

THE ALPERIN AND DADE CONJECTURES FOR SOME FINITE GROUPS

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1. Alperin's Weight Conjecture

Let G be a finite group, p a prime, and $O_p(G)$ the largest normal p -subgroup of G . In addition, let B be a p -block, R a p -subgroup of G , φ an irreducible ordinary character of the factor group $N(R)/R$. Then a pair (R, φ) is called a *B-weight* (character version) if the p -defect of φ is 0 and if the block $B(\varphi)$ of the normalizer $N(R)$ containing φ induces the block B (in the sense of Brauer), where φ is also viewed as a character of $N(R)$ and the p -defect of φ is the largest integer a such that p^a divides $\frac{|G|}{\varphi(1)}$. A weight is always identified with its conjugates in G .

Alperin's Weight Conjecture (1987): *The number of B-weights equals the number of irreducible Brauer characters in the block B.*

In 1989, Knörr and Robinson translated the conjecture into one involving only ordinary irreducible characters.

A p -subgroup chain

$$C : Q_0 < Q_1 < \dots < Q_n$$

of length $|C| = n$ is called a *normal p-chain* if each subgroup Q_i is a proper normal subgroup of Q_n for $1 \leq i \leq n-1$. Let $\mathcal{N}$ be the set of all normal p -chains. Then G acts on $\mathcal{N}$ by conjugation, and the stabilizer

$$N(C) = \cap_{i=1}^n N(Q_i)$$

of the chain C in G is called the normalizer of C . Denote by $k(N(C), B)$ the number of irreducible characters ψ of $N(C)$ such that the block $B(\psi)$ of $N(C)$ containing ψ induces the given block B . Alperin's weight conjecture is equivalent to the following.

The Knörr-Robinson Form (1989): *Whenever G is a finite group and B is a p -block, we have*

$$\sum_C (-1)^{|C|} k(N(C), B) = 0,$$

where C runs over a set $\mathcal{N}/G$ of representatives for G -orbits in $\mathcal{N}$.

2. Dade's Ordinary Conjecture

A p -subgroup R of G is called a *radical* subgroup if R is the largest normal p -subgroup of its normalizer $N(R)$, that is, $R = O_p(N(R))$.

A p -subgroup chain

$$C : P_0 < P_1 < \cdots < P_u$$

is called a *radical p -chain* if it satisfies the following two conditions

- (a) $P_0 = O_p(G)$.
- (b) P_k is a radical subgroup of the subgroup $\cap_{\ell=0}^{k-1} N(P_\ell)$

for each $1 \leq k \leq u$. Let $\mathcal{R} = \mathcal{R}(G)$ be the set of all radical p -chains of G .

Given a non-negative integer d , a p -block B of G and a radical p -chain C , let $k(N(C), B, d)$ be the number of irreducible characters ψ of the normalizer $N(C)$ such that

$B(\psi)$ induces B and the defect of ψ is d .

Dade's Ordinary Conjecture (1990): *If $O_p(G) = 1$ and B is a block with non-trivial defect groups, then for any integer d ,*

$$\sum_C (-1)^{|C|} k(N(C), B, d) = 0 \tag{2.1}$$

where C runs over a set $\mathcal{R}/G$ of representatives for the G -orbits in $\mathcal{R}$.

It was shown by Dade [D1] that

$$\sum_{C \in \mathcal{N}/G} (-1)^{|C|} k(N(C), B, d) = \sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N(C), B, d).$$

Thus Dade's ordinary conjecture implies the Knörr-Robinson form of Alperin's weight conjecture. It is also mentioned in Dade's paper [D1] that the ordinary

conjecture is equivalent to the final conjecture if the group G has both trivial Schur multiplier $\text{Mult}(G)$ and trivial outer automorphism group $\text{Out}(G)$. These conditions are satisfied by the following 11 sporadic simple groups:

$$J_1, J_4, M_{11}, M_{23}, M_{24}, Ly, \\ Co_2, Co_3, Fi_{23}, Th, M.$$

3. Dade's Invariant Conjecture

Suppose the center $Z(G)$ of G is trivial. Then we can identify G with its inner automorphism group $\text{Inn}(G)$. So the automorphism group $A = \text{Aut}(G)$ acts naturally on each p -chain C , and moreover, the stabilizer $N_A(C)$ of C in A acts on each irreducible character ψ of $N_G(C)$. So $N_G(C)$ is a normal subgroup of the stabilizer $N_A(C, \psi)$ of ψ in $N_A(C)$. The factor group $N_A(C, \psi)/N_G(C)$ is isomorphic to the subgroup of an outer automorphism group $O = \text{Out}(G)$ of G .

Given a radical p -chain C , a p -block $B \in \text{Blk}(G)$, a non-negative integer d , and a subgroup U of $O = \text{Out}(G)$, let $k(N(C), B, d, U)$ be the number of irreducible characters ψ of $N_G(C)$ such that the block of $N(C)$ containing ψ induces the block B , the defect of ψ is d , and $N_A(C, \psi)/N_G(C) = U$. The Dade invariant conjecture is stated as follows:

Dade's Invariant Conjecture [D3]: *If $Z(G) = O_p(G) = 1$ and B is a p -block of G with non-trivial defect group, then for any integer $d \geq 0$ and any subgroup $U \leq \text{Out}(G)$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N(C), B, d, U) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G -orbits in $\mathcal{R}$.

Dade's invariant conjecture is equivalent to his final conjecture whenever G has trivial Schur multiplier $\text{Mult}(G)$ and an outer automorphism group all of whose Sylow subgroups are cyclic. A lot of finite simple groups satisfy these conditions, for example,

$$He, HN, R_1(q), R_2(q), {}^3D_4(q), \\ G_2(q) \text{ (with } q \neq 3, 4), F_4(q) \text{ (with } q \neq 4), E_8(q).$$

4. Current Works

1. Alperin's weight conjecture has been verified for the following groups and blocks:

Blocks:

- (a) Cyclic and tame blocks (by Dade, Uno).
- (b) Abelian defect blocks with small inertial index (by Puig and Usami).
- (c) Abelian defect principal 2-blocks (by Fong and Harris).
- (d) Abelian defect unipotent blocks of a finite reductive group (by Broué, Malle and Michel).

Groups:

- (a) p -solvable groups (by Okuyama, Isaacs, Navarro, Gres, Barker).
- (b) Groups of Lie type in the defining characteristic (by Alperin, Cabanes, and reproved by Lehrer and Thévenaz).
- (c) S_n (by Alperin and Fong).
- (d) Classical groups in non-defining characteristics (by Alperin, Fong, Conder and An). In this case, the numbers of irreducible Brauer characters for blocks of symplectic and even-dimensional orthogonal groups are unknown (when $p \neq 2$).
- (e) $Sz(2^{2n+1})$, ${}^2G_2(q^2)$, ${}^2F_4(q^2)$, $G_2(q)$, ${}^3D_4(q)$ (by Dade, An).
- (f) M_{11} , M_{12} , M_{22} , M_{23} , M_{24} , He , Co_3 , J_1 (by Dade, Conder, An).
- (g) The covering groups of S_n and A_n ($p \neq 2$) (by Michler and Olsson).
- (h) Wrath product groups $G \wr S_n$ provided the conjecture holds for that finite group G (by Ewert)

2. Dade's final conjecture has been verified for the following cases:

- (a) 10 sporadic simple groups:
 M_{11} , M_{12} , M_{22} , M_{23} , M_{24} , He , J_1 , J_2 , J_3 , Co_3 (by Dade, Huang, Kotlica, Schwartz, Conder, An).
- (b) $L_2(q)$, $L_3(q)$ ($p|q$), $Sz(q)$, ${}^2G_2(3^{2n+1})$ ($p \neq 3$), $G_2(q)$ ($p \neq 3$ and $p \nmid q \neq 4$), the Tits group (by Dade, An).
- (c) Cyclic blocks (by Dade).

3. The invariant conjecture has been verified for all tame blocks, and for the group McL ($p \neq 2$) (by Uno, Murray).

4. The ordinary conjecture has been verified for the following cases:

- (a) $\text{GL}_n(q)$ ($p|q$), ${}^2F_4(2^{2n+1})$ ($p \neq 2$), $G_2(q)$ ($p \nmid q$) (by Olsson, Uno, An).
- (b) S_n (by Olsson and Uno when p odd, An when $p = 2$).
- (c) Ru (by Dade).
- (d) Unipotent abelian defect blocks (by Broué, Malle and Michel).
- (e) Abelian defect principal 2-blocks (by Fong and Harris).
- (f) All abelian defect blocks with small inertia index (by Usami).

References

- [A] J. L. Alperin, 'Weights for finite groups', in "The Arcata Conference on Representations of Finite Groups" Proc. of Symposia in Pure Math. 47 (1987) 369-379.
- [AF] J. L. Alperin and P. Fong, Weights for symmetric and general linear groups, *J. Algebra* **131** (1990), 2-22.
- [A1] Jianbei An, 2-weights for general linear groups, *J. Algebra* **149** (1992), 500-527.
- [A2] Jianbei An, 2-weights for unitary groups, *Trans. Amer. Math. Soc.* **339** (1993), 251-278.
- [A3] Jianbei An, 2-weights for classical groups, *J. reine angew. Math.* **439** (1993), 159-204.
- [A4] Jianbei An, Weights for the simple Ree groups ${}^2G_2(q^2)$, *New Zealand J. of Math.* **22** (1993), 1-8.
- [A5] Jianbei An, Weights for classical groups, *Trans. Amer. Math. Soc.* **342**(1994), 1-42.
- [A6] Jianbei An, Weights for the Chevalley groups $G_2(q)$, *Proc. of London Math. Soc.* **69** (1994), 22-46.
- [A7] Jianbei An, Weights for the Steinberg triality group ${}^3D_4(q)$, *Math. Z.* **218** (1995), 273-290.

- [A8] Jianbei An, Dade's conjecture for Chevalley groups $G_2(q)$ in non-defining characteristics, *Canad. J. Math.* **48** (1996), 673-691.
- [A9] Jianbei An, Dade's conjecture for the Tits Group, *New Zealand J. Math.* **25** (1996), 107-131.
- [A10] Jianbei An, Dade's conjecture for the simple Ree groups ${}^2G_2(q^2)$ in non-defining characteristics, *Indian J. Math.* **36** (1994), 7-27.
- [A11] Jianbei An, The Alperin and Dade conjectures for the simple Conway's third group, submitted.
- [A12] Jianbei An, Weights for the Ree groups ${}^2F_4(q^2)$, submitted.
- [A13] Jianbei An, The Alperin and Dade conjectures for the simple Held group, *J. Algebra* (to appear).
- [A14] Jianbei An, Dade's conjecture for 2-blocks of symmetric groups, submitted.
- [AC1] Jianbei An and Marston Conder, On the numbers of 2-weights, unipotent conjugacy classes, and irreducible Brauer 2-characters of finite classical groups, *Proc. Amer. Math. Soc.*, **123** (1995), 2297-2304.
- [AC2] Jianbei An and Marston Conder, The Alperin and Dade conjectures for simple Mathieu groups, *Comm. Algebra* **23** (1995), 2797-2823.
- [BMM] M. Broué, G. Malle, J. Michel, Generic blocks of finite reductive groups, *Astérisque*, **212** (1993) 1-92.
- [D1] E. C. Dade, Counting characters in blocks, I, *Invent. math.* **109** (1992) 187-210.
- [D2] E. C. Dade, Counting characters in blocks, II, *J. reine angew. Math.* **448** (1994) 97-190.
- [D3] E. C. Dade, Counting characters in blocks, 2.9, preprint.
- [D4] E. C. Dade, Counting characters in blocks with cyclic defect groups, I, preprint.
- [E1] H. Ellers, The defect groups of a clique, p -solvable groups, and Alperin's conjecture, *J. reine angew. Math.* **468** (1995), 1-48.
- [E2] H. Ellers, Cliques of irreducible representations of p -solvable groups and a theorem analogous to Alperin's conjecture, *Math. Z.* **217** (1994), 607-634.

- [Ew] R. Ewert, Die Alperin-Vermutung für Kranzprodukte der form $GwrS_n$, *J. Algebra* **162** (1993), 225-258.
- [FH] P. Fong and M. Harris, On perfect isometries and isotypies in finite groups, *Invent. math.* **114** (1993) 139-191.
- [IN] I. M. Isaacs and G. Navarro, Weights and vertices for characters of π -separable groups, *J. Algebra* **177** (1995), 339-366.
- [KR] R. Knörr and G. R. Robinson, Some remarks on a conjecture of Alperin, *J. London Math. Soc.* **39**(1989), 48-60.
- [R] G. R. Robinson, Alperin's conjecture, numbers of characters, and Euler characteristics of quotients of p -group complexes, *J. London Math. Soc.* **52**(1995), 88-96.
- [RS] G. R. G. R. Robinson and R. Staszewski, More on Alperin's conjecture, *Astérisque* **181-182**(1990), 237-255.
- [LT] G. Lehrer et J. Thévenaz, Sur la conjecture d'Alperin pour les groupes réductifs finis, *C. R. Acad. Sci. Paris Sér I Math.* **315** (1992), 1347-1351.
- [MO] G. Michler and J. Olsson, Weights for covering groups of symmetric and alternating groups, $p \neq 2$, *Canad. J. Math.* **43** (1991) 792-813.
- [N] G. Navarro, Weights, vertices and a correspondence of characters in groups of odd order, *Math. Z.* **212** (1993), 536-546.
- [OU1] J. B. Olsson and K. Uno, Dade's conjecture for general linear groups in the defining characteristic, *Proc. London Math. Soc.* **72** (1996) 359-384.
- [OU2] J. B. Olsson and K. Uno, Dade's conjecture for symmetric groups, *J. Algebra* **176** (1995) 534-560.
- [PU1] L. Puig and Y. Usami, Perfect isometries for blocks with Abelian defect groups and Klein four inertial quotients, *J. Algebra* **160** (1993), 192-225.
- [PU2] L. Puig and Y. Usami, Perfect isometries for blocks with Abelian defect groups and cyclic inertial quotients of order 4, *J. Algebra* **172** (1995), 205-213.
- [T] Y. Tsushima, Notes on trivial source modules, *Osaka J. Math.* **32**(1995), 475-482.

- [T1] J. Thévenaz, Locally determined functions and Alperin's conjecture *J. London Math. Soc* **45** (1992), 446-468.
- [T2] J. Thévenaz, Equivariant K -theory and Alperin's conjecture *J. Pure and Appl. Algebra* **85** (1993), 185-202.
- [U] K. Uno, Dade's conjecture for tame blocks, *Osaka J. Math.* **31** (1994) 747-772.
- [U1] Y. Usami, Perfect isometries for blocks with Abelian defect groups and dihedral inertial quotients of order 6, *J. Algebra* **782** (1995), 113-125.
- [U2] Y. Usami, Perfect isometries for blocks with Abelian defect groups and dihedral inertial quotients isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$, *J. Algebra* **181** (1996), 727-759.
- [U3] Y. Usami, Perfect isometries for blocks with Abelian defect groups and dihedral inertial quotients isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, *J. Algebra* **182** (1996), 140-164.