

Recent progress on Takhtajan-Zograf and Weil-Petersson metrics

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Abstract

We will survey recent progress on Weil-Petersson and Takhtajan-Zograf metric. After reviewing the backgrounds and the known results for those metrics, a new estimate of the asymptotic behavior of the Takhtajan-Zograf metric near the boundary of the moduli space of punctured Riemann surfaces is stated without proof.

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1 Backgrounds on Weil-Petersson and Takhtajan-Zograf metrics

$T_{g,n}$ denotes the **Teichmüller space** of Riemann surfaces of genus g with n marked points ($2g - 2 + n > 0$). Let $C_{g,n}$ be the **Teichmüller curve** over $T_{g,n}$ with the projection $\pi : C_{g,n} \rightarrow T_{g,n}$ which has n sections $\mathbf{P}_1, \dots, \mathbf{P}_n$ corresponding to n marked points. Consider $\Omega_{C_{g,n}}^1$ (resp. $\Omega_{T_{g,n}}^1$) the sheaf of holomorphic 1-forms on $C_{g,n}$ (resp. $T_{g,n}$). The sheaf of **relative differential forms** on $C_{g,n}$ is defined as

$$\omega_{C_{g,n}/T_{g,n}} := \Omega_{C_{g,n}}^1 / \pi^* \Omega_{T_{g,n}}^1. \quad (1.1)$$

Then the **determinant line bundle** λ_l on $T_{g,n}$ ($l \in \mathbf{N}$) is defined as

$$\lambda_l := \bigwedge^{\max} R^0 \pi_* \omega_{C_{g,n}/T_{g,n}}^{\otimes l} ((l-1)(\mathbf{P}_1 + \cdots + \mathbf{P}_n)). \quad (1.2)$$

For a point $s \in T_{g,n}$, $S := \pi^{-1}(s)$ is a compact Riemann surface. Set $S^0 := S - \{\mathbf{P}_1(s), \dots, \mathbf{P}_n(s)\}$ and $P_p := \mathbf{P}_p(s)$ ($p = 1, \dots, n$).

Here we can see

$$R^0 \pi_* \omega_{C_{g,n}/T_{g,n}}^{\otimes l} ((l-1)(\mathbf{P}_1 + \cdots + \mathbf{P}_n))|_s = \Gamma(S, K_S^{\otimes l} \otimes \mathcal{O}_S(P_1 + \cdots + P_n)^{\otimes (l-1)})$$

$\simeq \{\text{meromorphic } l \text{ differentials on } S \text{ with possibly poles of order at most } l-1 \text{ only at the marked points}\}.$

Pick a basis of local holomorphic sections $\phi_1, \dots, \phi_{d(l)}$ for $R^0 \pi_* \omega_{C_{g,n}/T_{g,n}}^{\otimes l} ((l-1)(\mathbf{P}_1 + \cdots + \mathbf{P}_n))$, where

$$d(l) = \begin{cases} g & (l = 1) \\ (2l-1)(g-1) + (l-1)n & (l > 1). \end{cases}$$

$$\langle \phi_i, \phi_j \rangle := \iint_{S^0} \phi_i \overline{\phi_j} \rho_{S^0}^{-(l-1)} \quad (i, j = 1, \dots, d(l)) \quad (1.3)$$

is called the **Petersson product**, where ρ_{S^0} is the hyperbolic area element on S^0 .

We set

$$\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2} := |\det(\langle \phi_i, \phi_j \rangle)|^{1/2}, \quad (1.4)$$

$$\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_Q := \|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2} Z_{S^0}(l)^{-\frac{1}{2}} \quad (1.5)$$

($l \geq 2$. For $l = 1$, employ $Z'_{S^0}(1)$ in place of $Z_{S^0}(1) = 0$). Here, $Z_{S^0}(l)$ denotes the special value of $Z_{S^0}(\cdot)$ on S^0 at l integer, which will be defined below. Then $\lambda_l \rightarrow T_{g,n}$ is a Hermitian holomorphic line bundle equipped with the **Quillen metric** $\|\cdot\|_Q$ (see [7]). Here

$$Z_{S^0}(s) := \prod_{\{\gamma\}} \prod_{m=1}^{\infty} (1 - e^{-(s+m)L(\gamma)}) \quad (1.6)$$

is the **Selberg Zeta function** for S^0 , $\text{Re}(s) > 1$, where γ runs over all oriented primitive closed geodesics on S^0 , and $L(\gamma)$ denotes the hyperbolic length of γ . It extends meromorphically to the whole plane in s .

In the late 80's, we have discovered the following important formulas for the curvature forms of the determinant line bundles with respect to the Quillen metrics.

Theorem 1.1 (Belavin-Knizhnik+Wolpert(1986), [1], [8]).

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2} \omega_{WP} \quad (n = 0).$$

Theorem 1.2 (Takhtajan-Zograf (1988, 1991), [7]).

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2} \omega_{WP} - \frac{1}{9} \omega_{TZ} \quad (n > 0).$$

Here, ω_{WP}, ω_{TZ} are the Kähler forms of the Weil-Petersson, the Takhtajan-Zograf metrics respectively.

Here remind us of the definitions of the Weil-Petersson and the Takhtajan-Zograf metrics. By the deformation theory of Kodaira-Spencer and the Hodge theory, for $[S^0] \in T_{g,n}$, we have

$$T_{[S^0]}T_{g,n} \simeq HB(S^0), \quad (1.7)$$

where $HB(S^0)$ is the space of harmonic Beltrami differentials on S^0 .

By the Serre duality, one has

$$T_{[S^0]}^*T_{g,n} \simeq Q(S^0), \quad (1.8)$$

where $Q(S^0)$ is the space of holomorphic quadratic differentials on S^0 with finite the Petersson-norm, which is dual to $HB(S^0)$.

The inner product of the **Weil-Petersson metric** at $T_{[S^0]}T_{g,n}$ is defined to be

$$\langle \alpha, \beta \rangle_{WP}([S^0]) := \iint_{S^0} \alpha \bar{\beta} \rho_{S^0}, \quad (1.9)$$

where α, β are in $HB(S^0) \simeq T_{[S^0]}T_{g,n}$.

The inner products of the **Takhtajan-Zograf metrics** are defined to be

$$\langle \alpha, \beta \rangle_p([S^0]) := \iint_{S^0} \alpha \bar{\beta} E_p(\cdot, 2) \rho_{S^0}, \quad (p = 1, \dots, n). \quad (1.10)$$

Here, $E_p(\cdot, 2)$ is the Eisenstein series associated with the p -th marked point with index 2. Moreover, we set

$$\langle \alpha, \beta \rangle_{TZ}([S^0]) := \sum_{p=1}^n \langle \alpha, \beta \rangle_p([S^0]). \quad (1.11)$$

The **Eisenstein series** associated with the p -th marked point with index 2 is defined to be

$$E_p(z, 2) := \sum_{A \in \Gamma_p \backslash \Gamma} \{ \text{Im}(\sigma_p^{-1} A(z)) \}^2, \quad \text{for } z \in \mathbf{H}, \quad (1.12)$$

where $\mathbf{H}$ is the upper-half plane, Γ is a uniformizing Fuchsian group for S^0 and Γ_p is the parabolic subgroup associated with the p -th marked point, and $\sigma_p \in \text{PSL}(2, \mathbf{R})$ is a normalizer. $E_p(z, 2)$ assumes the infinity at the p -th marked point and vanishes at the other marked points. In addition, the Eisenstein series satisfy

$$\Delta_{hyp} E_p(z, 2) = 2E_p(z, 2), \quad (1.13)$$

where Δ_{hyp} is the negative hyperbolic Laplacian on S^0 . Especially $E_p(z, 2)$ is a positive subharmonic function on S^0 .

$\text{Mod}_{g,n}$ denotes the **mapping class group** of surfaces of genus g with n marked points. Then the **moduli space** $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points is described as $\mathcal{M}_{g,n} = T_{g,n} / \text{Mod}_{g,n}$. λ_l and all metrics we defined are compatible with the action of $\text{Mod}_{g,n}$, thus they all naturally descend down to $\mathcal{M}_{g,n}$ as orbifold line sheaves and orbifold metrics respectively.

Let $\overline{\mathcal{M}}_{g,n}$ denote the **Deligne-Mumford compactification** of $\mathcal{M}_{g,n}$. We have known the relations of the L^2 -cohomology of $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson metric and the second cohomology of $\overline{\mathcal{M}}_{g,n}$.

Theorem 1.3 (Saper (1993) [6]). *For $g > 1, n = 0$,*

$$H_{(2)}^*(\mathcal{M}_g, \omega_{WP}) \simeq H^*(\overline{\mathcal{M}}_g, \mathbf{R}).$$

Here, the left hand side is the L^2 -cohomology with respect to the Weil-Petersson metric.

2 Known results for the asymptotic behaviors of the Weil-Petersson and Takhtajan-Zograf metrics

The proof of Theorem 1.3 is based on the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space which we will review now.

Here we set $D := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ the compactification divisor. Now take $X_0 \in D$ a degenerate Riemann surface of genus g with n marked points and k nodes (we regard the marked points as deleted from the surface).

Each node q_i ($i = 1, 2, \dots, k$) has a neighborhood

$$N_i = \{(z_i, w_i) \in \mathbb{C}^2 \mid |z_i|, |w_i| < 1, z_i w_i = 0\}.$$

X_t denotes the smooth surface gotten from X_0 after cutting and pasting N_i under the relation $z_i w_i = t_i$, $|t_i|$ small. Then, D is locally described as $\{t_1 \cdots t_k = 0\}$ (see 3. in more details).

D has locally the pinching coordinate $(t, s) = (t_1, \dots, t_k, s_{k+1}, \dots, s_{3g-3+n})$ around $[X_0]$. Set $\alpha_i = \partial/\partial t_i, \beta_\mu = \partial/\partial s_\mu \in T_{(t,s)}(T_{g,n})$. We define the Riemannian tensors for the Weil-Petersson metric

$$g_{i\bar{j}}(t, s) := \langle \alpha_i, \alpha_j \rangle_{WP}(t, s),$$

$$g_{i\bar{\mu}}(t, s) := \langle \alpha_i, \beta_\mu \rangle_{WP}(t, s),$$

$$g_{\mu\bar{\nu}}(t, s) := \langle \beta_\mu, \beta_\nu \rangle_{WP}(t, s),$$

$$(i, j = 1, 2, \dots, k, \mu, \nu = k+1, \dots, 3g-3+n).$$

Furthermore, we define the Riemannian tensors for the Takhtajan-Zograf metric

$$h_{i\bar{j}}(t, s) := \langle \alpha_i, \alpha_j \rangle_{TZ}(t, s),$$

$$h_{i\bar{\mu}}(t, s) := \langle \alpha_i, \beta_\mu \rangle_{TZ}(t, s),$$

$$h_{\mu\bar{\nu}}(t, s) := \langle \beta_\mu, \beta_\nu \rangle_{TZ}(t, s),$$

$$(i, j = 1, 2, \dots, k, \mu, \nu = k+1, \dots, 3g-3+n).$$

The following theorem is a pioneering result for the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space.

Theorem 2.1 (Masur (1976), [2]). As $t_i, s_\mu \rightarrow 0$,

- i) $g_{i\bar{i}}(t, s) \approx \frac{1}{|t_i|^2(-\log |t_i|)^3}$ for $i \leq k$,
- ii) $g_{i\bar{j}}(t, s) = O\left(\frac{1}{|t_i||t_j|(\log |t_i|)^3(\log |t_j|)^3}\right)$
for $i, j \leq k, i \neq j$,
- iii) $g_{i\bar{\mu}}(t, s) = O\left(\frac{1}{|t_i|(-\log |t_i|)^3}\right)$
for $i \leq k, \mu \geq k+1$,
- iv) $g_{\mu\bar{\nu}}(t, s) \rightarrow g_{\mu\bar{\nu}}(0, 0)$ for $\mu, \nu \geq k+1$.

Recently, we updated Masur's result by improving Wolpert's formula for the asymptotic of the hyperbolic metric for degenerating Riemann surfaces.

Theorem 2.2 (Obitsu and Wolpert (2008), [5]). We can improve iv) in Theorem 2.1 as follows;

$$\begin{aligned} \text{iv)'} \quad g_{\mu\bar{\nu}}(t, s) &= g_{\mu\bar{\nu}}(0, s) + \frac{4\pi^4}{3} \sum_{i=1}^k (\log |t_i|)^{-2} \left\langle \beta_\mu, (E_{i,1} + E_{i,2}) \beta_\nu \right\rangle_{WP}(0, s) \\ &+ O\left(\sum_{i=1}^k (\log |t_i|)^{-3}\right) \\ &\text{as } t \rightarrow 0, \text{ for } \mu, \nu \geq k+1. \end{aligned}$$

Here, $E_{i,1}, E_{i,2}$ denote a pair of the Eisenstein series with index 2 associated with the i -th node of the limit surface X_0 .

That is, the Takhtajan-Zograf metrics have appeared from degeneration of the Weil-Petersson metric. On the other hand, we have a result for asymptotics of the Takhtajan-Zograf metric near the boundary of the moduli space. Before stating the result, we need the following definition.

Definition 2.3. Let X_0 be a degenerate Riemann surface with n punctures $p_1, \dots, p_n$ and m nodes $q_1, \dots, q_m$.

A node q_i is said to be **adjacent to punctures** (resp. a puncture p_j) if the component of $X_0 \setminus \{q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_m\}$ containing q_i also contains at least one of the p_j 's (resp. the puncture p_j). Otherwise, it is said to be **non-adjacent to punctures** (resp. the puncture p_j).

Theorem 2.4 (Obitsu-To-Weng (2008), [3]). As $(t, s) \rightarrow 0$, we observe the followings:

- i) For any $\epsilon > 0$, there exists a constant $C_{1,\epsilon}$ such that

$$h_{i\bar{i}}(t, s) \leq \frac{C_{1,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4-\varepsilon}} \quad \text{for } i \leq k;$$

For any $\varepsilon > 0$, there exists a constant $C_{2,\varepsilon}$ such that

$$h_{i\bar{i}}(t, s) \geq \frac{C_{2,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4+\varepsilon}} \quad \text{for } i \leq k$$

and the node q_i adjacent to punctures;

$$ii) \ h_{i\bar{j}}(t, s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$$

for $i, j \leq k, i \neq j$;

$$iii) \ h_{i\bar{\mu}}(t, s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)$$

for $i \leq k, \mu \geq k+1$;

$$iv) \ h_{\mu\bar{\nu}}(t, s) \longrightarrow h_{\mu\bar{\nu}}(0, 0) \quad \text{for } \mu, \nu \geq k+1.$$

3 Degenerate families of punctured Riemann surfaces and A test Eisenstein series

First of all, let us review the construction of degenerating punctured hyperbolic surfaces. We recall the construction of the plumbing family (see 2 [5]). Considerations begin with the *plumbing variety* $\mathcal{V} = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\}$. The defining function $zw - t$ has differential $zdw + wdz - dt$. Consequences are that $\mathcal{V}$ is a smooth variety, (z, w) are global coordinates, while (z, t) and (w, t) are not. Consider the projection $\Pi : \mathcal{V} \rightarrow D$ onto the t -unit disc. The projection Π is a submersion, except at $(z, w) = (0, 0)$; we consider $\Pi : \mathcal{V} \rightarrow D$ as a (degenerate) family of open Riemann surfaces. The t -fiber, $t \neq 0$, is the hyperbola germ $zw = t$ or equivalently the annulus $\{|t| < |z| < 1, w = t/z\} = \{|t| < |w| < 1, z = t/w\}$. The 0-fiber is the intersection of the unit ball with the union of the coordinate axes in $\mathbb{C}^2$; on removing the origin the union becomes $\{0 < |z| < 1\} \cup \{0 < |w| < 1\}$. Each fiber of $\mathcal{V}_0 = \mathcal{V} - \{0\} \rightarrow D$ has a complete hyperbolic metric.

Consider X_0 a finite union of hyperbolic surfaces with cusps. A plumbing family is the fiberwise gluing of the complement of cusp neighborhoods in X_0 and the plumbing variety $\mathcal{V} = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\}$. For a positive constant $c_* < 1$ and initial surface X_0 , with puncture p with cusp coordinate z and puncture q with cusp coordinate w , we construct a family $\{X_t\}$. For $|t| < c_*^4$ the resulting surface X_t will be independent of c_* ;

the constant c_* will serve to specify the overlap of coordinate charts and to define a *collar* in each X_t .

We first describe the gluing of fibers. For $|t| < c_*^4$, remove from X_0 the punctured discs $\{0 < |z| \leq |t|/c_*\}$ about p and $\{0 < |w| \leq |t|/c_*\}$ about q to obtain a surface X_{t/c_*}^* . For $t \neq 0$, form an identification space X_t , by identifying the annulus $\{|t|/c_* < |z| < c_*\} \subset X_{t/c_*}^*$ with the annulus $\{|t|/c_* < |w| < c_*\} \subset X_{t/c_*}^*$ by the rule $zw = t$. The resulting surface X_t is the *plumbing* for the prescribed value of t . We note for $|t| < |t'|$ that there is an inclusion of X_{t'/c_*}^* in X_{t/c_*}^* ; the inclusion maps provide a way to compare structures on the surfaces. The inclusion maps are a basic feature of the plumbing construction. We next describe the plumbing family. Consider the variety $\mathcal{V}_{c_*} = \{(z, w, t) \mid zw = t, |z|, |w| < c_*, |t| < c_*^4\}$ and the disc $D_{c_*} = \{|t| < c_*^4\}$. The complex manifolds $M = X_{t/c_*}^* \times D_{c_*}$ and $\mathcal{V}_{c_*}$ have holomorphic projections to the disc D_{c_*} . The variables z, w denote prescribed coordinates on X_{t/c_*}^* and on $\mathcal{V}_{c_*}$. There are holomorphic maps of subsets of M to $\mathcal{V}_{c_*}$, commuting with the projections to D_{c_*} , as follows

$$(z, t) \xrightarrow{\hat{F}} (z, t/z, t) \text{ and } (w, t) \xrightarrow{\hat{G}} (w, t/w, t).$$

The identification space $\mathcal{F} = M \cup \mathcal{V}_{c_*} / \{\hat{F}, \hat{G} \text{ equivalence}\}$ is the *plumbing family* $\{X_t\}$ with projection to D_{c_*} (an analytic fiber space of Riemann surfaces in the sense of Kodaira. For $0 < |t| < c_*^4$, the t -fiber of $\mathcal{F}$ is the surface X_t constructed by overlapping annuli N_t .

We set two annuli

$$\Omega_t^1 := \left\{ z \in \mathbb{C} \mid \frac{|t|}{e^{a_0} c_*} < |z| < e^{a_0} c_* \right\} \quad \text{for } |t| < (c_*)^4, \quad (3.1)$$

$$\Omega_t^2 := \left\{ w \in \mathbb{C} \mid \frac{|t|}{e^{a_0} c_*} < |w| < e^{a_0} c_* \right\} \quad \text{for } |t| < (c_*)^4. \quad (3.2)$$

Here $0 < c_* < 1, a_0 < 0$ are the constants in [5].

When $t \neq 0$, one can identify as an annulus via coordinate projections as

$$N_t \longleftrightarrow \Omega_t^1 \longleftrightarrow \Omega_t^2. \quad (3.3)$$

And we may write $N_t = N_t^1 \cup N_t^2$, where

$$N_t^1 = \{z \in \mathbb{C} \mid |t|^{\frac{1}{2}} \leq |z| < e^{a_0} c_*\}, N_t^2 = \{w \in \mathbb{C} \mid |t|^{\frac{1}{2}} \leq |w| < e^{a_0} c_*\}. \quad (3.4)$$

For $t = 0$, define the cusp neighborhood

$$N_0 := \Omega_0^1 \cup \Omega_0^2. \quad (3.5)$$

In another word, we may consider that Ω_t^1 embed into X_t holomorphically for t, z . (See 2 in [5])

Here, remember the test function which is defined in [3]. For $t \neq 0$ one defines for $z \in \Omega_t^1$,

$$E_t^*(z) := \frac{-\pi}{\log |t| \sin \left(\frac{\pi \log |z|}{\log |t|} \right)}, \quad \rho_t^*(z) := \frac{\pi^2}{|z|^2 \log^2 |t| \sin^2 \left(\frac{\pi \log |z|}{\log |t|} \right)},$$

for $t = 0, z \in \Omega_t^1$,

$$E_0^*(z) := \frac{-1}{\log |z|}, \quad \rho_0^*(z) := \frac{1}{|z|^2 \log^2 |z|}.$$

It is easy to see that for $t \neq 0$, E_t^*, ρ_t^* have similar expressions for w in Ω_t^2 via the rule $zw = t$. Thus, E_t^*, ρ_t^* can be considered as functions on the manifolds N_t for $t \neq 0$. And one defines for $w \in \Omega_0^2$, $E_0^*(w), \rho_0^*(w)$ as the same expression as $E_0^*(z), \rho_0^*(z)$. Furthermore, we can easily observe that

$$\rho_0^* \leq \rho_t^* \quad \text{on } N_t \quad \text{for } |t| < (c^*)^4. \quad (3.6)$$

Masur showed in (6.5) [2] that there exists a positive constant K such that

$$\rho_t^* \leq K \rho_0^* \quad \text{on } N_t \quad \text{for } |t| < (c^*)^4. \quad (3.7)$$

From now, we always assume that the smooth surfaces X_t have at least one punctures. We are ready to consider a function

$$\varphi_t := \frac{E_t}{E_t^*}, \quad \text{on } N_t, \quad \text{for } |t| < (c^*)^4,$$

where E_t is the intrinsic Eisenstein series on a punctured hyperbolic surface X_t associated with a puncture.

We have already seen in the proof of Proposition 4.2.2 in [3] that on Ω_t^1 ,

$$\Delta E_t(z) = 2\rho_t(z)E_t(z), \quad (3.8)$$

$$\Delta E_t^*(z) = \left(1 + \cos^2 \left(\frac{\pi \log |z|}{\log |t|} \right) \right) \rho_t^*(z)E_t^*(z), \quad (3.9)$$

where $\Delta := 4 \frac{\partial^2}{\partial z \partial \bar{z}}$, $\rho_t(z)$ is the intrinsic hyperbolic area element on X_t , and $\rho_t^*(z)$ is the restriction to Ω_t^1 of the complete hyperbolic metric $r(z)|dz|^2$ of an annulus $\{z \in \mathbb{C} \mid |t| < |z| < 1\}$. It should be noted that $\rho_t^*(z)$ on Ω_t^1 is strictly smaller than the complete hyperbolic metric of Ω_t^1 . Now a straightforward calculation leads the following proposition (see [4] for the proof).

Proposition 3.1. *The function $\varphi_t(z)$ satisfies the following equation on Ω_t^1*

$$\begin{aligned} & -\Delta\varphi_t(z) + \frac{\pi}{\log|t|} \cot\left(\frac{\pi \log|z|}{\log|t|}\right) \left(\frac{2}{z} \frac{\partial\varphi_t(z)}{\partial\bar{z}} + \frac{2}{\bar{z}} \frac{\partial\varphi_t(z)}{\partial z} \right) \\ & + \left\{ 2\rho_t(z) - \left(1 + \cos^2\left(\frac{\pi \log|z|}{\log|t|}\right) \right) \rho_t^*(z) \right\} \varphi_t(z) = 0. \end{aligned}$$

We need the following result which is a special case of [5] Theorem 1.

Theorem 3.2. *On N_t , ρ_t has the expansion for $t \rightarrow 0$,*

$$\rho_t = \rho_t^* \left(1 + \frac{4\pi^4}{3} \left(E_{t,1}^\dagger + E_{t,2}^\dagger \right) \frac{1}{(\log|t|)^2} + Q(t) \right),$$

where $Q(t)$ has the estimate

$$Q(t) = O\left(\frac{1}{(\log|t|)^3} \right) \quad \text{for } t \rightarrow 0.$$

The function $E_{t,1}^\dagger, E_{t,2}^\dagger$ is the modified Eisenstein series. The O -term refers to the intrinsic C^1 -norm of a function on X_t . The bounds depend on the choice of c^*, a_0 and a lower bound for the injectivity radius for the complement of the cusp regions in X_0 .

The functions $E_{t,1}^\dagger, E_{t,2}^\dagger$ are constructed as follows (see Definition 1 in [5]). First, consider the case where the pinching curve is non-dividing. Now we may assume that for $t = 0$, our coordinates z, w are so-called the *standard* coordinate (see Remark-Definition 2.1.2 in [3]). Take the two Eisenstein series $E_{0,1}, E_{0,2}$ on X_0 associated with the node. Set $E_{0,1}^\sharp = E_{0,1} - (\log|z|)^2$ on Ω_0^1 , $E_{0,1}^\sharp = E_{0,1}$ otherwise. $E_{0,2}^\sharp = E_{0,2} - (\log|w|)^2$ on Ω_0^2 , $E_{0,2}^\sharp = E_{0,2}$ otherwise. Set $E_{t,1}^\dagger = E_{0,1}^\sharp(z) + E_{0,1}^\sharp(\frac{t}{z})$ on N_t . Similarly set $E_{t,2}^\dagger = E_{0,2}^\sharp(w) + E_{0,2}^\sharp(\frac{t}{w})$ on N_t . These functions are smooth, bounded and strictly positive on N_t for $|t| < (c^*)^4$. In the dividing case, we consider $E_{0,1}$ be just 0 on the other component, follow the construction in the non-dividing case. It should be noted that $E_{0,1}^\sharp, E_{0,2}^\sharp$ on N_t is independent of t . Furthermore, we should remark that in the construction of [5], $E_{0,1}^\sharp, E_{0,2}^\sharp$ are modified except for the factor $(\log|z|)^2, (\log|w|)^2$ just on $\{e^{a_0}c^* < |z| < c^*\} \simeq \{\frac{|t|}{c^*} < |w| < \frac{|t|}{e^{a_0}c^*}\}$ and $\{\frac{|t|}{c^*} < |z| < \frac{|t|}{e^{a_0}c^*}\} \simeq \{e^{a_0}c^* < |w| < c^*\}$ so that the modified function be smooth, thus in our case, $E_{0,1}^\sharp, E_{0,2}^\sharp$ is exactly $E_{0,1}, E_{0,2}$ on X_0 except for the factor $(\log|z|)^2, (\log|w|)^2$ respectively.

Remark 3.3. *As mentioned before, ρ_t^* is strictly smaller than the complete hyperbolic metric of Ω_t^1 . Thus, the claim of Theorem 3.2 does not contradict the implication of the classical Schwarz lemma.*

4 A new estimate for the Takhtajan-Zograf metric

We are ready to state a new estimate of the intrinsic Eisenstein series which is an improvement of Proposition 4.2.2 in [3]. Detailed proofs will appear in [4]. Here we quote a lemma (Lemma 1[5]).

Lemma 4.1. *There exist a positive constant C^* such that*

$$E_0 \leq C^* E_0^* \quad \text{on } \Omega_0^1.$$

We are now in a position to generalize Lemma 4.1 for any t .

Proposition 4.2. *Assume that in the family $\{X_t\}$, N_0 has the intersection with the component attached to the cusp where the Eisenstein series E_0 has a singularity. Then there exists a positive constant C, C' independent of t such that*

$$E_t \leq C E_t^* \quad \text{on } N_t \quad \text{for } |t| \text{ sufficiently small,} \quad (4.1)$$

$$E_t \leq C' E_0^* \quad \text{on } N_t \quad \text{for } |t| \text{ sufficiently small.} \quad (4.2)$$

Applying Proposition 4.2, we can improve (i) of Theorem 1 in [3].

Theorem 4.3. *For the simplicity of description, we assume that the degenerating family of a punctured hyperbolic surface X_t has only one pinching curve. Then there exists a positive constant C such that the Takhtajan-Zograf inner product has the estimate*

$$g^{TZ}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \leq \frac{C}{|t|^2(\log |t|)^4} \quad \text{for } t \rightarrow 0.$$

That is, we have removed , in (i) of Theorem 1 in [3].

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