancies and inversion of adjunction*, Invent. Math. **153** (2003) 519-535.
10. L. Ein and M. Mustață. *Inversion of Adjunction for local complete intersection varieties*, Amer. J. Math. **126** (2004), 1355–1365.
11. L. Ein, R. Lazarsfeld and M. Mustață, *Contact loci in arc spaces*, Compositio Math. **140** (2004) 1229–1244.
12. G. Gonzalez-Sprinberg and M. Lejeune-Jalabert, *Families of smooth curves on surface singularities and wedges*, Annales Polonici Mathematici, LXVII.2, (1997) 179–190.
13. M. Hickel, *Fonction de Artin et germes de courbes tracées sur un germe d'espace analytique*, Amer. J. Math. **115**, (1993) 1299–1334.
14. S. Ishii and J. Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** No.3 (2003) 601-620.
15. S. Ishii, *The arc space of a toric variety*, J. Algebra, **278** (2004) 666–683
16. S. Ishii, *Arcs, valuations and the Nash map*, J. reine angew. Math, **588** (2005) 71–92.
17. S. Ishii, *The local Nash problem on arc families of singularities*, to appear in Ann. Inst. Fourier, Grenoble.
18. E. R. Kolchin, *Differential algebra and algebraic groups*, Pure and Applied Mathematics, Vol. **54**, Academic Press, New York-London, 1973.
19. M. Kontsevich, *Lecture at Orsay* (December 7, 1995)
20. M. Lejeune-Jalabert, *Arcs analytiques et résolution minimale des surfaces quasihomogènes*. in: Lecture Notes in Math. **777**, (1980) 303–336.
21. M. Lejeune-Jalabert, *Courbes tracées sur un germe d'hypersurface*. Amer. J. Math. **112**, (1990) 525–568.
22. M. Lejeune-Jalabert and A. J. Reguera-Lopez, *Arcs and wedges on sandwiched surface singularities*, Amer. J. Math. **121**, (1999) 1191–1213.
23. M. Mustață, *Singularities of pairs via jet schemes*, J. Am. Math. Soc. **15**, (2002) 599–615.
24. M. Mustață, *Jet schemes of locally complete intersection canonical singularities*, with an appendix by David Eisenbud and Edward Frenkel, Invent. Math. **145** (2001) 397–424.
25. J. F. Nash, *Arc structure of singularities*, Duke Math. J. **81**, (1995) 31–38.

- 26. A. Nobile, *On Nash theory of arc structure of singularities*, Ann. Mat. Pura Appl. **160** (1991), 129–146.
- 27. C. Plénat and P. Popescu Pampu, *A class of non-rational surface singularities for which the Nash map is bijective* math.AG/0410145.
- 28. C. Plénat and P. Popescu-Pampu, *Families of higher dimensional germs with bijective Nash map*, math.AG/0605566.
- 29. B. Poonen, *The Grothendieck ring of varieties is not a domain*, Math. Res. Letters **9** (2002) 493–498.
- 30. A. J. Reguera-Lopez, *Families of arcs on rational surface singularities*, Manuscr. Math. **88**, (1995) 321–333.
- 31. A. J. Reguera-Lopez, *Image of the Nash map in terms of wedge*, C.R. Acad. Sci. Paris, Ser. I **338** (2004) 385–390.
- 32. W. Veys, *Arc spaces, motivic integration and stringy invariants*, math.AG/0401374.
- 33. P. Vojta, *Jets via Hasse-Schmidt derivations*, math.AG/0407113.