

Hyperbolic automorphisms of a Lie algebra

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Let $\mathfrak{g}$ be a Lie algebra (of finite dimension) over $\mathbb{R}$. An automorphism σ of $\mathfrak{g}$ is said to be hyperbolic if no eigenvalue of σ is of absolute value one. Here we shall give a simple proof of the following theorem, which was stated in [2] in a slightly stronger form:

THEOREM. If $\mathfrak{g}$ has a hyperbolic automorphism, then $\mathfrak{g}$ is nilpotent.

Let σ be an automorphism of $\mathfrak{g}$, and let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$. Then σ can be extended naturally to an automorphism of $\tilde{\mathfrak{g}}$, and if $\tilde{\mathfrak{g}}$ is nilpotent then so is $\mathfrak{g}$. Therefore in order to prove the theorem it suffices to consider a complex Lie algebra $\mathfrak{g}$. After

this, by a Lie algebra we mean one over $\mathbb{C}$.

Known facts

(1) (S. Lie) Any representation of a solvable Lie algebra can be triangularized simultaneously.

(2) Let $\mathfrak{g}$ be a Lie algebra, and σ an automorphism of $\mathfrak{g}$. For $\alpha \in \mathbb{C}$, if α is an eigenvalue of σ we let $\mathfrak{g}(\alpha)$ denote the eigenspace of σ with eigenvalue α , and we put $\mathfrak{g}(\alpha) = \{0\}$ otherwise. Then

$$[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] \subset \mathfrak{g}(\alpha\beta).$$

(3) (M. Goto [1]) If a Lie algebra $\mathfrak{g}$ contains two nilpotent subalgebras $\mathfrak{N}_1$ and $\mathfrak{N}_2$ with $\mathfrak{N}_1 + \mathfrak{N}_2 = \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

Proof of THEOREM

Let σ be a hyperbolic automorphism of $\mathfrak{g}$. Let $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l\}$ be the totality of eigenvalues of σ , where

$$|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_k| > 1 > |\beta_h| \geq \dots \geq |\beta_1|.$$

We put

$$\mathfrak{N}_1 = \mathfrak{g}(\alpha_1) + \mathfrak{g}(\alpha_2) + \dots + \mathfrak{g}(\alpha_k),$$

$$\mathfrak{N}_2 = \mathfrak{g}(\beta_1) + \mathfrak{g}(\beta_2) + \dots + \mathfrak{g}(\beta_h),$$

$$\mathfrak{g} = \mathfrak{N}_1 + \mathfrak{N}_2.$$

By (2), $[\mathfrak{g}(\alpha_i), \mathfrak{g}(\alpha_j)] \subset \mathfrak{g}(\alpha_i \alpha_j) = \{0\}$ or $\mathfrak{g}(\alpha_l)$ with $l < \min(i, j)$. Hence $\mathfrak{N}_1$ is a nilpotent subalgebra, and so is $\mathfrak{N}_2$. Therefore $\mathfrak{g}$ is solvable by (3).

We adopt the notation $(\text{ad } X)Y = [X, Y]$ for $X, Y \in \mathfrak{g}$. For $X \in \mathfrak{g}(\gamma)$ and $Y \in \mathfrak{g}(\delta)$ where γ and δ are either α_i or β_j , we have

$$(\text{ad } X)^s Y \in \mathfrak{g}(\gamma^s \delta) \quad s = 1, 2, \dots$$

Since γ is not a root of unity,

$$\gamma\delta, \gamma^2\delta, \dots, \gamma^s\delta, \dots$$

are all distinct to each other, and we have that $\mathfrak{g}(\gamma^s\delta) = \{0\}$ for a sufficiently large s . This implies that $\text{ad } X$ is nilpotent.

By a suitable choice of basis of $\mathfrak{g}$ $\text{ad } X$ ($X \in \mathfrak{g}$) is represented by a matrix of the form

$$\begin{pmatrix} \lambda_1(X) & & & \\ & \lambda_2(X) & & \times \\ & & \ddots & \\ 0 & & & \lambda_n(X) \end{pmatrix}$$

simultaneously, by (1), where $n = \dim \mathfrak{g}$. But for $X \in \mathfrak{g}(X)$, $\text{ad } X$ is nilpotent and we have $\lambda_1(X) = \dots = \lambda_n(X) = 0$. Hence $\lambda_1 = \dots = \lambda_n = 0$ identically. Therefore $(\text{ad } \mathfrak{g})^n = \{0\}$, that is $\mathfrak{g}$ is nilpotent. Q. E. D.

References

- [1] M. Goto, Note on a characterization of solvable Lie algebras, J. Sci. Hiroshima Univ. 26 (1962), pp. 1-2.
- [2] S. Smale, Differentiable dynamical systems, Bull. A. M. S. 73 (1967), pp. 747-817.

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