

THE ITERATED REMAINDERS OF THE RATIONALS

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ABSTRACT. Repeat taking remainders of Stone-Čech compactifications of the rationals

$$Q^{(1)} = Q^* = \beta Q \setminus Q, \quad Q^{(2)} = \beta Q^{(1)} \setminus Q^{(1)}, \quad Q^{(3)} = \beta Q^{(2)} \setminus Q^{(2)}, \quad Q^{(4)} \dots$$

We point out that they have similar structures, but, are topologically different. In particular we prove here that $Q^{(1)} \not\approx Q^{(3)}$. This result will be generalized to show that $Q^{(n)} \not\approx Q^{(n+2)}$ for any $n \geq 1$ in the forthcoming paper [4].

1. INTRODUCTION

Consider the space of rationals Q , and repeat taking its remainders of Stone-Čech compactifications $Q^{(n+1)} = (Q^{(n)})^* = \beta Q^{(n)} \setminus Q^{(n)}$ ($n \geq 0$) where $Q^{(0)} = Q$, i.e.,

$$Q^{(1)} = Q^*, \quad Q^{(2)} = Q^{**}, \quad Q^{(3)} = Q^{***}, \dots$$

Van Douwen [2] asked whether or not $Q^{(n)} \approx Q^{(n+2)}$ for $n \geq 1$, remarking that $Q^{(m)}$ for even m is never homeomorphic to $Q^{(n)}$ for odd n , because the former is σ -compact but the latter is not.

In this paper we point out that both $Q^{(n)}$ and $Q^{(n+2)}$ have a similar structure of “fiber bundle” for every $n \geq 1$, but they are topologically different. In particular we here show that $Q^{(1)} \not\approx Q^{(3)}$, which we can generalize in the forthcoming paper [4] to show that $Q^{(n)} \not\approx Q^{(n+2)}$ for any $n \geq 1$, answering van Douwen’s question.

The precise connections of the remainders can be seen by the following construction. Viewing βQ as a compactification of $Q^{(1)}$, let

$$\Phi_0 : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \rightarrow Q \cup Q^{(1)} = \beta Q$$

be the Stone extension of the identity map $id : Q^{(1)} \rightarrow Q^{(1)}$. Denote by

$$\phi_0 : Q^{(2)} \rightarrow Q^{(0)}$$

the restriction of Φ_0 . Next let

$$\Phi_1 : \beta Q^{(2)} = Q^{(2)} \cup Q^{(3)} \rightarrow Q^{(1)} \cup Q^{(2)} = \beta Q^{(1)}$$

be the Stone extension of the identity map $id : Q^{(2)} \rightarrow Q^{(2)}$, and let

$$\phi_1 : Q^{(3)} \rightarrow Q^{(1)}$$

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denote the restriction of Φ_1 . In this way, for every $n \geq 0$ we can generally get the Stone extension

$$\Phi_n : \beta Q^{(n+1)} = Q^{(n+1)} \cup Q^{(n+2)} \rightarrow Q^{(n)} \cup Q^{(n+1)} = \beta Q^{(n)}$$

of the identity map $id : Q^{(n+1)} \rightarrow Q^{(n+1)}$, and its restriction map

$$\phi_n : Q^{(n+2)} \rightarrow Q^{(n)}.$$

Since every Φ_n ($n \in \omega$) is perfect, so is every ϕ_n . Hence every $Q^{(n)}$ ($n \in \omega$) is Lindeöf since both $Q^{(0)} = Q$, $Q^{(1)}$ are Lindeöf. We can also see that $Q^{(n)}$ is σ -compact for even n , but $Q^{(n)}$ is not for odd n , because $Q^{(0)}$ is σ -compact but $Q^{(1)}$ is not since $Q^{(1)}$ is a perfect pre-image of the irrationals $\mathbb{P}$ as we see below.

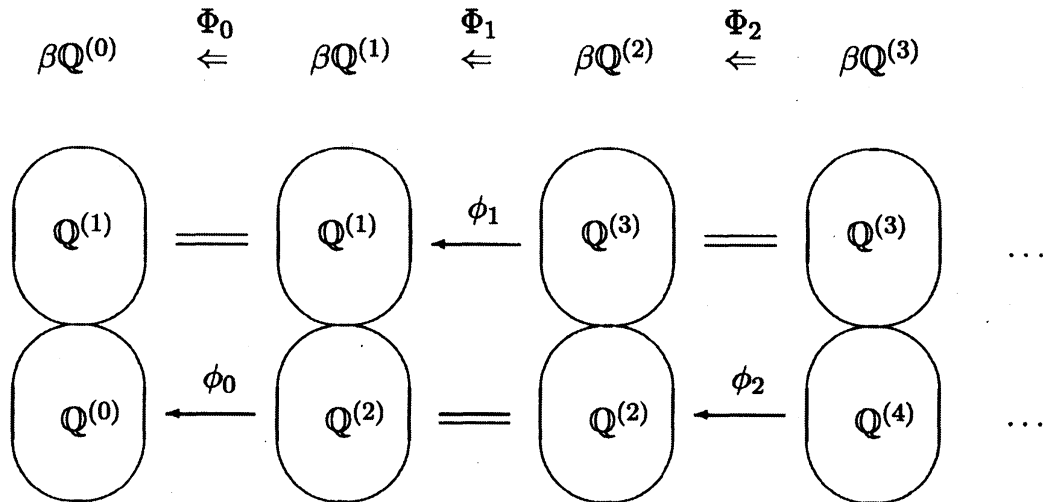


FIG. 1

A collection $\mathcal{B}$ of nonempty open sets of X is called a π -base for X if every nonempty open set in X includes some member of $\mathcal{B}$. The minimal cardinality of such a π -base is called the π -weight of X . Note that any dense subspace of X has the same π -weight as X , and any space of countable π -weight is separable. Consequently, any dense subset of a space of countable π -weight is also of countable π -weight, and hence separable. So, all of $\beta Q^{(n)}$, $Q^{(n)}$ ($n \in \omega$) are of countable π -weight, and hence separable.

Recall that an onto map $g : X \rightarrow Y$ is called *irreducible* if every nonempty open subset U of X includes some fiber $g^{-1}(y)$, and it is well known and easy to see that

- (1) every extension of a homeomorphism is irreducible, and
- (2) the restriction of a closed irreducible map to any dense subset is irreducible.

Therefore we can see that all of the maps Φ_n, ϕ_n ($n \in \omega$) are perfect irreducible. Consider the partition of the closed interval $[0, 1] = Q \cup P$ where

$$Q = [0, 1] \cap \mathbb{Q} \approx \mathbb{Q} \text{ and } P = [0, 1] \setminus \mathbb{Q} \approx \mathbb{P},$$

and let $f : \beta\mathbb{Q} \rightarrow [0, 1]$ be the Stone extension of the homeomorphism $\mathbb{Q} \approx Q$. Then the restriction $f_0 = f \upharpoonright \mathbb{Q}^{(1)} : \mathbb{Q}^{(1)} \rightarrow P \approx \mathbb{P}$ is perfect irreducible. Thus we get the following sequence of perfect irreducible maps:

$$\mathbb{Q} \leftarrow \mathbb{Q}^{(2)} \leftarrow \mathbb{Q}^{(4)} \leftarrow \dots; \quad \mathbb{P} \leftarrow \mathbb{Q}^{(1)} \leftarrow \mathbb{Q}^{(3)} \leftarrow \mathbb{Q}^{(5)} \leftarrow \dots$$

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. "Partition" is synonymous with "disjoint union." For a subset A of some compact space K we use the notation A^* to denote the remainder $\text{cl}_K A \setminus A$ when K is clear from the context. Our terminologies are based upon [3].

2. SIMILAR STRUCTURES

We first show that both $\mathbb{Q}^{(n)}$ and $\mathbb{Q}^{(n+2)}$ have a similar structure for every $n \geq 1$. In general, for any space Y let us denote by $\mathbf{H}(Y)$ the collection of all homeomorphisms $h : Y \approx Y$. Let X be a nowhere compact, dense-in-itself space, where *nowhere compact* (or *nowhere locally compact*) means that X contains no compact neighborhood, or equivalently, that X is a dense subset of some/any compact space K such that the remainder $K \setminus X$ is also dense in K . Let cX be some compactification of X and let $\mathcal{H}_* \subseteq \mathbf{H}(X)$ denote the collection of all $h \in \mathbf{H}(X)$ such that

($\star$) h is extendable to $c(h) \in \mathbf{H}(cX)$.

(Of course, $\mathcal{H}_* = \mathbf{H}(cX)$ if $cX = \beta X$.) Let $X^{(1)} = cX \setminus X$ be the remainder, and for every $h \in \mathcal{H}_*$ define $h^{(1)} \in \mathbf{H}(X^{(1)})$ to be the restriction of $c(h)$ to $X^{(1)}$. Next consider the Stone-Ćech compactification $\beta X^{(1)}$ of $X^{(1)}$ and the Stone extension $\beta h^{(1)} \in \mathbf{H}(\beta X^{(1)})$ of $h^{(1)}$. Let $X^{(2)} = \beta X^{(1)} \setminus X^{(1)}$ be the remainder, and define $h^{(2)} \in \mathbf{H}(X^{(2)})$ to be the restriction of $\beta h^{(1)}$ to the remainder $X^{(2)}$; hence

$$h : X \approx X, \quad h^{(1)} : X^{(1)} \approx X^{(1)}, \quad h^{(2)} : X^{(2)} \approx X^{(2)}.$$

Note that $X^{(1)}$ is dense in βX , and $X^{(2)}$ is dense in $\beta X^{(1)}$, since we assume that X is nowhere compact. Viewing that βX is a compactification of $X^{(1)}$, we can consider the Stone extension $\Phi : \beta X^{(1)} \rightarrow \beta X$ of the identity map $\text{id}_{X^{(1)}} : X^{(1)} = X^{(1)}$. Let $\phi : X^{(2)} \rightarrow X$ be the restriction of Φ . Then both Φ and ϕ are perfect irreducible maps. We can show that the correspondence $\mathbf{H}(X) \supseteq \mathcal{H}_* \ni h \mapsto h^{(2)} \in \mathbf{H}(X^{(2)})$ is compatible with the perfect irreducible map ϕ , i.e.,

Lemma 2.1. $h \circ \phi = \phi \circ h^{(2)} : X^{(2)} \rightarrow X$.

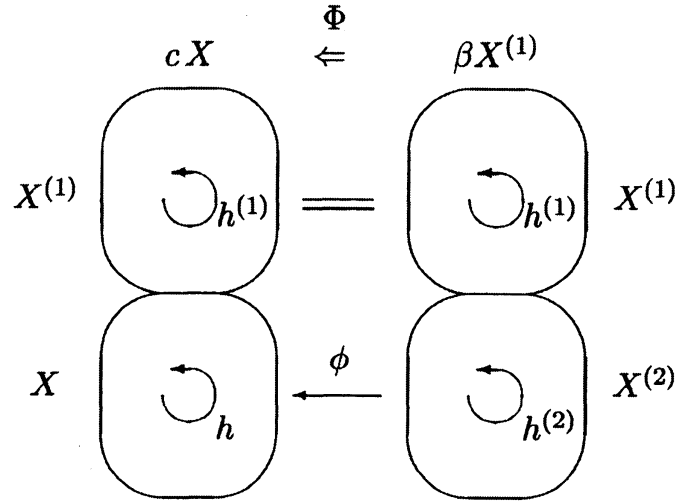


FIG. 2

Proof. To show this equality, it suffices to prove the equality

$$c(h) \circ \Phi = \Phi \circ \beta h^{(1)} : \beta X^{(1)} \rightarrow cX,$$

which follows from the obvious equality

$$h^{(1)} \circ id_{X^{(1)}} = id_{X^{(1)}} \circ h^{(1)} : X^{(1)} \rightarrow X^{(1)}$$

on the dense subset $X^{(1)}$ of $\beta X^{(1)}$. $\square$

Corollary 2.2. *If $h(x) = y$ for $x, y \in X$, then $h^{(2)}(\phi^{-1}(x)) = \phi^{-1}(y)$.*

Proof. The inclusion $h^{(2)}(\phi^{-1}(x)) \subseteq \phi^{-1}(y)$ follows from 2.1. Since h is a homeomorphism, we can replace h by h^{-1} to get the reverse inclusion. $\square$

Taking $X = \mathbb{Q}$, $cX = \beta\mathbb{Q}$, $\mathcal{H}_\star = \mathbf{H}(\mathbb{Q})$, we can deduce from 2.1 that

$$(2-1) \quad h \circ \phi_0 = \phi_0 \circ h^{(2)} : \mathbb{Q}^{(2)} \rightarrow \mathbb{Q} \quad \text{for every } h \in \mathbf{H}(\mathbb{Q}).$$

Let $[0, 1] = Q \cup P$, $Q \approx \mathbb{Q}$, $P \approx \mathbb{P}$ be as at the end of §1, and take $X = P$, $cX = [0, 1]$; then $X^{(1)} = Q$, $X^{(2)} = Q^{(1)}$, and the corresponding map ϕ in Fig. 2 is identical to the map $f_0 : \mathbb{Q}^{(1)} \rightarrow P$ at the end of §1. Note that $\mathcal{H}_\star \subseteq \mathbf{H}(P)$ is the collection of all homeomorphisms of P extendable to homeomorphisms of $[0, 1]$. Then we can deduce from 2.1 that

$$(2-2) \quad h \circ f_0 = f_0 \circ h^{(2)} : \mathbb{Q}^{(1)} \rightarrow P \quad \text{for every } h \in \mathcal{H}_\star.$$

Note that for every pair of irrationals $p_1 < p_2$ in $P = [0, 1] \setminus \mathbb{Q}$ we can find an $h \in \mathcal{H}_\star$ such that $h(p_1) = p_2$; for example, we can take as $c(h)$ in $(\star)$ a strictly increasing function $c(h) : [0, 1] \rightarrow [0, 1]$ such that $c(h)(Q) = Q$, $c(h)(0) = 0$, $c(h)(p_1) = p_2$, $c(h)(1) = 1$. For $m \geq 1$ define g_{2m} and f_{2m-1} by

$$g_{2m} = \phi_0 \circ \phi_2 \circ \cdots \circ \phi_{2m-2} : \mathbb{Q}^{(2m)} \rightarrow \mathbb{Q},$$

$$f_{2m-1} = f_0 \circ \phi_1 \circ \phi_3 \circ \cdots \circ \phi_{2m-3} : \mathbb{Q}^{(2m-1)} \rightarrow P.$$

Then, using 2.1 we can extend the above (2-1), (2-2) to the followings, respectively, for $m \geq 1$.

$$(2-3) \quad h \circ g_{2m} = g_{2m} \circ h^{(2m)} : \mathbb{Q}^{(2m)} \rightarrow \mathbb{Q} \quad \text{for every } h \in \mathbf{H}(\mathbb{Q}),$$

$$(2-4) \quad h \circ f_{2m-1} = f_{2m-1} \circ h^{(2m-1)} : \mathbb{Q}^{(2m-1)} \rightarrow P \quad \text{for every } h \in \mathcal{H}_*,$$

where $h^{(n)} \in \mathbf{H}(\mathbb{Q}^{(n)})$. Combining these results with 2.2 we can summarize that

Theorem 2.3. *Let $m \geq 1$. Then every $\mathbb{Q}^{(2m)}$ admits a perfect irreducible projection g_{2m} onto $\mathbb{Q}$, and every $\mathbb{Q}^{(2m-1)}$ admits a perfect irreducible projection f_{2m-1} onto $P \approx \mathbb{P}$, with the additional property that they are "fiber-wise" homogeneous in the following sense:*

(1) *For any $q_1 < q_2 \in \mathbb{Q}$ there exists a homeomorphism of $\mathbb{Q}^{(2m)}$, induced by a homeomorphism of $\mathbb{Q}$, carrying the fiber $g_{2m}^{-1}(q_1)$ to $g_{2m}^{-1}(q_2)$.*

(2) *For any $p_1 < p_2 \in P$ there exists a homeomorphism of $\mathbb{Q}^{(2m-1)}$, induced by a homeomorphism of P , carrying the fiber $f_{2m-1}^{-1}(p_1)$ to $f_{2m-1}^{-1}(p_2)$.*

Moreover, under CH (=the Continuum Hypothesis) every fiber $g_{2m}^{-1}(q)$ of $q \in \mathbb{Q}$ as well as every fiber $f_{2m-1}^{-1}(p)$ of $p \in P$ is homeomorphic to $\omega^ = \beta\omega \setminus \omega$.*

This last assertion follows from the well-known

Fact 2.4. (see 1.2.6 in [8] or 3.37 in [9]) (CH) *Let Y be a 0-dimensional, locally compact, σ -compact, non-compact space of weight at most $\mathfrak{c}$. Then $Y^* = \beta Y \setminus Y$ and ω^* are homeomorphic.*

Indeed, put $Z = g_{2m}^{-1}(q)$ and $Y = \beta\mathbb{Q}^{(2m-1)} \setminus Z$. Then Z is a zero-set of the 0-dimensional $\beta\mathbb{Q}^{(2m-1)}$ included in the remainder $\mathbb{Q}^{(2m)} = \beta\mathbb{Q}^{(2m-1)} \setminus \mathbb{Q}^{(2m-1)}$, so that $Y^* = \beta Y \setminus Y = Z$. Since Y is a cozero-set and separable, Y satisfies the condition in 2.4. Hence $Z \approx \omega^*$. Similarly we can prove that $f_{2m-1}^{-1}(p) \approx \omega^*$.

3. REMOTE POINTS AND EXTREMALLY DISCONNECTED POINTS

To analyze further the structure of $\mathbb{Q}^{(n)}$'s, we need the notion of remote points and extremally disconnected points. A point $p \in \beta X \setminus X$ is called a *remote point* of X if $p \notin \text{cl}_{\beta X} F$ for every nowhere dense closed subset F of X . Van Douwen [2], Chae, Smith [1], showed

Fact 3.1. *Every non-pseudocompact space of countable π -weight has $2^{\mathfrak{c}}$ many remote points.*

An easy consequence of this fact is

Fact 3.2. *Let X be a non-compact, Lindelöf space of countable π -weight. Then remote points of X form a G_δ -dense subset of $X^* = \beta X \setminus X$.*

Proof. Choose any point $p \in X^*$ and a zero-set Z of βX containing p . Since X is Lindelöf, we can suppose that Z misses X . Put $Y = \beta X \setminus Z$; then $\beta Y = \beta X$, and Y is of countable π -weight since X is. Hence 3.1 implies that $Y^* = Z$ contains remote points of Y , which are also remote points of X . $\square$

A space T is said to be *extremally disconnected at a point* $p \in T$ (see [2]) if $p \notin \text{cl}_T U_1 \cap \text{cl}_T U_2$ for every pair of disjoint open sets U_1, U_2 in T . Let us call such a point p as an *extremally disconnected point* of T , or simply, an *e.d.* point of T , and denote the set of all such e.d. points by $\text{Ed}(T)$. A space T is *extremally disconnected* if every point of T is an e.d. point, i.e., $\text{Ed}(T) = T$. If S is dense in T , we always have $\text{cl}_T U = \text{cl}_T(U \cap S)$ for every open set U of T ; hence a point $p \in S$ is an e.d. point of S if and only if it is an e.d. point of T , i.e., $\text{Ed}(S) = S \cap \text{Ed}(T)$.

Fact 3.3. ([2]) (1) *Any remote point of X is an e.d. point of βX .*
 (2) *Suppose X is first countable and hereditarily separable, and $p \in \beta X \setminus X$. Then p is a remote point of X if and only if p is an e.d. point of βX .*

Let us call a point $p \in T$ a *common boundary point* of T if p is not an e.d. point of T , i.e., if $p \in \text{cl}_T U_1 \cap \text{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T . Similarly, we call a subset $A \subseteq T$ a *common boundary set* in T if $A \subseteq \text{cl}_T U_1 \cap \text{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T . We abbreviate “common boundary” to “co-boundary.” (Such p, A are called “2-point,” “2-set,” respectively, in [2].) Note that any co-boundary set in T is nowhere dense in T , but the converse need not be true. Let $\text{Cob}(T) = T \setminus \text{Ed}(T)$ denote the set of all co-boundary points of T . Note also that if A is a co-boundary set, then every point of A is obviously a co-boundary point, but the converse need not be true except the case A is a countable discrete subset:

Lemma 3.4. *Suppose A is a countable discrete subset consisting of co-boundary points of T . Then A , and hence also $\text{cl}_T A$, is a co-boundary set in T . Therefore, if T is compact, $\text{Cob}(T)$ is always countably compact in the strong sense that every countable discrete subset has compact closure in $\text{Cob}(T)$.*

Proof. Let $A = \{a_n\}_{n \in \omega} \subseteq \text{Cob}(T)$ be discrete in T , and choose disjoint open sets $\{W_n\}_{n \in \omega}$ in T such that $a_n \in W_n$. In each W_n choose disjoint open sets U_n, V_n with $a_n \in \text{cl}_T U_n \cap \text{cl}_T V_n$. Put $U = \bigcup_{n \in \omega} U_n$ and $V = \bigcup_{n \in \omega} V_n$. Then these disjoint open sets U, V satisfy $A \subseteq \text{cl}_T U \cap \text{cl}_T V$, and hence $\text{cl}_T A \subseteq \text{cl}_T U \cap \text{cl}_T V$. $\square$

For an open set $U \subseteq X$ its maximal open extension $\text{Ex}(U) \subseteq \beta X$ is defined by

$$\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X}(X \setminus U).$$

Suppose W is an open set in βX ; then

$$\text{cl}_{\beta X} W = \text{cl}_{\beta X}(W \cap X) = \text{cl}_{\beta X} \text{Ex}(W \cap X).$$

Therefore we see

Fact 3.5. *Suppose $p \in \beta X \setminus X$. Then p is a co-boundary point of βX if and only if $p \in \text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V)$ for some disjoint open sets U, V in X .*

We denote the boundary of a subset W in Y by $\text{Bd}_Y W$ so that $\text{Bd}_Y W = \text{cl}_Y W \setminus W$ if W is open in Y . Van Douwen [2] proved the equality

$$(*) \quad \text{Bd}_{\beta X} \text{Ex}(U) = \text{cl}_{\beta X} \text{Bd}_X(U)$$

for every open set U of X . (Note that 3.3 (1) follows from this equality since $\text{Bd}_X(U)$ is a nowhere dense subset of X .) Using this $(*)$ and 3.5 we get an "inner" characterization of co-boundary points, hence of e.d. points also, of βX for a normal space X :

Lemma 3.6. *Assume X is normal, and $p \in \beta X \setminus X$. Then p is a co-boundary point of βX if and only if $p \in \text{cl}_{\beta X} F$ for some co-boundary set F in X . In other words, p is an e.d. point of βX if and only if*

$$p \notin \text{cl}_{\beta X} F \text{ for every co-boundary set } F \text{ in } X.$$

Proof. By 3.5 it suffices to show the equality

$$\text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) = \text{cl}_{\beta X} (\text{cl}_X U \cap \text{cl}_X V)$$

for disjoint open sets U, V in X , since $\text{cl}_X U \cap \text{cl}_X V$ is a co-boundary set in X . Using $(*)$ we get

$$\begin{aligned} \text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) &= \text{Bd}_{\beta X} \text{Ex}(U) \cap \text{Bd}_{\beta X} \text{Ex}(V) \\ &= (\text{cl}_{\beta X} \text{Bd}_X U) \cap (\text{cl}_{\beta X} \text{Bd}_X V). \end{aligned}$$

Since X is normal, this set is equal to $\text{cl}_{\beta X} (\text{Bd}_X U \cap \text{Bd}_X V)$, where $\text{Bd}_X U \cap \text{Bd}_X V = \text{cl}_X U \cap \text{cl}_X V$. □

Lemma 3.7. *Suppose A is a closed subset of a normal space X . Then $A \subseteq \text{Ed}(X)$ implies $\text{cl}_{\beta X} A \subseteq \text{Ed}(\beta X)$.*

Proof. Let A be a closed subset of a normal space X , and that $A \subseteq \text{Ed}(X)$. Let F be any co-boundary closed set in X . By 3.6 it suffices to show that $\text{cl}_{\beta X} F \cap \text{cl}_{\beta X} A = \emptyset$. Since $F \subseteq \text{Cob}(X)$ and $A \subseteq \text{Ed}(X)$, we know that F, A are disjoint closed subsets of X . Hence the normality of X implies that $\text{cl}_{\beta X} F \cap \text{cl}_{\beta X} A = \emptyset$. □

The next lemma shows how co-boundary points or e.d. points behave w.r.t. closed irreducible maps. Let g be a map from X onto Y . For a subset $U \subseteq X$ define $g^\circ(U) \subseteq Y$, a *small image* of U , by

$$y \in g^\circ(U) \text{ if and only if } g^{-1}(y) \subseteq U,$$

i.e., $g^\circ(U) = Y \setminus g(X \setminus U) \subseteq g(U)$; so, g is irreducible if $g^\circ(U) \neq \emptyset$ for every non-empty open set U . Note an obvious useful formula

$$g^\circ(U \cap V) = g^\circ(U) \cap g^\circ(V)$$

for any sets $U, V \subseteq X$, which especially implies that $g^\circ(U) \cap g^\circ(V) = \emptyset$ whenever $U \cap V = \emptyset$. Suppose g is closed irreducible. Then it is well known that $g^\circ(U)$ is non-empty and open whenever U is, and

$$\text{cl}_Y g^\circ(U) = \text{cl}_Y g(U) = g(\text{cl}_X U)$$

for every open subset $U \subseteq X$.

Lemma 3.8. *Let $g : X \rightarrow Y$ be any closed irreducible map. Then g maps co-boundary points to co-boundary points, i.e., $g(\text{Cob}(X)) \subseteq \text{Cob}(Y)$. Furthermore, for every $x \in X$*

$g(x) \in \text{Cob}(Y)$ if and only if $x \in \text{Cob}(X)$ or $|g^{-1}(g(x))| > 1$, i.e.,

$g(x) \in \text{Ed}(Y)$ if and only if $x \in \text{Ed}(X)$ and $g^{-1}(g(x)) = \{x\}$.

Consequently, $g^{-1}(\text{Ed}(Y)) \subseteq \text{Ed}(X)$, and the restriction of g to

$$g^{-1}(\text{Ed}(Y)) \rightarrow \text{Ed}(Y)$$

is a homeomorphism.

Proof. Let U_1, U_2 be any disjoint open sets in X . Then

$$g(\text{cl}_X U_1 \cap \text{cl}_X U_2) \subseteq g(\text{cl}_X U_1) \cap g(\text{cl}_X U_2) = \text{cl}_Y g^\circ(U_1) \cap \text{cl}_Y g^\circ(U_2),$$

and $g^\circ(U_1), g^\circ(U_2)$ are disjoint open. Hence g maps co-boundary points to co-boundary points. Similarly, we can show that

$$|g^{-1}(g(x))| > 1 \text{ implies } g(x) \in \text{Cob}(Y).$$

Indeed, if we take two points $x_1 \neq x_2$ in $g^{-1}(g(x))$, we can choose disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$ (using the Hausdorffness of X), getting $g(x) \in g(\text{cl}_X U_1) \cap g(\text{cl}_X U_2) = \text{cl}_Y g^\circ(U_1) \cap \text{cl}_Y g^\circ(U_2)$.

So, to complete our proof, assume $g(x) \in \text{Cob}(Y)$ and $|g^{-1}(g(x))| = 1$; then we need to show $x \in \text{Cob}(X)$. The condition $g(x) \in \text{Cob}(Y)$ implies that $g(x) \in \text{cl}_Y V_1 \cap \text{cl}_Y V_2$ for some disjoint open sets V_1, V_2 in Y . Since g is a closed map, $g(x) \in \text{cl}_Y V_i$ implies $g^{-1}(g(x)) \cap \text{cl}_X g^{-1}(V_i) \neq \emptyset$ for $i = 1, 2$. Hence the condition $g^{-1}(g(x)) = \{x\}$ implies $x \in \text{cl}_X g^{-1}(V_1) \cap \text{cl}_X g^{-1}(V_2)$, showing $x \in \text{Cob}(X)$. $\square$

4. TOPOLOGICAL DIFFERENCE OF $\mathbb{Q}^{(1)}$ AND $\mathbb{Q}^{(3)}$

Now let us apply the general theory in §3 to our spaces

$$\beta\mathbb{Q}^{(n)} = \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} \quad (n \geq 0).$$

Recall that every $\mathbb{Q}^{(n)}$ is of countable π -weight and Lindelöf, hence normal. Put $C_n = \text{Cob}(\mathbb{Q}^{(n)})$ and $E_n = \text{Ed}(\mathbb{Q}^{(n)})$; then this gives a partition of $\mathbb{Q}^{(n)}$

$$\mathbb{Q}^{(n)} = C_n \cup E_n.$$

It is obvious that $E_0 = \emptyset$, i.e., $\mathbb{Q}^{(0)} = C_0$. Lemma 3.4 implies that each C_n ($n \geq 1$) is dense in $\mathbb{Q}^{(n)}$, and Fact 3.2 with 3.3 (1) implies that each

E_n ($n \geq 1$) is dense in $Q^{(n)}$. Note in particular that E_1 coincides with the set of all remote points of Q , by 3.3 (2).

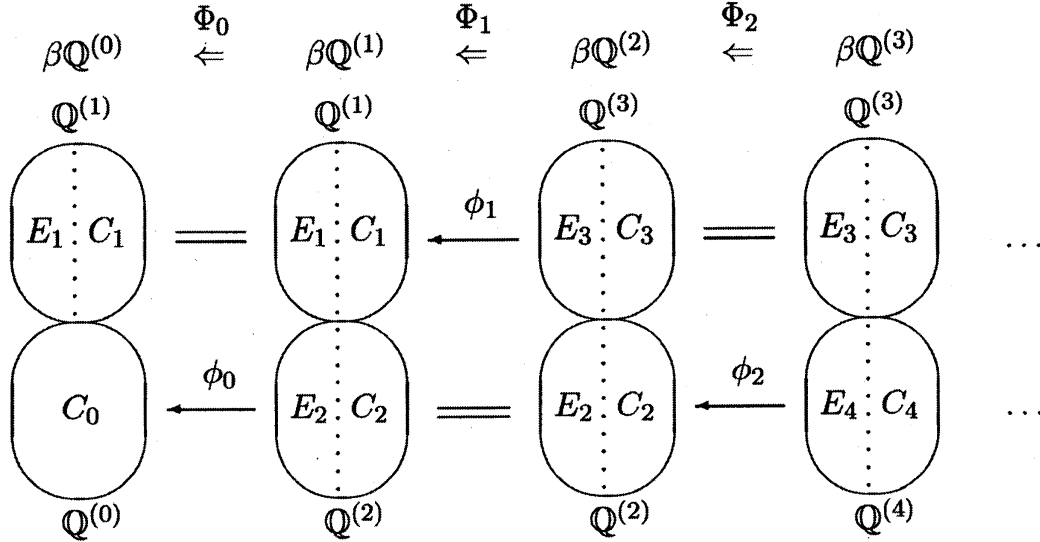


FIG. 3

Property 4.1. Let A be any countable discrete subset of E_2 which is closed in $Q^{(2)}$. Then

(1) $\text{cl } A \subseteq E_2 \cup C_1$ in $\beta Q^{(1)}$, while (2) $\text{cl } A \subseteq E_2 \cup E_3$ in $\beta Q^{(2)}$.

Proof. (2) follows from 3.7. To prove (1), let A be as above. Then, since $\phi_0 : Q^{(2)} \rightarrow Q^{(0)}$ is perfect, $\phi_0(A)$ is also a countable discrete closed subset of $Q^{(0)} = C_0$. Since $C_0 \cup C_1 = \text{Cob}(\beta Q^{(0)})$ is countably compact in the strong sense as stated in 3.4, we have $\text{cl } \phi_0(A) \subseteq C_0 \cup C_1$ in $\beta Q^{(0)}$. Pulling back by the map Φ_0 , we get $\text{cl } A \subseteq Q^{(2)} \cup C_1$ in $\beta Q^{(1)}$. This is the same as the assertion (1) since $A \subseteq E_2$. $\square$

Now we can prove the following strong assertion which in particular implies that $Q^{(1)} \not\approx Q^{(3)}$.

Theorem 4.2. $Q^{(1)}$ admits no perfect irreducible map onto $Q^{(3)}$.

Proof. Suppose there existed a perfect irreducible map $\psi : Q^{(1)} \rightarrow Q^{(3)}$. Then, since $\beta Q^{(2)}$ can be seen as a compactification of $Q^{(3)}$, ψ extends to a perfect irreducible map

$$\Psi : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \rightarrow \beta Q^{(2)} = Q^{(3)} \cup Q^{(2)}.$$

Lemma 3.8 implies then that

$$E_2 \cup E_1 \supseteq \Psi^{-1}(E_2 \cup E_3) \approx E_2 \cup E_3.$$

Choose any countable discrete subset $B \subseteq E_2 \subseteq Q^{(2)} \subseteq \beta Q^{(2)}$ which is closed in $Q^{(2)}$. (We can do this because E_2 is dense in $Q^{(2)}$, and $Q^{(2)}$ is

Lindelöf.) Put $A = \Psi^{-1}(B)$, then this A is also a countable discrete subset of E_2 which is closed in $\mathbb{Q}^{(2)}$. Property 4.1 (2) shows $\text{cl } B \subseteq E_2 \cup E_3$ in $\beta\mathbb{Q}^{(2)}$, and so, pulling back by Ψ , we get

$$\text{cl } A \subseteq \Psi^{-1}(E_2 \cup E_3) \subseteq E_2 \cup E_1$$

in $\beta\mathbb{Q}^{(1)}$. But this contradicts 4.1 (1). $\square$

We will be able to show in [4] that for any $n \geq 1$, $\mathbb{Q}^{(n)}$ admits no perfect irreducible map onto $\mathbb{Q}^{(n+2)}$ by analyzing further the behavior of limit points of countable discrete subsets in $\mathbb{Q}^{(m)}$. Some of the basic techniques in this paper can be found also in [5, 6, 7].

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