

Density matrices and LR transforms (Genesis of Orthogonal Functions)

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1 Introduction

The two kinds of Chebyshev polynomials $T_n(\cos \theta) = \cos n\theta$ and $U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$ are linearly related to each other by the formulae

$$\cos n\theta = \frac{1}{2} \left\{ \frac{\sin(n+1)\theta}{\sin \theta} - \frac{\sin(n-1)\theta}{\sin \theta} \right\}$$

Both polynomials satisfy the difference equations

$$xu_n = \frac{1}{2}(u_{n+1} + u_{n-1})$$

This is a simplest case of LR -transforms associated with difference operators for orthogonal functions.

A system of orthogonal functions are intimately related with eigenfunctions for a self-adjoint operator through density matrices. Once a family of self-adjoint operators are given, we can discuss the interplay among LR -transforms of self-adjoint operators, linear transforms of density matrices and connection relations between two system of eigen-functions for the operators. This mechanism enables us to give a new orthogonal system from the previous one, and so on.

Let A be an infinite real tri-diagonal matrix $(a_{n,m})_{n,m=-\infty}^{\infty}$ which defines a bounded self-adjoint operator on $l^2(\mathbf{Z})$. There exist the spectral kernels $d\Theta(n, m|\lambda)$ which are the Stieltjes measures on $\mathbf{R}$ such that

$$\delta_{n,m} = \int_{-\infty}^{\infty} d\Theta(n, m; \lambda) \quad (1.1)$$

$$a_{n,m} = \int_{-\infty}^{\infty} \lambda d\Theta(n, m; \lambda) \quad (1.2)$$

The eigenfunction expansion for A is an expression of $d\Theta(n, m; \lambda)$, by using generalized eigenfunctions $\psi^{(\epsilon)}(n; \lambda)$ ($\epsilon = \pm$) of A satisfying

$$A\psi^{(\epsilon)}(n; \lambda) = \lambda\psi^{(\epsilon)}(n; \lambda) \quad (1.3)$$

and Stieltjes measures called density matrices $d\rho_{\epsilon, \epsilon'}(\lambda)$, as

$$d\Theta(n, m; \lambda) = \sum_{\epsilon=\pm, \epsilon'=\pm} \psi^{(\epsilon)}(n; \lambda) \overline{\psi^{(\epsilon')}(m; \lambda)} d\rho_{\epsilon, \epsilon'}(\lambda) \quad (1.4)$$

Let $f(\lambda)$ be a positive continuous function such that $f(A)$ defines a positive definite operator on $l^2(\mathbf{Z})$.

Assume that there exists a Gauss decomposition of $f(A)$ of the following type

$$f(A) = B_- \cdot B_+ \quad (1.5)$$

where B_+ (or $B_- = {}^t B_+$ the transpose of B_+) denotes an upper triangular (or lower triangular) matrix such that the inverses $B_{\pm}^{-1}$ are also well-defined.

Then the LR transform of A can be defined as follows.

$$A \rightarrow A' = B_-^{-1} \cdot A \cdot B_- = B_+ \cdot A \cdot B_+^{-1} \quad (1.6)$$

In this note we show that this transform is equivalent to a certain linear or projective transform of the density matrices $d\rho_{\epsilon, \epsilon'}(\lambda)$ and evaluate it explicitly in the following four cases

- (1) Orthogonal polynomials in a single variable
- (2) Inverse scattering case
- (3) Periodic case
- (4) Orthogonal polynomials in multi-variables respectively.

This note has been written in collaboration with Dr. Masahiko Ito. Especially the computations for proving Proposition 8 are mostly due to him.

2 Orthogonal polynomials in a single variable

We consider a Stieltjes measure $d\rho(\lambda)$ with infinite increments and whose support is contained in the finite interval $[a, b]$ ($a < b$) in $\mathbf{R}$. There exist the unique orthonormal polynomials in λ

$$p_0(\lambda), p_1(\lambda), p_2(\lambda), \dots$$

(we put $p_{-1}(\lambda) = 0$) such that they satisfy

$$p_n(\lambda) = k_n \lambda^n + (\text{lower degree terms}) \quad k_n > 0 \quad (2.1)$$

$$\int_a^b p_n(\lambda) p_m(\lambda) d\rho(\lambda) = \delta_{n,m} \quad (2.2)$$

The three term recurrence equations hold

$$\lambda p_n(\lambda) = b_{n-1} p_{n-1}(\lambda) + a_n p_n(\lambda) + b_n p_{n+1}(\lambda) \quad (n \geq 0) \quad (2.3)$$

Let A denote the corresponding tri-diagonal matrix $(a_{n,m})_{n,m=-\infty}^{\infty}$ such that

$$a_{n,n} = a_n, a_{n,n+1} = a_{n+1,n} = b_n \quad n \geq 0 \quad (2.4)$$

The matrix A defines a self-adjoint operator on $l^2(\mathbf{Z}_{\geq 0})$. A has the spectral decomposition (1.1), (1.2) where $d\Theta(n, m; \lambda)$ is represented simply by

$$d\Theta(n, m; \lambda) = p_n(\lambda) p_m(\lambda) d\rho(\lambda) \quad (2.5)$$

Let $f(x)$ be a positive continuous function on $[a, b]$. Then $f(A)$ and $f(A)^{-1}$ define bounded self-adjoint operators. There exist the unique upper triangular and lower triangular matrices B_+ and B_- with positive diagonal elements satisfying (1.5). All $B_{\pm}$ and $B_{\pm}^{-1}$ are bounded operators.

The LR transform of A associated with the function $f(\lambda)$ is defined by the correspondence (1.6). A' is again a tri-diagonal self-adjoint operator on

Y.Nakamura and Y.Kodama, and also V.Spiridonov and A.Zhedanov have investigated LR -transforms associated with finite matrices and orthogonal polynomials (see [23],[24],[30]). Here we want to relate them to linear (or projective) transforms of density matrices $d\rho(\lambda)$.

In section 7-8 we extend LR -transforms to the case of orthogonal polynomials in multi-variables. In the final section we shall obtain explicit formulae for LR -transforms associated with Koornwinder polynomials.

Proposition 1 *Let $d\rho'(\lambda)$ be the density corresponding to the operator A' . The LR transform (1.6) is equivalent to the linear correspondence*

$$d\rho'(\lambda) = f(\lambda)d\rho(\lambda) \quad (2.6)$$

If $d\rho(\lambda)$ and $d\rho'(\lambda)$ are normalized such that

$$\int_a^b d\rho(\lambda) = \int_a^b d\rho'(\lambda) = 1 \quad (2.7)$$

then (2.6) should be modified as

$$d\rho(\lambda) \rightarrow d\rho'(\lambda) = \frac{f(\lambda)d\rho(\lambda)}{\int_a^b f(\lambda)d\rho(\lambda)} \quad (2.8)$$

In fact, (2.6) implies the formulae

$$(f(A))_{n,m} = \int_{-\infty}^{\infty} p_n(\lambda)p_m(\lambda)d\rho'(\lambda) \quad (2.9)$$

Let $\{p'_n(\lambda)\}$ be the orthonormal polynomials with respect to the density $d\rho'(\lambda)$. $p_n(\lambda)$ can be expressed uniquely as a linear combination of $p'_m(\lambda)$

$$p_n(\lambda) = \sum_{m=0}^n b_{m,n}p'_m(\lambda) \quad (2.10)$$

Let B_+ be the upper triangular matrix $(b_{n,m})_{n,m=0}^{\infty}$. Then (1.5) holds from (2.9). On the other hand

$$(A')_{n,m} = \int_{-\infty}^{\infty} \lambda p'_n(\lambda) p'_m(\lambda) d\rho'(\lambda) \quad (2.11)$$

From (2.9)-(2.11), we deduce (1.6).

In particular, if A is itself positive definite and $f(\lambda) = \lambda$, (1.6) reduces to the original Rutishauser's LR algorithm.

Examples 1. Jacobi polynomials.

Let $d\rho(\lambda) = (1 - \lambda)^\alpha(1 + \lambda)^\beta d\lambda$ on $[-1, 1]$, for $\alpha, \beta > -1$. The Jacobi polynomials $P_n^{(\alpha, \beta)}(\lambda)$ are defined by the equations

$$(1 - \lambda)^\alpha(1 + \lambda)^\beta P_n^{(\alpha, \beta)}(\lambda) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{d\lambda} \right)^n \{ (1 - \lambda)^{\alpha+n}(1 + \lambda)^{\beta+n} \} \quad (2.12)$$

Then

$$P_n^{(\alpha, \beta)}(\lambda) = l_n^{(\alpha, \beta)} \lambda^n + \dots$$

$$l_n^{(\alpha, \beta)} = 2^{-n} \frac{\Gamma(2n + \alpha + \beta + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}$$

and

$$\int_{-1}^1 (1 - \lambda)^\alpha(1 + \lambda)^\beta P_n^{(\alpha, \beta)}(\lambda) P_m^{(\alpha, \beta)}(\lambda) d\lambda = 0, \quad n \neq m$$

$$\int_{-1}^1 (1 - \lambda)^\alpha(1 + \lambda)^\beta \{P_n^{(\alpha, \beta)}(\lambda)\}^2 d\lambda = h_n^{(\alpha, \beta)}$$

$$h_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}$$

The recurrence equations for $P_n^{\alpha, \beta}(x)$ are as follows.

$$\begin{aligned} & 2(n + 1)(n + 1 + \alpha + \beta)(2n + \alpha + \beta)P_{n+1}^{(\alpha, \beta)}(\lambda) \\ = & (2n + \alpha + \beta + 1)\{(2n + \alpha + \beta + 2)(2n + \alpha + \beta)\lambda + \alpha^2 - \beta^2\}P_n^{(\alpha, \beta)}(\lambda) \\ & - 2(n + \alpha + 1)(n + \beta + 1)(2n + \alpha + \beta + 2)P_{n-1}^{(\alpha, \beta)}(\lambda) \end{aligned} \quad (2.13)$$

$$p_n(\lambda) = \{h_n^{(\alpha, \beta)}\}^{-\frac{1}{2}} P_n^{(\alpha, \beta)}(\lambda)$$

then $p_n(\lambda)$ is the orthonormal polynomials with respect to $d\rho(\lambda)$. We denote by A the tri-diagonal operator on $l^2(\mathbf{Z})_{\geq 0}$ derived from (2.13).

The shift $\alpha \rightarrow \alpha + 1$ induces the transform of the densities

$$d\rho(\lambda) \rightarrow d\rho'(\lambda) = (1 - \lambda)d\rho(\lambda) \quad (2.14)$$

Since $1 - A$ is positive definite, the Gauss decomposition

$$1 - A = B_- \cdot B_+ \quad (2.15)$$

is uniquely determined. Likewise we have

$$1 + A = B_- \cdot B_+ \quad (2.16)$$

These are Christoffel-Darboux transforms of contiguity relation.

In fact, if we put

$$\psi_n(\alpha, \beta) = \frac{1}{l_n^{(\alpha, \beta)}} P_n^{(\alpha, \beta)}(\lambda) = \lambda^n + \frac{n(\alpha - \beta)}{2n + \alpha + \beta} \lambda^{n-1} + \dots$$

then

$$\begin{aligned} \psi_n(\alpha, \beta) &= \psi_n(\alpha + 1, \beta) + v_n \psi_{n-1}(\alpha + 1, \beta) \\ v_n &= -\frac{2n(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)} \end{aligned} \quad (2.17)$$

More exactly saying, $B_{\pm}^{-1}$ are not bounded on $l^2(\mathbf{Z})$, although $B_{\pm}$ are bounded. We must modify the operators $B_{\pm}$ as follows.

We denote by $\mathcal{H}$ the Hilbert space $l^2(\mathbf{Z}_{\geq 0})$ consisting of sequences $u = (u_n)_{n=0}^{\infty}$, $v = (v_n)_{n=0}^{\infty}$ etc with the inner product $(u, v) = \sum_{n=0}^{\infty} u_n \overline{v_n}$. We define another Hilbert space $\mathcal{H}_0$, the closed linear subspace spanned by $B_+ u'$ ($u' \in l^2(\mathbf{Z}_{\geq 0})$). $\mathcal{H}_0$ is isomorphic to the Hilbert space consisting of the sequences $u = (u_n)_{n \geq 0}$ such that $((1 - A)^{-1} u, u) < \infty$. B_+^{-1} is a bounded operator from

$\mathcal{H}_0$ to $\mathcal{H}$, so that $B_+AB_+^{-1}$ is bounded as a linear mapping from $\mathcal{H}_0$ to $\mathcal{H}$ which is extendable to a bounded operator on $\mathcal{H}$.

Example 2. Askey-Wilson polynomials (see [5]).

Let q be the real modulus such that $0 < q < 1$, and c_1, c_2, c_3, c_4 be real numbers. Askey-Wilson polynomials are defined by using the basic hypergeometric series of order m

$${}_m\varphi_{m-1}\left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_{m-1} \end{matrix}; \lambda\right) = \sum_{\nu=0}^{\infty} \frac{(a_1; q)_{\nu} \cdots (a_m; q)_{\nu}}{(b_1; q)_{\nu} \cdots (b_{m-1}; q)_{\nu} (q; q)_{\nu}} \lambda^{\nu} \quad (2.18)$$

as

$$\begin{aligned} p_n(\lambda; c_1, c_2, c_3, c_4) &= c_1^{-n} (c_1 c_2; q)_n \cdot (c_1 c_3; q)_n \cdot (c_1 c_4; q)_n \cdot {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n-1} c_1 c_2 c_3 c_4, c_1 e^{i\theta}, c_1 e^{-i\theta} \\ c_1 c_2, c_3 c_4, c_1 c_4 \end{matrix}; q\right) \\ &= l_n \lambda^n + \cdots \quad (l_n = 2^n (c_1 c_2 c_3 c_4 q^n; q)_n) \end{aligned} \quad (2.19)$$

where $\lambda = \cos\theta$. The weight function $w(\lambda)$ ($d\rho(\lambda) = \frac{w(\lambda)}{\sqrt{1-\lambda^2}} d\lambda$) is given by

$$w(\lambda) = \frac{\prod_{k=0}^{\infty} (1 - 2(2\lambda^2 - 1)q^k + q^{2k})}{h(\lambda, c_1)h(\lambda, c_2)h(\lambda, c_3)h(\lambda, c_4)} \quad (2.20)$$

where

$$h(\lambda, a) = \prod_{k=0}^{\infty} (1 - 2a\lambda q^k + q^{2k} a^2) = (ae^{i\theta}; q)_{\infty} (ae^{-i\theta}; q)_{\infty} \quad (2.21)$$

Then the orthogonality relations are

$$\frac{1}{2\pi} \int_{-1}^1 p_n(\lambda; c_1, c_2, c_3, c_4) p_m(\lambda; c_1, c_2, c_3, c_4) \frac{w(\lambda)}{\sqrt{1-\lambda^2}} d\lambda = \delta_{n,m} h_n \quad (2.22)$$

$$h_n = \frac{(c_1 c_2 c_3 c_4 q^{2n}; q)_\infty (c_1 c_2 c_3 c_4 q^{n-1}; q)_\infty (q^{n+1}; q)_\infty^{-1} (c_1 c_2 q^n; q)_\infty^{-1}}{(c_1 c_3 q^n; q)_\infty (c_1 c_4 q^n; q)_\infty (c_2 c_3 q^n; q)_\infty (c_2 c_4 q^n; q)_\infty (c_3 c_4 q^n; q)_\infty} \quad (2.23)$$

The three term recurrence relations for $p_n(\lambda; c_1, c_2, c_3, c_4)$ are expressed as

$$2\lambda p_n(\lambda) = b_{n-1} p_{n-1}(\lambda) + a_n p_n(\lambda) + b'_n p_{n+1}(\lambda) \quad (2.24)$$

$$\begin{aligned} b_{n-1} &= (1 - q^n)(1 - c_1 c_2 q^{n-1})(1 - c_1 c_3 q^{n-1})(1 - c_1 c_4 q^{n-1}) \\ &\quad \times \frac{(1 - c_2 c_3 q^{n-1})(1 - c_2 c_4 q^{n-1})(1 - c_3 c_4 q^{n-1})}{(1 - c q^{2n-2})(1 - c q^{2n-1})}, \\ b'_n &= \frac{1 - c q^{n-1}}{(1 - c q^{2n-1})(1 - c q^{2n})}, \\ a_n &= \frac{q^{n-1}[(1 + c q^{2n-1})(s q + s' c) - q^{n-1}(1 + q)(s + s' q)c]}{(1 - c q^{2n-2})(1 - c q^{2n})} \end{aligned}$$

($s = c_1 + c_2 + c_3 + c_4$, $s' = c_1^{-1} + c_2^{-1} + c_3^{-1} + c_4^{-1}$, $c = c_1 c_2 c_3 c_4$).

$d\rho(\lambda)$ depends on c_1, c_2, c_3, c_4 . In fact each shift

$$T_1 : c_1 \rightarrow c_1 q; \quad T_2 : c_2 \rightarrow c_2 q; \quad T_3 : c_3 \rightarrow c_3 q; \quad T_4 : c_4 \rightarrow c_4 q \quad (2.25)$$

multiplies $w(\lambda)$ by

$$1 + c_1^2 - 2c_1 \lambda, \quad 1 + c_2^2 - 2c_2 \lambda, \quad 1 + c_3^2 - 2c_3 \lambda, \quad 1 + c_4^2 - 2c_4 \lambda \quad (2.26)$$

times respectively.

The corresponding LR transforms of A are defined as the Gauss decompositions of each positive operator

$$1 + A^2 - 2c_1 A > 0, \quad 1 + A^2 - 2c_2 A > 0, \quad 1 + A^2 - 2c_3 A > 0, \quad 1 + A^2 - 2c_4 A > 0$$

Put

$$\psi_n(\lambda; c_1, c_2, c_3, c_4) = \frac{1}{l_n} p_n(\lambda; c_1, c_2, c_3, c_4) \quad (2.27)$$

then, as for T_1 for example, the transform B_+ is equivalent to the following contiguity relation

$$\psi_n(x; c_1, c_2, c_3, c_4) = \psi_n(x; c_1 q, c_2, c_3, c_4) + v_n \psi_{n-1}(x; c_1 q, c_2, c_3, c_4)$$

$$v_n = -\frac{2(1-q)c_1}{(1-aq^{2n-2})(1-aq^{2n-1})(1-c_2c_3q^{n-1})(1-c_2c_4q^{n-1})(1-c_3c_4q^{n-1})}$$

likewise for T_2, T_3, T_4 .

3 Inverse scattering, Application of H.Flaschka theory

A be a tri-diagonal matrix which defines a bounded self-adjoint operator on $\mathcal{H} = l^2(\mathbb{Z})$.

We put $a_{n,n} = a_n$ and $a_{n,n+1} = a_{n+1,n} = b_n$ for $-\infty < n < \infty$ and assume the following condition

$$(C) \quad \sum_{n=-\infty}^{\infty} |a_n| |n| < \infty, \quad \sum_{n=-\infty}^{\infty} |b_n - \frac{1}{2}| |n| < \infty \quad (3.1)$$

The inverse scattering theory for the difference operator A was developed by H.Flaschka (see [10],[32]). We put the spectral parameter $z = \frac{1}{2}(\zeta + \zeta^{-1})$.

If $|\zeta| \leq 1$, then the Jost solutions $\psi^\pm(n; z)$ (minimal solutions in the sense of S.Elaydi [9]) are uniquely determined as the eigenfunctions (1.3) with the asymptotic behaviours

$$\psi^\pm(n; z) \asymp \zeta^{\pm n} \quad n \rightarrow \pm\infty \quad (3.2)$$

The connection relations between $\psi^\pm(n; z)$ are

$$\psi^-(n; z) = \alpha(z) \tilde{\psi}^\pm(n; z) + \beta(z) \psi^\pm(n; z) \quad (3.3)$$

where $\tilde{\psi}^\pm(n; z)$ are defined to be the conjugates of $\psi^\pm(n; z)$ when ζ ($|\zeta| = 1$) is replaced by ζ^{-1} . $\alpha(z)$, $\beta(z)$ can be holomorphically extended to the domain $|\zeta| \leq 1$.

The Wronskian and the reflection coefficients are expressed as

$$R(z) = \frac{\beta(z)}{\alpha(z)} \quad (3.4)$$

$$W(\psi_+, \psi_-) = \frac{1}{2}(\zeta^{-1} - \zeta)\alpha(z) \quad (3.5)$$

respectively.

For $\lambda \in [-1, 1]$, $\psi^\pm(n; \lambda + i0)$, $\alpha(\lambda + i0)$, $\beta(\lambda + i0)$ do exist. Moreover, $\alpha(z)$ has a finite number of simple poles λ_k , $k = 1, 2, 3, \dots, s$ such that $|\lambda_k| > 1$.

Under this circumstance, it holds the following two expansion formulae which are equivalent to each other.

Proposition 2 (1)

$$\begin{aligned} d\Theta(n, m; \lambda) &= \frac{\chi_{[-1,1]}(\lambda)d\lambda}{2\pi\sqrt{1-\lambda^2}|\alpha(\lambda+i0)|^2} \{ \psi^+(n; \lambda+i0)\overline{\psi^+(m; \lambda+i0)} \\ &+ \psi^-(n; \lambda+i0)\overline{\psi^-(m; \lambda+i0)} \} \\ &+ \sum_{k=1}^s \psi^+(n; \lambda_k)\psi^+(m; \lambda_k)c_k^2\delta(\lambda - \lambda_k)d\lambda \end{aligned} \quad (3.6)$$

where $c_k^2 = \frac{\beta(\lambda_k)}{\alpha'(\lambda_k)\sqrt{\lambda_k^2-1}}$ and $\chi_{[-1,1]}(\lambda)$ denotes the indicator function of $[-1, 1]$.
(2)

$$\begin{aligned} d\Theta(n, m; \lambda) &= \frac{\chi_{[-1,1]}(\lambda)d\lambda}{2\pi\sqrt{1-\lambda^2}} \{ \psi^+(n; \lambda+i0)\overline{\psi^+(m; \lambda+i0)} \\ &+ \psi^+(n; \lambda-i0)\overline{\psi^+(m; \lambda-i0)} \\ &+ R(\lambda+i0)\psi^+(n; \lambda+i0)\overline{\psi^+(m; \lambda-i0)} \\ &+ R(\lambda-i0)\psi^+(n; \lambda-i0)\overline{\psi^+(m; \lambda+i0)} \} \\ &+ \sum_{k=1}^s \psi^+(n; \lambda_k)\psi^+(m; \lambda_k)c_k^2\delta(\lambda - \lambda_k)d\lambda \end{aligned} \quad (3.7)$$

For the proof see [3],[6].

We can rewrite (3.6),(3.7) by using the Fourier expnasions of $\psi^\pm(n; z)$

$$\psi^+(n; z) = \sum_{m \geq n} K(n, m) \zeta^m \quad K(n, n) > 0 \quad (3.8)$$

$$F(m) = F_c(m) + F_p(m), \quad (3.9)$$

$$F_c(m) = \frac{1}{2\pi i} \int_{|\zeta|=1} R(z) \zeta^{m-1} d\zeta \quad (3.10)$$

$$F_p(m) = \sum_{k=1}^s c_k^2 \zeta_k^m \quad (3.11)$$

We denote by $\hat{F}$, $\hat{K}$ the operators defined by the kernel functions $\{F(n+m)\}_{n,m=-\infty}^\infty$ and $\{K(n, m)\}_{n,m=-\infty}^\infty$. $\hat{F}$ is of Fredholm type and of Hankel type. $\hat{K}$ has a bounded inverse.

Then (3.7) imply the following Gelfand-Levitan-Marchenko decomposition (abbreviated by GLM decomposition)

Proposition 3 (1.1), (1.2) can be expressed in operator form as

$$1 = \hat{K}(1 + \hat{F})^t \hat{K} \quad (3.12)$$

$$A = \hat{K} A_0 (1 + \hat{F})^t \hat{K} = \hat{K} A_0 \hat{K}^{-1} \quad (3.13)$$

where ${}^t \hat{K}$ denotes the transpose of $\hat{K}$. We denote by A_0 the symmetric tri-diagonal matrix such that $b_n = \frac{1}{2}$, $a_n = 0$.

$1 + \hat{F}$ is positive definite so that $\hat{K}$ is uniquely determined by (3.12).

A_0 has the unique decomposition

$$A_0 = A_{0,+} + A_{0,-} \quad (3.14)$$

where $A_{0,+}$ and $A_{0,-}$ are upper triangular and lower triangular matrices respectively. $2A_{0,\pm}$ are unitary operators which shift the indices by ± 1 respec-

Now let us discuss how the LR transform of A can be expressed in terms of $\hat{F}$.

Since A, A_0 are bounded, there exists a positive number c such that all 4 operators $A(c) = A + c, A_0(c) = A_0 + c$ and $A(c)^{-1}, A_0(c)^{-1} > 0$ are positive definite.

We want to find the upper triangular bi-diagonal matrix $A_+(c)$, with (n, n) th entries ξ_n and $(n, n + 1)$ th entries η_n such that $\xi_n > 0$, and its transpose $A_-(c) = {}^t A_+(c)$, such that the following Gauss decomposition holds.

$$A(c) = A_-(c) \cdot A_+(c) \quad (3.15)$$

i.e.,

$$\xi_n^2 + \eta_{n-1}^2 = a_n + c, \quad \xi_n \eta_n = b_n \quad (3.16)$$

The equations (3.16) have the unique solution such that ξ_0^2 equals the convergent continued fraction

$$\xi_0^2 = \frac{b_0^2}{a_1 + c} - \frac{b_1^2}{a_2 + c} - \dots = -b_0 \frac{\psi^+(1; -c)}{\psi^+(0; -c)} \quad (3.17)$$

because, if $z \notin \sigma(A)$, we have

$$b_0 \frac{\psi(1; z)}{\psi(0; z)} = \frac{b_0^2}{z - a_1} - \frac{b_1^2}{z - a_2} - \dots \quad (3.18)$$

We shall call the Gauss decomposition (3.15) thus obtained canonical. The LR -transform is then defined as

$$A \rightarrow A' = A_+(c) \cdot A_-(c) = A_+(c) \cdot A \cdot A_-(c)^{-1} \quad (3.19)$$

A' is also tri-diagonal.

We can now state

Theorem 1 *Let the GLM decompositon of A' be*

$$1 = K' \cdot (1 + \hat{F}') \cdot {}^t\hat{K}' \quad (3.20)$$

$$A' = K' \cdot A_0 \cdot (1 + \hat{F}') \cdot {}^t\hat{K}' \quad (3.21)$$

then A' is the LR-transform of A if and only if

$$\hat{F}' = \hat{F} \cdot A_{0,-}(c) \cdot A_{0,+}(c)^{-1} = A_{0,+}(c) \cdot \hat{F} \cdot A_{0,+}(c)^{-1} \quad (3.22)$$

(Remark that $\hat{F} \cdot A_{0,\pm}(c) = A_{0,\mp}(c) \cdot \hat{F}$.)

If we put

$$g(\zeta) = \frac{\sqrt{c+1} - \sqrt{c-1}}{2} \zeta + \frac{\sqrt{c+1} + \sqrt{c-1}}{2}$$

i.e.,

$$z + c = g(\zeta)g(\zeta^{-1})$$

then (3.22) can be restated as

$$R'(z) = R(z)g(\zeta)^{-1}g(\zeta^{-1}) \quad (3.23)$$

which is nothing else than dressing transformation in the sense of Zakhalov-Shabat. (This fact has been pointed out to the author by S. Kakei.)

Proof 1 *First we show that (3.22) implies (3.19). From (3.20), (3.22) and because of the uniqueness of Gauss decomposition, we have*

$$\hat{K}' = A_+(c) \cdot \hat{K} \cdot g(2A_{0,+}) \quad (3.24)$$

Hence, from (3.21)

$$\begin{aligned} A' &= \hat{K}' \cdot A_0 \cdot (1 + \hat{F}') \cdot {}^tK' = A_+(c) \hat{K} g(2A_{0,+}) A_0 g(2A_{0,-})^{-1} \hat{K}^{-1} A_+(c)^{-1} \\ &= A_+(c) \hat{K} A_0 \hat{K}^{-1} A_+(c)^{-1} = A_+(c) \cdot A \cdot A_+(c)^{-1} \end{aligned}$$

(3.19) has thus been obtained.

Next we show the converse. We remark first that any bounded upper triangular operator which commutes $A_{0,+}$ is a holomorphic function of $2A_{0,+}$. As is seen from (3.12) and (3.19), there exists a holomorphic function $\tilde{g}(\zeta)$ of ζ ($|\zeta| < 1$) such that

$$\hat{K}' = A_+(c) \cdot \hat{K} \cdot \tilde{g}(2A_{0,+}) \quad (3.25)$$

Hence from (3.13), (3.20) and (3.21)

$$\tilde{g}(2A_{0,+})\tilde{g}(2A_{0,-}) + \tilde{g}(2A_{0,+})^2\hat{F}' = A_0(c)^{-1}(1 + \hat{F}) \quad (3.26)$$

By uniqueness of this matrix expression, we have

$$\tilde{g}(2A_{0,+})\tilde{g}(2A_{0,-}) = A_0(c)^{-1} \quad (3.27)$$

$$\tilde{g}(2A_{0,+})^2\hat{F}' = A_0(c)^{-1}\hat{F} \quad (3.28)$$

which imply

$$\tilde{g}(2A_{0,+}) = A_{0,+}(c)^{-1} \quad (3.29)$$

and

$$A_{0,+}(c)^{-2}\hat{F}' = A_0(c)^{-1}\hat{F} \quad (3.30)$$

which are nothing else than (3.22).

4 Periodic Toda lattice

Let A be a periodic tri-diagonal matrix with period N ,

$$a_{n+N} = a_n, b_{n+N} = b_n \quad (4.1)$$

We assume that it is positive definite on $l^2(\mathbf{Z})$. Let h be the Floquet multiplier and $A_h = (\tilde{a}_{n,m})_{n,m=0}^{N-1}$ be the $N \times N$ matrix defined by

$$\begin{aligned}
\tilde{a}_{n,m} &= hb_{N-1} \quad (n, m) = (N-1, 0), \\
&= h^{-1}b_{N-1} \quad (n, m) = (0, N-1), \\
&= a_{n,m} \quad \text{otherwise}
\end{aligned}$$

The determinant of $z - A_h$ can be written as

$$\det[z - A_h] = -b_0 b_1 \cdots b_{N-1} (h + h^{-1} - \Delta) \quad (4.2)$$

where Δ denotes the polynomial of degree N such that

$$b_0 b_1 \cdots b_{N-1} \Delta = z^N - (a_0 + a_1 + \cdots + a_{N-1}) z^{N-1} + \cdots$$

The function h annihilating (4.2) is obtained by the equation

$$h = \frac{\Delta - \sqrt{\Delta^2 - 4}}{2} \quad (4.3)$$

which defines the hyperelliptic curve X of genus $N-1$.

Let $\lambda_1, \dots, \lambda_{2N}$ be the roots of the equation $\Delta^2 - 4 = 0$, such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{2N-1} < \lambda_{2N}$$

$|h| = 1$ i.e., $|\Delta| < 4$ holds if and only if

$$\lambda \in [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \cup \cdots \cup [\lambda_{2N-1}, \lambda_{2N}] \quad (4.4)$$

In other words, the spectra $\sigma(A)$ are continuous and given by the bands (4.4). When $\lambda \notin \sigma(A)$, we have $|h| < 1$.

Let $\psi^\pm(n; z)$ be the Bloch solutions to (1.3) satisfying

$$\psi^\pm(n + N; z) = h^{\pm 1} \psi^\pm(n; z) \quad (4.5)$$

which are obtained by solving the finite equations

$$(z - A_h)\tilde{\psi} = 0 \quad (4.6)$$

Let $K^\pm(n; z)$ be the normalized Bloch solutions such that $K^\pm(0; z) = 1$.

We denote by $D(i, j)$ the subdeterminant corresponding to the (n, m) th entries $(i \leq n, m \leq j)$ of $z - A$.

Then $K^\pm(n; z)$ can be expressed in terms of $D(i, j)$, in particular

$$K^+(1; z) = -\frac{(-1)^N h b_1 \cdots b_{N-1} + b_0 D(2, N-1)}{D(1, N-1)} \quad (4.7)$$

$$K^-(1; z) = -\frac{(-1)^N h^{-1} b_1 \cdots b_{N-1} + b_0 D(2, N-1)}{D(1, N-1)} \quad (4.8)$$

Proposition 4 *We have*

$$K^-(n; \lambda + i0) = K^+(n; \lambda - i0) = \overline{K^+(n; \lambda + i0)} \quad \text{for } \lambda \in \sigma(A) \quad (4.9)$$

Put

$$d\rho_+(\lambda) = d\rho_-(\lambda) = \frac{1}{2\pi} \frac{|D(1, N-1)|}{|b_0 b_1 \cdots b_{N-1}| \sqrt{4 - \Delta^2}}, \quad \lambda \in \sigma(A) \quad (4.10)$$

Then the spectral kernels of A can be expressed as

$$d\Theta(n, m; \lambda) = 2\Re\{K^+(n; \lambda + i0) \overline{K^+(m; \lambda + i0)}\} d\rho_+(\lambda) \quad (4.11)$$

We may put $D(1, N-1) = \prod_{k=1}^{N-1} (z - \mu_k)$ where $\mu_1, \mu_2, \dots, \mu_{N-1}$ denote the auxiliary spectra such that

$$\lambda_2 < \mu_1 < \lambda_3 < \lambda_4 < \cdots < \mu_{N-1} < \lambda_{2N-1} < \lambda_{2N} \quad (4.12)$$

We want to find the Gauss decomposition of A as in (3.15) (we put $c = 0$).

$$A = A_- \cdot A_+, \quad A_- = {}^t A_+ \quad (4.13)$$

such that $\xi_{n+N} = \xi_n$, $\eta_{n+N} = \eta_n$ hold.

We can find uniquely ξ_n , η_n such that (3.16), (3.17) hold with $c = 0$, i.e.,

$$\xi_0^2 = -b_0 K^+(1; 0) \quad (4.14)$$

Remark that (3.17) is a periodic continued fraction in this case.

The LR -transform is now defined by

$$A = A_- \cdot A_+ \rightarrow A' = A_+ \cdot A_- = A_+ \cdot A \cdot A_+^{-1} \quad (4.15)$$

The following Proposition 5 is most fundamental.

Proposition 5 *Let $\{\mu'_1, \mu'_2, \dots, \mu'_{N-1}\}$ be the auxiliary spectra for A' . Then A' is the LR -transform of A if and only if*

$$z \frac{\prod_{k=1}^{N-1} (z - \mu'_k)}{\prod_{k=1}^{N-1} (z - \mu_k)} = (\xi_0 + \eta_0 K^+(1; z))(\xi_0 + \eta_0 K^-(1; z)) \quad (4.16)$$

The matrices A' i.e., $\xi_n, \eta_n, \mu'_1, \dots, \mu'_{N-1}$ can be uniquely obtained by solving (4.16).

Proof 2 *The Bloch solutions for A' are given by*

$$K'_+(n; z) = \frac{\xi_n K^+(n; z) + \eta_n K^+(n+1; z)}{\xi_0 + \eta_0 K_+(1; z)} \quad (4.17)$$

We want to show first that (4.16) implies (4.15).

At $z = \infty$, $K^\pm(n; z)$ are meromorphic and satisfy

$$K^+(n; z) = O(z^{-n}), \quad K'^+(n; z) = O(z^{-n})$$

There exists the unique upper triangular real matrix $\Xi = (\xi_{n,m})_{n,m=-\infty}^\infty$ such that

$$K'^+(n; z) = \sum_{m=n}^{\infty} \xi_{n,m} K^+(m; z) \quad (4.18)$$

From (4.16), (1.1) and (1.2) we have the relations of operators

$$1 = \Xi \cdot \frac{\prod_{k=1}^{N-1} (A - \mu'_k)}{\prod_{k=1}^{N-1} (A - \mu_k)} \cdot {}^t\Xi \quad (4.19)$$

$$A' = \Xi \cdot A \cdot \frac{\prod_{k=1}^{N-1} (A - \mu'_k)}{\prod_{k=1}^{N-1} (A - \mu_k)} \cdot {}^t\Xi = \Xi \cdot A \cdot \Xi^{-1} \quad (4.20)$$

Moreover, there exists an upper triangular matrix $Y = (\eta_{n,m})_{n,m=-\infty}^{\infty}$ such that

$$(\xi_0 + \eta_0 K^+(1; z)) K^+(n; z) = \sum_{m=n}^{\infty} \eta_{n,m} K^+(m; z) \quad (4.21)$$

which is equivalent to the relations

$$\eta_{n,m} = 2 \int_{-\infty}^{\infty} \Re\{(\xi_0 + \eta_0 K^+(1; \lambda + i0)) K^+(n; \lambda + i0) \overline{K^+(m; \lambda + i0)}\} d\rho_+(\lambda) \quad (4.22)$$

in view of (1.1) and (4.11). Therefore by substitution of A into $K^+(1; z)$, we have

$$\xi_0 + \eta_0 K^+(1; A) = Y \quad (4.23)$$

In the same way,

$$\xi_0 + \eta_0 K^-(1; A) = {}^tY \quad (4.24)$$

From (4.16), these two equalities imply

$$A \frac{\prod_{k=1}^{N-1} (A - \mu'_k)}{\prod_{k=1}^{N-1} (A - \mu_k)} = Y \cdot {}^tY = {}^tY \cdot Y \quad (4.25)$$

Since Y and tY commute each other,

$$A = {}^tY \cdot \frac{\prod_{k=1}^{N-1}(A - \mu_k)}{\prod_{k=1}^{N-1}(A - \mu'_k)} \cdot Y = {}^tY \cdot {}^t\Xi \cdot \Xi \cdot Y$$

which is nothing else than the Gauss decomposition of A , i.e.,

$$A_- = {}^tY \cdot {}^t\Xi, \quad A_+ = \Xi \cdot Y \quad (4.26)$$

From (4.20)

$$A' = A_+ \cdot Y^{-1} \cdot A \cdot Y \cdot A_+^{-1} = A_+ \cdot A \cdot A_+^{-1}$$

which leads to (4.15).

Next we show that (4.15) implies (4.16).

Put

$$\psi' = A_+(K^+) \quad (4.27)$$

and normalize it such that $K'^+(0; z) = 1$ as follows.

$$K'^+(n; z) = \frac{\psi'(n; z)}{\psi'(0; z)} \quad (4.28)$$

which gives (4.17) for A' . Then there exists the unique upper triangular matrix Ξ satisfying (4.18). Hence,

$$\{A_+(K^+)\}(n; z) = \{(\xi_0 + \eta_0 K^+(1; z))\Xi(K^+)\}(n; z) = \{\Xi \cdot Y(K^+)\}(n; z)$$

In other words,

$$A_+ = \Xi \cdot Y \quad (4.29)$$

As a consequence,

$$\begin{aligned}
\{A'\}_{n,m} &= \{A_+ \cdot A_-\}_{n,m} = \{\Xi \cdot Y \cdot {}^tY \cdot {}^t\Xi\}_{n,m} \\
&= 2 \int_{-\infty}^{\infty} \Re\{(\xi_0 + \eta_0 K^+(1, \lambda + i0))(\xi_0 + \eta_0 K^+(1, \lambda - i0)) \\
&\quad K'^+(n, \lambda + i0)K'^+(m, \lambda - i0)\} d\rho_+(\lambda) \tag{4.30}
\end{aligned}$$

On the other hand, by definition

$$\{A'\}_{n,m} = 2 \int_{-\infty}^{\infty} \lambda \Re\{\lambda K'^+(n; \lambda + i0)K'^+(m; \lambda - i0)\} d\rho'_+(\lambda)$$

Therefore by uniqueness of expression

$$\lambda d\rho'_+(\lambda) = (\xi_0 + \eta_0 K^+(1, \lambda + i0))(\xi_0 + \eta_0 K^+(1, \lambda - i0)) d\rho_+(\lambda)$$

Seeing that

$$d\rho'_+(\lambda) = \frac{\prod_{k=1}^{N-1} (\lambda - \mu'_k)}{\prod_{k=1}^{N-1} (\lambda - \mu_k)} d\rho_+(\lambda)$$

we have (4.16).

The hyperelliptic curve X defined by (4.3) has two sheets, physical and unphysical, which correspond to $|h| < 1$ (> 1) respectively, for $\lambda \notin \sigma(A)$.

Since $K^\pm(n; z)$ are meromorphic functions on X , we can represent the functions $K^\pm(n; z)$ by using divisors in X . Since $z = 0, \infty$ are not branch points of X , there are two points in X in each case, lying over $z = 0$, and $z = \infty$ $\langle 0 \rangle, \langle \infty \rangle$ in the physical sheet, $\langle 0^* \rangle, \langle \infty^* \rangle$ in the unphysical sheet respectively. X has the canonical involution

$$\iota : h \rightarrow h^{-1} \tag{4.31}$$

Obviously $\iota(\langle 0 \rangle) = \langle 0^* \rangle$ and $\iota(\langle \infty \rangle) = \langle \infty^* \rangle$. We denote by D^* the conjugate $\iota(D)$ of a divisor D . Then

Lemma 1 Fix $n \geq 0$. $K^+(n; z)$ has simple poles at the physical points in X , lying over $z = \mu_1, \mu_2, \dots, \mu_{N-1}$ which do not depend on n . We denote the corresponding positive divisor of degree $N - 1$ by D_0 . It has also a pole of

order n at $\langle \infty^* \rangle$. Similarly it has simple zeros at the unphysical points lying over $z = \mu_1, \mu_2, \dots, \mu_{N-1}$ (its divisor of degree $N - 1$ is denoted by D_n) and a zero of order n at $\langle \infty \rangle$.

In other words, in terms of divisors,

$$(K^+(n; z)) = n\langle \infty \rangle - n\langle \infty^* \rangle - D_0 + D_n \quad (4.32)$$

$$(K^-(n; z)) = n\langle \infty^* \rangle - n\langle \infty \rangle - D_0^* + D_n^* \quad (4.33)$$

Furthermore

$$\left(\prod_{k=1}^{N-1} (z - \mu_k) \right) = -(N-1)\{\langle \infty \rangle + \langle \infty^* \rangle\} + D_0 + D_0^* \quad (4.34)$$

$$(h) = N(\langle \infty \rangle - \langle \infty^* \rangle) \quad (4.35)$$

As for the zeros and poles of $\xi_0 + \eta_0 K^+(1; z)$, we have

Theorem 2 *There exist a positive divisor of degree $N - 1$, D'_0 and its conjugate $D_0'^*$, such that*

$$(\xi_0 + \eta_0 K^+(1; z)) = \langle 0 \rangle - \langle \infty^* \rangle - D_0 + D'_0 \quad (4.36)$$

$$(\xi_0 + \eta_0 K^-(1; z)) = \langle 0^* \rangle - \langle \infty \rangle - D_0^* + D_0'^* \quad (4.37)$$

Hence, there exists a positive divisor of degree $N - 1$, D'_1 such that

$$(K'^+(1; z)) = \langle \infty \rangle - \langle \infty^* \rangle - D'_0 + D'_1 \quad (4.38)$$

The set of divisor classes of degree $N - 1$ in X makes the Jacobi variety of X denoted by $Jac(X)$. As is seen from (4.36), we have the equality as a point of $Jac(X)$

$$D'_0 - D_0 \equiv -\langle 0 \rangle + \langle \infty^* \rangle \quad (4.39)$$

The new tri-diagonal operator A' has the same spectra as A and therefore we can take the LR -transform of A' again. By repeating this procedure, we get a sequence of tri-diagonal operators

$$A \rightarrow A' \rightarrow A'' \rightarrow \dots \quad (4.40)$$

and a sequence of corresponding divisor classes

$$D_0 \rightarrow D'_0 \rightarrow D''_0 \rightarrow \dots \quad (4.41)$$

such that

$$D'_0 - D_0 \equiv D''_0 - D'_0 \equiv \dots \equiv -\langle 0 \rangle + \langle \infty^* \rangle \quad (4.42)$$

As a conclusion,

Theorem 3 *The sequence of LR-transforms (4.40) is realized in $\text{Jac}(X)$, by the discrete paralell displacement of $\mathbf{p}_m$ by the constant divisor class $-\langle 0 \rangle + \langle \infty^* \rangle$, starting from $\mathbf{p}_0 = D_0$ such that*

$$\mathbf{p}_m = \mathbf{p}_0 + m\{-\langle 0 \rangle + \langle \infty^* \rangle\}, \quad m = 0, 1, 2, 3, \dots \quad (4.43)$$

Corollary 1 *The sequence of LR-transforms is periodic with period $M > 0$ if and only if*

$$M\{-\langle 0 \rangle + \langle \infty^* \rangle\} \equiv 0 \quad (4.44)$$

Remark 1 *When A is finite or semi-infinite, the sequence (4.40) never become periodic. In fact, in a finite case, A tends to a diagonal matrix, so that the eigenvalues of A are approximated by these procedure([25],[26],[27]). I do not know how they behave, when A is semi-finite.*

Remark 2 *$f(z)$ is a polynomial of degree r , it is possible to extend (4.16) to a more general transform (1.6). In this situation (4.16) must be replaced by the equation*

$$f(z) \frac{\prod_{k=1}^{N-1} (z - \mu'_k)}{\prod_{k=1}^{N-1} (z - \mu_k)} = (\xi_0 + \sum_{k=1}^r \eta_{0,k} K^+(k; z)) (\xi_0 + \sum_{k=1}^r \eta_{0,k} K^-(k; z))$$

Since $f(A)$ is no more tri-diagonal, we cannot find tri-diagonal matrices $B_{\pm}$ satisfying (1.5).

Suppose that $f(A)$ is positive definite and multiple-diagonal of width $2m+1$. Then $f(A)$ is a tri-diagonal matrix in block form, consisting of matrices $A_{n,n}$, ($A_{n,n} = {}^t A_{n,n} > 0$) $A_{n,n+1}, A_{n+1,n} = {}^t A_{n,n+1}$ of size $m+1$. One can find an upper block bi-triangular matrix B_+ consisting of triangular matrices $B_{n,n}$ and $B_{n,n+1}$ of size $m+1$ such that

$$A_{n,n} = {}^t B_{n,n} \cdot B_{n,n} + {}^t B_{n-1,n} \cdot B_{n-1,n}, \quad A_{n,n+1} = {}^t B_{n,n} \cdot B_{n,n+1}$$

If we put $Z_n = {}^t B_{n,n} \cdot B_{n,n}$, then we have the recurrence relations

$$Z_n = A_{n,n+1} \cdot (A_{n+1,n+1} - Z_{n+1})^{-1} \cdot {}^t A_{n,n+1}$$

which give the matrix version of the convergent continued fraction (3.17) such that

$$Z_n \leq A_{n,n+1} \cdot A_{n+1,n+1}^{-1} \cdot {}^t A_{n+1,n}$$

$B_{n,n}$ can be solved uniquely from Z such that all the diagonal elements are positive.

In the next section, in case of $N = 2$, we shall give explicit computation in terms of the sigma functions on the elliptic curve X .

5 Case of period $N = 2$

It is sufficient to give $\{a_0, a_1, b_0, b_1\}$ to define the operator A .

We put $W(z) = b_0^2 b_1^2 (\Delta^2 - 4)$, then

$$W(z) = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4), \quad 0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \quad (5.1)$$

Moreover

$$d\rho_{\pm} = \frac{1}{4\pi} \frac{|\lambda - a_1|}{\sqrt{|W(\lambda)|}}, \quad \lambda_2 < a_1 < \lambda_3 \quad (5.2)$$

$$K^+(1; z) = \frac{b_0 + b_1 h}{z - a_1}, \quad K^-(1; z) = \frac{b_0 + b_1 h^{-1}}{z - a_1}, \quad (5.3)$$

(4.16) reduces to

$$z \frac{z - a'_1}{z - a_1} = (\xi_0 + \eta_0 K^+(1; z))(\xi_0 + \eta_0 K^-(1; z)) \quad (5.)$$

Put

$$\begin{aligned} u &= \int_{\lambda_4}^z \frac{dz}{\sqrt{W(z)}}, v = \int_{\lambda_4}^{\infty} \frac{dz}{\sqrt{W(z)}} > 0, w = \int_{\lambda_4}^0 \frac{dz}{\sqrt{W(z)}} > 0 \\ v - c &= \int_{\lambda_4}^{a_1} \frac{dz}{\sqrt{W(z)}}, \quad v > \Re c > 0, \Im c < 0 \\ \omega_1 &= \int_{\lambda_2}^{\lambda_3} \frac{dz}{\sqrt{W(z)}} > 0, \omega_2 = i \int_{\lambda_3}^{\lambda_4} \frac{dz}{\sqrt{|W(z)|}} \in i\mathbf{R}_{>0} \end{aligned}$$

then, $2\omega_1, 2\omega_2$ are double periods, and $\langle 0 \rangle, \langle 0^* \rangle, \langle \infty \rangle, \langle \infty^* \rangle$ correspond to

$$u = w, \quad u = -w, \quad u = v, \quad u = -v$$

respectively. Furthermore,

$$4v = 2\omega_1 \equiv 0$$

i.e.,

$$D_2 - D_0 \sim 0$$

$\sigma(u)$ has the zero $u = 0$, and quasi-periodic

$$\begin{aligned} \sigma(u + 2\omega_1) &= -e^{2(\eta_1 u + \omega_1)} \sigma(u) \\ \sigma(u + 2\omega_2) &= -e^{2(\eta_2 u + \omega_2)} \sigma(u) \end{aligned}$$

(where η_1, η_2 denote constants). We have

$$\begin{aligned}
z &= -\frac{\sigma(u+w)\sigma(u-w)\sigma(2v)}{\sigma(u-v)\sigma(u+v)\sigma(v+w)\sigma(v-w)} \\
h &= C_1 \frac{\sigma^2(u-v)}{\sigma^2(u+v)} \\
K^+(1; z) &= C_2 \frac{\sigma(u-v)\sigma(u+v+c)}{\sigma(u+v)\sigma(u-v+c)} \\
K^{+'}(1; z) &= C_3 \frac{\sigma(u-v)\sigma(u+v+c')}{\sigma(u+v)\sigma(u-v+c')}
\end{aligned}$$

If we put

$$c' - c = v + w = \int_{0^*}^{\infty} \frac{dz}{\sqrt{W(z)}}$$

then the LR -transform represents the parallell displacement on the 1 dimensional complex torus $\mathbf{C}/(\mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2)$

$$c \rightarrow c + v + w \rightarrow c + 2(v + w) \rightarrow \dots$$

In order that it is periodic, there exists a positive integer M such that

$$M(v + w) \equiv 0 \quad (2\omega_1, 2\omega_2)$$

6 Multi-Index Hankel Matrices and Orthogonal Polynomials in Multi-Variables

In the next three sections we shall make a multi-dimensional extension of LR -transforms developed in the previous sections. Multi-dimensional LR -transforms are related with eigenfunction expansions for commuting self-adjoint operators.

We restrict ourselves to orthogonal polynomials case. The problem of finding LR -transforms reduces to obtaining the connection formula between two systems of orthogonal polynomials. Our main result in this section is

Theorem 4. In the course of proof, we shall give a formula for the connection matrix which is a lower triangular matrix, in terms of determinants of the associated multi-dimensional Hankel matrix.

Let $d\rho = d\rho(x)$, $x = (x_1, \dots, x_n)$ be a Radon measure on the n dimensional Euclidean space $\mathbf{R}^n$ whose support is a bounded closed set $\mathcal{D}$. We assume that all multi-index moments

$$c_{i_1, \dots, i_n} = \int_{\mathbf{R}^n} x_1^{i_1} \cdots x_n^{i_n} d\rho(x) \quad (6.1)$$

are finite.

Let $\mathcal{H}_\rho$ be the Hilbert space completed by the inner product $(f, g)_\rho$

$$(f, g)_\rho = \int_{\mathbf{R}^n} f(x)g(x)d\rho(x) \quad (6.2)$$

for real continuous functions $f(x), g(x)$ on $\mathbf{R}^n$.

For two sequences of indices $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_n)$, we define the sum $I + J$ by the sequence of indices $(i_1 + j_1, \dots, i_n + j_n)$.

We define the lexicographic ordering $\mathcal{O}$ for the set of multi-indices as follows.

$(i_1, \dots, i_n)$ is greater than $(j_1, \dots, j_n)$ if and only if there exists a number r ($1 \leq r \leq n$) such that $i_1 = j_1, \dots, i_{r-1} = j_{r-1}, i_r > j_r$. In this case, we also say the monomial $x_1^{i_1} \cdots x_n^{i_n}$ is greater than the monomial $x_1^{j_1} \cdots x_n^{j_n}$.

Thus we have the sequence of monomials in increasing order

$$1 < x_1 < \cdots < x_n < x_1^2 < x_1 x_2 < \cdots < x_n^2 < x_1^3 < \cdots$$

Let N be the unique bijective mapping from the set of positive integers onto the set of multi-indices such that $N(l_1) < N(l_2)$ for two positive integers $l_1 < l_2$.

We have $N(1) = (0, \dots, 0)$, $N(2) = (1, \dots, 0)$, $N(3) = (0, 1, 0, \dots, 0)$, $\dots$, $N(n+1) = (0, \dots, 0, 1)$, and $N(n+2) = (2, 0, \dots, 0)$ etc.

We assume that $d\rho(x)$ is non-degenerate in the sense that

$$\int_{\mathbf{R}^n} f(x)^2 d\rho(x) > 0$$

for any polynomial $f(x)$ which is not identically zero on $\mathcal{D}$.

Let $\mathcal{C}$ be the generalized Hankel matrix with the $N(l), N(m)$ th entries $c_{N(l)+N(m)}$ for $l, m = 1, 2, 3, \dots$. It is a positive definite matrix, so that all the determinants

$$D_{N(r)} = \det((c_{N(l)+N(m)})_{l,m=1}^r) > 0$$

for $(0 \leq r < \infty)$. Here we put $D_{N(0)} = 1$.

Gram-Schmit orthonormalization with respect to the lexicographic ordering gives the orthonormalized polynomials $\{p_{i_1, \dots, i_n}\}_{i_1, \dots, i_n \geq 0}$ such that

$$p_{i_1, \dots, i_n} = \xi_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} + (\text{lower order terms}) \quad (6.3)$$

where $\xi_{i_1, \dots, i_n}$ denote normalizing positive constants. Hence we have the orthonormality

$$(p_{i_1, \dots, i_n}, p_{j_1, \dots, j_n})_\rho = \delta_{i_1, j_1} \cdots \delta_{i_n, j_n} \quad (6.4)$$

Let $N(l)$ denote the multi-index $(i_1, \dots, i_n)$. We denote the monomial $x^{N(l)} = x_1^{i_1} \cdots x_n^{i_n}$. Then the polynomials $\tilde{p}_{i_1, \dots, i_n}(x)$ defined by the determinant

$$\tilde{p}_{i_1, \dots, i_n}(x) = \frac{1}{D_{N(l-1)}} \begin{vmatrix} c_{N(1)} & c_{N(2)} & \cdots & c_{N(l)} \\ c_{N(2)} & c_{N(1)+N(2)} & \cdots & c_{N(2)+N(l)} \\ \dots & \dots & \dots & \dots \\ c_{N(l-1)} & c_{N(l-1)+N(2)} & \cdots & c_{N(l-1)+N(l)} \\ x^{N(1)} & x^{N(2)} & \cdots & x^{N(l)} \end{vmatrix} \quad (6.5)$$

are monic orthogonal polynomials such that the following equations hold.

$$(\tilde{p}_{i_1, \dots, i_n}(x), \tilde{p}_{j_1, \dots, j_n}(x)) = 0 \quad \text{for } (i_1, \dots, i_n) \neq (j_1, \dots, j_n) \quad (6.6)$$

$$(\tilde{p}_{i_1, \dots, i_n}(x), \tilde{p}_{i_1, \dots, i_n}(x)) = \frac{D_{N(l)}}{D_{N(l-1)}} \quad \text{for } N(l) = (i_1, \dots, i_n) \quad (6.7)$$

so that we have the orthonormalized polynomials

$$p_{N(l)}(x) = \sqrt{\frac{D_{N(l-1)}}{D_{N(l)}}} \tilde{p}_{N(l)}(x) \quad (6.8)$$

(The above computation can be done in the same way as in [31].)

We have

$$p_{N(l)}(x) = \sqrt{\frac{D_{N(l-1)}}{D_{N(l)}}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + (\text{lower order terms})$$

Let $A_1, A_2, \dots, A_n$ be the bounded linear operators on $\mathcal{H}_\rho$ defined by

$$A_j \varphi(x) = x_j \varphi(x) \quad \varphi(x) \in \mathcal{H}_\rho \quad (6.9)$$

They can be expressible in matrix form $a_{N(l), N(m)}^{(j)}$ in terms of the basis $x^{N(l)}$ $l = 1, 2, 3, \dots$

$$x_j p_{N(l)}(x) = \sum_{m \geq 1 \text{ (finite sum)}} a_{N(l), N(m)}^{(j)} p_{N(m)}(x) \quad (1 \leq j \leq n) \quad (6.10)$$

We have

$$a_{N(l), N(m)}^{(j)} = (x_j p_{N(l)}(x), p_{N(m)}(x))_\rho \quad (6.11)$$

A_j are self-adjoint bounded operators and commute each other.

Let $L^2(\mathbf{Z}_{\geq 0}^n)$ denote the Hilbert space consisting of real sequences $u = (u_{i_1, \dots, i_n})_{i_1, \dots, i_n \geq 0}$ with the inner product

$$(u, v) = \sum_{i_1, \dots, i_n \geq 0} u_{i_1, \dots, i_n} v_{i_1, \dots, i_n} \quad u, v \in L^2(\mathbf{Z}_{\geq 0}^n)$$

The correspondence from the set of real sequences $(u_{i_1, i_2, \dots, i_n})_{i_1, \dots, i_n \geq 0}$ to continuous functions $\varphi(x)$

$$(u_{i_1, i_2, \dots, i_n})_{i_1, \dots, i_n \geq 0} \rightarrow \varphi(x) = \sum_{i_1, \dots, i_n \geq 0} u_{i_1, i_2, \dots, i_n} p_{i_1, \dots, i_n}(x) \quad (6.12)$$

give rise to the isomorphism between the space $L^2(\mathbf{Z}_{\geq 0}^n)$ and $\mathcal{H}_\rho$.

Consider the shifts τ_ν for the sequences $i_1 \geq \dots, i_n \geq 0$ as

$$\tau_\nu^\pm : (i_1, \dots, i_n) \rightarrow (i_1, \dots, i_\nu \pm 1, \dots, i_n) \quad (6.13)$$

For $N(l) = (i_1, \dots, i_n)$, we denote by $\tau_\nu^\pm l$ the number $l^\pm$ such that $N(l^\pm) = \tau_\nu^\pm N(l)$ by abuse of notation (Remark that l^- does not exist when $i_\nu = 0$.)

From the relations (6.5), (6.8) and (6.10) the following Proposition holds.

Proposition 6 Assume $1 \leq m \leq l$. We can represent explicitly the matrix elements $a_{N(l),N(m)}^{(j)}$ as

$$a_{N(l),N(m)}^{(j)} = \sum_{m \leq r \leq l} (-1)^{l+m+r} \frac{1}{\sqrt{D_{N(l)} D_{N(l-1)} D_{N(m)} D_{N(m-1)}}}$$

$$\cdot \begin{vmatrix} c_{N(1)} & c_{N(2)} & \cdots & c_{N(m)} \\ c_{N(2)} & c_{N(1)+N(2)} & \cdots & c_{N(2)+N(m)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N(m-1)} & c_{N(m-1)+N(2)} & \cdots & c_{N(m-1)+N(m)} \\ c_{N(r)} & c_{N(r)+N(2)} & \cdots & c_{N(r)+N(m)} \end{vmatrix}$$

$$\cdot \begin{vmatrix} c_{N(1)} & c_{N(2)} & \cdots & c_{N(l-1)} \\ c_{N(2)} & c_{N(1)+N(2)} & \cdots & c_{N(2)+N(l-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \rangle c_{\tau_j^- N(r)} & c_{\tau_j^- N(r)+N(2)} & \cdots & c_{\tau_j^- N(r)+N(l-1)} \langle \\ \cdots & \cdots & \cdots & \cdots \\ c_{N(l)} & c_{N(l)+N(2)} & \cdots & c_{N(l)+N(l-1)} \end{vmatrix} \quad (6.14)$$

(The symbol $\rangle \cdots \langle$ denotes the deletion of a line)

Let $f(x)$ be a continuous function on $\mathbf{R}^n$ which is non-negative on $\mathcal{D}$. Consider the new density $d\rho'(x)$ on $\mathbf{R}^n$ with the same support as $\mathcal{D}$.

$$d\rho'(x) = f(x)d\rho(x) \quad (6.15)$$

Then we can define the multiplication operators

$$A'_j : \varphi(x) \rightarrow x_j \varphi(x) \quad (1 \leq j \leq n) \quad (6.16)$$

on the new Hilbert space $\mathcal{H}_{\rho'} = L^2(\mathbf{R}^n; d\rho')$ with the inner product $(\cdots, \cdots)_{\rho'}$.

Let $(c'_{i_1, \dots, i_n})_{i_1, \dots, i_n \geq 0}$ be the moments of the density $d\rho'$ and C' be the corresponding generalized Hankel matrix with the $N(l), N(m)$ th entries $c'_{N(l)+N(m)}$.

Then $f(A_1, \dots, A_n)$ is a self-adjoint operator on $\mathcal{H}_{\rho'}$, which is positive definite, because

$$(f(A_1, \dots, A_n)\varphi(x), \varphi(x))_{\rho} = \int_{\mathcal{D}} \varphi(x)^2 f(x) d\rho(x) > 0$$

for a continuous function $\varphi(x)$ which does not vanish identically in $\mathcal{D}$.

Let $(p'_{i_1, \dots, i_n}(x))_{i_1, \dots, i_n \geq 0}$ be the Gram-Schmidt orthonormalization according to the lexicographic ordering $\mathcal{O}$.

$(\tilde{p}'_{i_1, \dots, i_n}(x))_{i_1, \dots, i_n \geq 0}$ are defined similarly to (1.5), replacing $c_{i_1, \dots, i_n}$ by $c'_{i_1, \dots, i_n}$.

The operator $f(A_1, \dots, A_n)$ can be represented by the matrix with the $N(l), N(m)$ th elements $(f(A_1, \dots, A_n)p_{N(l)}(x), p_{N(m)}(x))_\rho$.

We are interested in the connection relations between the two set of orthogonal polynomials $(p_{i_1, \dots, i_n})_{i_1, \dots, i_n}$ and $(p'_{i_1, \dots, i_n})_{i_1, \dots, i_n}$.

$p_{i_1, \dots, i_n}$ can be represented as a linear combination of $p'_{j_1, \dots, j_n}$

$$p_{N(l)}(x) = \sum_{1 \leq m \leq l} R_{N(l)/N(m)} p'_{N(m)}(x) \quad (6.17)$$

We put further $R_{N(l)/N(m)}$ to be 0 for $l < m$, so that $R = (R_{N(l)/N(m)})_{l, m \geq 0}$ defines an invertible lower triangular matrix with respect to the lexicographic ordering. In particular the diagonal elements are expressed as

$$R_{N(l)/N(l)} = \sqrt{\frac{D'_{N(l)} D_{N(l-1)}}{D_{N(l)} D'_{N(l-1)}}} > 0 \quad (6.18)$$

As for the relations between $\tilde{p}_{i_1, \dots, i_n}$ and $\tilde{p}'_{i_1, \dots, i_n}$, we have similarly

$$\tilde{p}_{N(l)} = \sum_{1 \leq m \leq l} \tilde{R}_{N(l)/N(m)} \tilde{p}'_{N(m)} \quad (6.19)$$

for an invertible lower triangular matrix $\tilde{R} = (\tilde{R}_{N(l)/N(m)})_{1 \leq l, m < \infty}$. Remark that $\tilde{R}_{N(l)/N(l)} = 1$. In view of (6.8), (6.17) and (6.19), the following identities hold.

$$R_{N(l), N(m)} = \sqrt{\frac{D_{N(l-1)} D'_{N(l)}}{D'_{N(l-1)} D_{N(l)}}} \tilde{R}_{N(l), N(m)} \quad (6.20)$$

Theorem 4 *As a matrix expression, we have*

$$f(A_1, \dots, A_n) = R \cdot {}^t R \quad (6.21)$$

The matrix R is uniquely determined by (1.21).

For every j , we have the following LR -transforms

$$A'_j = R^{-1} \cdot A_j \cdot R \quad (6.22)$$

In particular,

$$f(A'_1, \dots, A'_n) = R^{-1} \cdot f(A_1, \dots, A_n) \cdot R = {}^t R \cdot R$$

which is just the interchange of R and ${}^t R$. R is an invertible matrix so that R^{-1} is well-defined.

For $u = (u_{i_1, \dots, i_n})_{i_1, \dots, i_n} \in L^2(\mathbb{Z}_{\geq 0}^n)$, (6.10) and (6.12) give the matrix expression

$$(A_j u)_{i_1, \dots, i_n} = \sum_{j_1, \dots, j_n \geq 0} a_{(i_1, \dots, i_n), (j_1, \dots, j_n)}^{(j)} u_{j_1, \dots, j_n} \quad (6.23)$$

Let $\mathcal{H}_0$ be the Hilbert space spanned by the sequences ${}^t R u$. $\mathcal{H}_0$ is isomorphic to the space of sequences $v = (v_{i_1, \dots, i_n})_{i_1, \dots, i_n}$ in $L^2(\mathbb{Z}_{\geq 0}^n)$ such that $(f(A_1, \dots, A_n)^{-1} v, v) < \infty$. Then the inverse R^{-1} is well-defined as a bounded operator from $\mathcal{H}_0$ to $L^2(\mathbb{Z}_{\geq 0}^n)$.

The matrix elements $\tilde{R}_{N(l), N(m)}$ can be expressed by using the following system of determinants $\psi_{l_1, \dots, l_r}$ for different positive integers $l_1, \dots, l_r, \dots$, from each other.

$$\psi_{l_1} = c_{N(l_1)}, \psi_{l_1, l_2} = \begin{vmatrix} c_{N(l_1)} & c_{N(l_2)} \\ c_{N(2)+N(l_1)} & c_{N(2)+N(l_2)} \end{vmatrix}$$

$$\psi_{l_1, l_2, \dots, l_r} = \begin{vmatrix} c_{N(l_1)} & c_{N(l_2)} & \cdots & c_{N(l_r)} \\ c_{N(2)+N(l_1)} & c_{N(2)+N(l_2)} & \cdots & c_{N(2)+N(l_r)} \\ \dots & \dots & \dots & \dots \\ c_{N(r)+N(l_1)} & c_{N(r)+N(l_2)} & \cdots & c_{N(r)+N(l_r)} \end{vmatrix}$$

(Remark that $N(1) = (0, 0, \dots, 0)$.)

In the same way we define the determinants $\psi'_{l_1, \dots, l_r}$ associated with the moments $c'_{N(l)}$

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Then we have

$$\begin{aligned} \tilde{R}_{N(l), N(m)} &= \frac{1}{D'_{N(m)} \cdots D'_{N(l-1)} D_{N(l-1)}} \\ &\cdot \sum \epsilon \psi'_{1,2,\dots,m-1,\alpha_{m,m}} \psi'_{1,2,\dots,m-1,\alpha_{m+1,m},\alpha_{m+1,m+1}} \\ &\cdots \psi'_{1,2,\dots,m-1,\alpha_{l-1,m},\dots,\alpha_{l-1,l-1}} \psi'_{1,2,\dots,m-1,\alpha_{l,m},\dots,\alpha_{l,l-1}} \end{aligned} \quad (6.24)$$
$$\begin{aligned} \{\alpha_{m,m}, \alpha_{m+1,m}\} &= \{m, m+1\}, \\ \{\alpha_{m+1,m+1}, \alpha_{m+2,m}, \alpha_{m+2,m+1}\} &= \{m, m+1, m+2\}, \\ &\dots = \dots \\ \{\alpha_{l-2,l-2}, \alpha_{l-1,m}, \dots, \alpha_{l-1,l-2}\} &= \{m, m+1, \dots, l-1\}, \\ \{\alpha_{l-1,l-1}, \alpha_{l,m}, \dots, \alpha_{l,l-1}\} &= \{m, m+1, \dots, l\} \end{aligned}$$
$$\alpha_{m+2,m} < \alpha_{m+2,m+1},$$

$$\dots = \dots$$

ϵ denotes the suitably chosen sign $\pm$ depending on the choices of α 's.

This Proposition follows by solving (6.19) term by term in view of (6.5). For example,

$$\begin{aligned}
\tilde{R}_{N(l),N(l)} &= 1, \\
\tilde{R}_{N(l),N(l-1)} &= \frac{1}{D'_{N(l-1)}D_{N(l-1)}} \\
&\quad \cdot (\psi'_{1,2,\dots,l-2,l}\psi_{1,2,\dots,l-2,l-1} - \psi'_{1,2,\dots,l-2,l-1}\psi_{1,2,\dots,l-2,l}) \\
\tilde{R}_{N(l),N(l-2)} &= \frac{1}{D'_{N(l-1)}D'_{N(l-2)}D_{N(l-1)}} \\
&\quad \cdot (\psi'_{1,2,\dots,l-2}\psi'_{1,2,\dots,l-1}\psi_{1,2,\dots,l-3,l-1,l} \\
&\quad - \psi'_{1,2,\dots,l-3,l-1}\psi'_{1,2,\dots,l-2,l}\psi_{1,2,\dots,l-1} \\
&\quad + \psi'_{1,2,\dots,l-2}\psi'_{1,2,\dots,l-3,l-1,l}\psi_{1,2,\dots,l-1} \\
&\quad - \psi'_{1,2,\dots,l-3,l-1}\psi'_{1,2,\dots,l-1}\psi_{1,2,\dots,l-2,l})
\end{aligned}$$

and so on.

Example. (Appell's Polynomials) Suppose the density

$$d\rho(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} (1 - x_1 - \cdots - x_n)^{\alpha_{n+1}} dx_1 \wedge \cdots \wedge dx_n \quad (6.25)$$

be defined on the simplex $\mathcal{D} : x_1 \geq 0, \dots, x_n \geq 0, x_1 + \cdots + x_n \leq 1$. We have

$$c_{i_1, \dots, i_n} = \frac{\Gamma(\alpha_1 + i_1 + 1) \cdots \Gamma(\alpha_n + i_n + 1) \Gamma(\alpha_{n+1} + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_{n+1} + i_1 + \cdots + i_n + n + 1)}$$

The ratio $D(N(l))/D(N(1))$ and the monic polynomial $\tilde{p}_{N(l)}$ for every l are rational functions of $\alpha_1, \dots, \alpha_{n+1}$. Whence every element $\tilde{R}_{N(l),N(m)}$ is a rational function of $\alpha_1, \dots, \alpha_{n+1}$.

7 Matrix Form of LR-Transforms and Proof of Theorem 4

Assume that the orthonormal polynomials $p_{N(l)}$ and $p'_{N(l)}$ $l = 1, 2, 3, \dots$ are expressed as linear combinations of monomials $x^{N(m)}$ $m = 1, 2, 3, \dots$ as

$$p_{N(l)} = \sum_{m=1}^l \xi_{N(l),N(m)} x^{N(m)}$$

$$p'_{N(l)} = \sum_{m=1}^{\infty} \xi'_{N(l), N(m)} x^{N(m)}$$

We put $\xi_{N(l), N(m)}$ and $\xi'_{N(l), N(m)}$ to be 0 for $l < m$. Let Ξ, Ξ' be the lower triangular matrices with the $N(l), N(m)$ th elements $\xi_{N(l), N(m)}, \xi'_{N(l), N(m)}$ respectively. Then the orthonormality and the spectral representations for A_j and A'_j imply the matrix relations

$$\Xi \cdot C \cdot {}^t\Xi = 1 \quad (7.1)$$

$$\Xi' \cdot C' \cdot {}^t\Xi' = 1 \quad (7.2)$$

and

$$A_j = \Xi \cdot \tau_j^+ C \cdot {}^t\Xi \quad (7.3)$$

$$A'_j = \Xi' \cdot \tau_j^+ C' \cdot {}^t\Xi' \quad (7.4)$$

respectively.

Lemma 2 *Let M_ν , $1 \leq \nu \leq n$ be the operator defined by the matrix whose $(i_1, \dots, i_n; j_1, \dots, j_n)$ th elements are equal to 1 if $(j_1, \dots, j_n) = (i_1, \dots, i_{\nu-1}, i_\nu + 1, i_{\nu+1}, \dots, i_n)$ and equal to 0 otherwise. Then we have*

$$\tau_\nu^+(C) = M_\nu \cdot C \quad (7.5)$$

This lemma shows that (7.3) and (7.4) are equivalent to the followings

$$A_j = \Xi \cdot M_j \cdot \Xi^{-1} \quad (7.6)$$

$$A'_j = \Xi' \cdot M_j \cdot \Xi'^{-1} \quad (7.7)$$

respectively. We have further

$$R = \Xi \cdot \Xi'^{-1} \quad (7.8)$$

(7.6)-(7.8) imply that

$$A'_j = R^{-1} \cdot A_j \cdot R \quad (7.9)$$

This proves the Theorem.

This is a discrete analog of the argument done in [34].

8 Symmetric Polynomials Case

LR -transforms can also be applied to symmetric orthogonal polynomials with respect to a non-degenerate symmetric Radon measure $d\rho(x)$ on $\mathbf{R}^n$ with support $\hat{\mathcal{D}}$ which is a bounded closed set.

Let $\lambda_1, \dots, \lambda_n$ be a partition, namely a sequence of non-increasing integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$.

Assume that $\lambda_1 = \dots = \lambda_{r_1} > \lambda_{r_1+1} = \dots = \lambda_{r_2} > \dots > \lambda_{r_{m-1}+1} = \lambda_{r_m}$ for an increasing sequence $0 < r_1 < r_2 < \dots < r_m$. Let $m_\lambda(x)$ be the symmetric polynomials defined by the symmetrization

$$m_\lambda = \frac{1}{r_1!(r_2 - r_1)! \dots (r_m - r_{m-1})!} \sum_{\sigma \in \mathcal{S}_n} \sigma(x_1^{\lambda_1} \dots x_n^{\lambda_n})$$

under the permutation group $\mathcal{S}_n$ of degree n .

The symmetric lexicographic ordering $\hat{\mathcal{O}}$ can be introduced for the partitions as follows. The partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is greater than the partition $\mu = (\mu_1, \dots, \mu_n)$ if there exists a positive integer r such that $\lambda_1 = \mu_1, \dots, \lambda_{r-1} = \mu_{r-1}$ and $\lambda_r > \mu_r$.

The symmetric moments are defined as

$$\hat{c}_\lambda = \frac{1}{n!} \int_{\mathbf{R}^n} m_\lambda(x) d\rho(x). \quad (8.1)$$

Let $\hat{\mathcal{H}}_\rho$ be the Hilbert space consisting of symmetric functions on $\mathbf{R}^n$ with the inner product $(f, g)_\rho$ and the norm $\|f\|_\rho = \sqrt{(f, f)_\rho}$,

$$(f, g)_\rho = \frac{1}{n!} \int_{\mathbf{R}^n} f(x)g(x) d\rho(x) \quad (8.2)$$

for functions $f(x), g(x)$ on $\hat{\mathcal{D}}$.

Let $\hat{N}$ be the bijective mapping from the set of positive integers onto the set of all partitions such that $\hat{N}(l) > \hat{N}(m)$ for $l > m$. Hence $\hat{N}(1) = (0, \dots, 0)$, $\hat{N}(2) = (1, 0, \dots, 0)$, $\hat{N}(3) = (1, 1, 0, \dots, 0) \dots$, $\hat{N}(n+1) = (1, 1, \dots, 1)$, $\hat{N}(n+2) = (2, 0, \dots, 0)$, $\hat{N}(n+3) = (2, 1, 0, \dots, 0)$, $\hat{N}(n+4) = (2, 1, 1, 0, \dots, 0)$, $\hat{N}(2n+1) = (2, 1, 1, \dots, 1) \dots$, so that we have

$m_{\hat{N}(1)}(x) = 1$, $m_{\hat{N}(2)}(x) = x_1 + \dots + x_n$, $m_{\hat{N}(3)}(x) = \sum_{1 \leq i < j \leq n} x_i x_j$, $m_{\hat{N}(n+1)}(x) = x_1 \dots x_n$, $m_{\hat{N}(n+2)}(x) = \sum_{j=1}^n x_j^2$, etc.

The generalized Hankel matrix $\hat{C}$ are defined with the $\hat{N}(l), \hat{N}(m)$ th elements $\hat{c}_{\hat{N}(l)+\hat{N}(m)}$.

We denote the determinants for each $\hat{N}(l) = \lambda$,

$$\hat{D}_{\hat{N}(l)} = \det((\hat{c}_{\hat{N}(r)+\hat{N}(s)})_{r,s=1}^l) \quad (8.3)$$

The symmetric orthogonal polynomials $\tilde{p}_\lambda(x)$ parametrized by the partitions $\hat{N}(l) = \lambda$ are given by the formulae

$$\tilde{p}_\lambda(x) = \frac{1}{\hat{D}_{N(l-1)}} \begin{vmatrix} \hat{c}_{\hat{N}(1)} & \hat{c}_{\hat{N}(2)} & \cdots & \hat{c}_{\hat{N}(l)} \\ \hat{c}_{\hat{N}(2)} & \hat{c}_{\hat{N}(1)+\hat{N}(2)} & \cdots & \hat{c}_{\hat{N}(2)+\hat{N}(l)} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{c}_{\hat{N}(l-1)} & \hat{c}_{\hat{N}(l-1)+\hat{N}(2)} & \cdots & \hat{c}_{\hat{N}(l-1)+\hat{N}(l)} \\ m_{\hat{N}(1)}(x) & m_{\hat{N}(2)}(x) & \cdots & m_{\hat{N}(l)}(x) \end{vmatrix} \quad (8.4)$$

$$= m_{\hat{N}(l)}(x) + (\text{lower order symmetric polynomials}) \quad (8.5)$$

The orthogonality and the norms are given by

$$(\tilde{p}_\lambda(x), \tilde{p}_\mu(x))_\rho = 0 \quad \lambda \neq \mu \quad (8.6)$$

$$= \frac{\hat{D}_{\hat{N}(l)}}{\hat{D}_{\hat{N}(l-1)}} \quad \lambda = \mu \quad (8.7)$$

so that

$$\hat{p}_{\hat{N}(l)}(x) = \sqrt{\frac{\hat{D}_{\hat{N}(l-1)}}{\hat{D}_{\hat{N}(l)}}} \tilde{p}_{\hat{N}(l)}(x) \quad (8.8)$$

are the orthonormal polynomials having the properties

$$\begin{aligned} (\hat{p}_\lambda(x), \hat{p}_\mu(x))_\rho &= 0 \quad \lambda \neq \mu \\ &= 1 \quad \lambda = \mu \end{aligned} \quad (8.9)$$

and

$$p_{\hat{N}(l)}(x) = \sqrt{\frac{\hat{D}_{\hat{N}(l-1)}}{\hat{D}_{\hat{N}(l)}}} m_{\hat{N}(l)}(x) + \text{lower order symmetric polynomials} \quad (8.10)$$

Let e_r ($1 \leq r \leq n$) be the elementary symmetric polynomials $e_r = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} x_{i_2} \dots x_{i_r}$. We define the bounded linear operators $\hat{A}_r$ (Pieri operators) on $\hat{\mathcal{H}}_\rho$

$$\hat{A}_r : f(x) \in \hat{\mathcal{H}}_\rho \rightarrow e_r(x)f(x) \in \hat{\mathcal{H}}_\rho \quad (8.11)$$

They can be expressed in matrix form as

$$e_r(x)\hat{p}_\lambda(x) \rightarrow \sum_{\mu} \hat{a}_{\lambda,\mu}^{(r)} \hat{p}_\mu(x) \quad (8.12)$$

Let $f(x)$ be a symmetric polynomial in x , such that $f(x)$ can be expressed as a polynomial F in $e_1, \dots, e_n$ $f(x) = F(e_1, \dots, e_n)$

The multiplication operator by $f(x)$ on $\hat{\mathcal{H}}_\rho$ can be expressed as $F(\hat{A}_1, \dots, \hat{A}_n)$.

We assume that $f(x)$ is positive in $\hat{\mathcal{D}}$ so that $F(A_1, \dots, A_n)$ is a positive definite operator on $\hat{\mathcal{H}}_\rho$.

Let $d\rho'(x) = f(x)d\rho(x)$ be another positive Radon measure on $\mathbf{R}^n$ with the same support $\mathcal{D}$ as $d\rho(x)$.

We denote by $\hat{D}'_{\hat{N}(l)}$ the determinant $\det((\tilde{c}'_{\hat{N}(r)+\hat{N}(s)})_{r,s=1}^l)$. We can define the orthogonal polynomials $\tilde{p}'_\lambda(x)$ and $\hat{p}'_\lambda(x)$ in the same way as (3.4), (3.7) respectively.

$$\hat{p}'_{N(l)}(x) = \sqrt{\frac{\hat{D}'_{N(l-1)}}{\hat{D}'_{N(l)}}} \tilde{p}'_{N(l)}(x) \quad (8.13)$$

and

$$\begin{aligned} (\hat{p}'_\lambda(x), \tilde{p}'_\mu(x))_{\rho'} &= 0 \quad \lambda \neq \mu \\ &= 1 \quad \lambda = \mu \end{aligned} \quad (8.14)$$

We have the connection relations between $\{\hat{p}_{\hat{N}(l)}(x)\}_{l \geq 1}$ and $\{\hat{p}'_{\hat{N}(l)}(x)\}_{l \geq 1}$ in the following form

$$\hat{p}_{\hat{N}(l)}(x) = \sum_{m=1}^l \hat{R}_{\hat{N}(l)/\hat{N}(m)} \hat{p}'_{\hat{N}(m)}(x) \quad (8.15)$$

$$\tilde{p}_{\hat{N}(l)}(x) = \sum_{m=1}^l \tilde{R}_{\hat{N}(l)/\hat{N}(m)} \tilde{p}'_{\hat{N}(m)}(x) \quad (8.16)$$

where

$$\hat{R}_{\hat{N}(l),\hat{N}(m)} = \sqrt{\frac{\hat{D}_{\hat{N}(l-1)} \hat{D}'_{\hat{N}(m)}}{\hat{D}_{\hat{N}(l)} \hat{D}'_{\hat{N}(m-1)}}} \tilde{R}_{\hat{N}(l)/\hat{N}(m)} \quad (8.17)$$

In particular,

$$\hat{R}_{\hat{N}(l),\hat{N}(l)} = \sqrt{\frac{\hat{D}'_{\hat{N}(l)} \hat{D}_{\hat{N}(l-1)}}{\hat{D}_{\hat{N}(l)} \hat{D}'_{\hat{N}(l-1)}}} > 0 \quad (8.18)$$

because $\tilde{R}_{\hat{N}(l),\hat{N}(l)} = 1$. Let $\hat{R}$ and $\tilde{R}$ be the corresponding lower triangular operators which are both invertible.

Theorem 5 *We have the LR transforms*

$$F(\hat{A}_1, \dots, \hat{A}_n) = \hat{R} \cdot {}^t \tilde{R} \quad (8.19)$$

$$\hat{A}_r' = \hat{R}^{-1} \cdot \hat{A}_r \cdot \hat{R} \quad (8.20)$$

Example 2. (Koornwinder polynomials, Heckman-Opdam BC_2 -type Polynomials) (see [12],[18].)

Let $d\rho(x) = (1-x_1)^\alpha(1-x_2)^\alpha(1+x_1)^\beta(1+x_2)^\beta(x_1-x_2)^{2\gamma+1}$ defined on $\mathcal{D} : -1 \leq x_1 \leq x_2 \leq 1$ in the 2-dimensional Euclidean space.

For a partition $\lambda_1 \geq \lambda_2 \geq 0$

$$\hat{c}_{\lambda_1, \lambda_2} = \int_{\mathcal{D}} (x_1^{\lambda_1} x_2^{\lambda_2} + x_2^{\lambda_1} x_1^{\lambda_2}) d\rho(x) \quad \lambda_1 > \lambda_2 \quad (8.21)$$

$$= \int_{\mathcal{D}} (x_1 x_2)^{\lambda_1} d\rho(x) \quad \lambda_1 = \lambda_2 \quad (8.22)$$

One can consider the symmetric orthogonal polynomials $\tilde{p}_{\hat{N}(l)}(x)$ as functions of $u = x_1 + x_2$, $v = x_1 x_2$ (We also denote them by $\tilde{p}_{\hat{N}(l)}^{\alpha, \beta, \gamma}(u, v)$ or simply

by $\tilde{p}_{N(l)}^\alpha(u, v)$, $\tilde{p}_{N(l)}^\gamma(u, v)$, $\tilde{p}_{N(l)}(u, v)$ etc according as we are interested in dependence on α, β or γ .)

$\mathcal{D}$ is defined by the inequalities

$$1 - u + v \geq 0, \quad 1 + u + v \geq 0, \quad u^2 - 4v \geq 0$$

The symmetric lexicographic ordering $\hat{\mathcal{O}}$ with respect to x_1, x_2 coincides with the lexicographic ordering $\mathcal{O}$ with respect to u, v .

Let

$$1, u, v, u^2, uv, v^2, u^3, u^2v, uv^2, v^3, \dots \quad (8.23)$$

be the sequence of monomials in the lexicographic order.

$$\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma}(u, v) = u^{\lambda_1 - \lambda_2} v^{\lambda_2} + (\text{lower order terms})$$

are the monic orthogonal polynomials in u, v , obtained by Gram-Schmidt orthogonalization with respect to the inner product

$$(f(u, v), g(u, v))_\gamma = \int_{\mathcal{D}} f(u, v) g(u, v) \mu^{\alpha, \beta, \gamma}(u, v) du dv \quad (8.24)$$

where $\mu^{\alpha, \beta, \gamma}(u, v)$ denotes the density

$$\mu^{\alpha, \beta, \gamma}(u, v) = 2^{2\alpha+2\beta+2\gamma+3} (1 - u + v)^\alpha (1 + u + v)^\beta (u^2 - 4v)^\gamma \quad (8.25)$$

The first moment can be evaluated by using the 2 dimensional Selberg integral formula :

$$\begin{aligned} c_{0,0} &= \hat{c}_{0,0} = 2^{2\alpha+2\beta+2\gamma+2} \\ &\cdot \frac{\Gamma(\alpha + \gamma + \frac{3}{2})\Gamma(\beta + \gamma + \frac{3}{2})\Gamma(\alpha + 1)\Gamma(\beta + 1)\Gamma(2\gamma + 2)}{\Gamma(\alpha + \beta + 2\gamma + 3)\Gamma(\alpha + \beta + \gamma + \frac{5}{2})\Gamma(\gamma + \frac{3}{2})} \end{aligned} \quad (8.26)$$

The moments $c_{i,j} = \int_{\mathcal{D}} u^i v^j \mu^{\alpha, \beta, \gamma} du dv$ can be expressed explicitly as follows (see [1].)

We put $U = 1 - u + v$ and $V = 1 + u + v$. Remark first that

$$\int_{\mathcal{D}} U^r V^s \mu^{\alpha, \beta, \gamma} du dv = c_{0,0} \frac{(\alpha + \gamma + \frac{3}{2})_r (\beta + \gamma + \frac{3}{2})_s (\alpha + 1)_r (\beta + 1)_s}{(\alpha + \beta + 2\gamma + 3)_{r+s} (\alpha + \beta + \gamma + \frac{5}{2})_{r+s}} \quad (8.27)$$

where $(\alpha)_r$ denotes the product $\alpha(\alpha+1)\cdots(\alpha+r-1)$. Hence

$$\begin{aligned} c_{i,j} &= \int_{\mathcal{D}} 2^{-i-j} (U-V)^i (U+V-2)^j \mu^{\alpha+\beta+\gamma} du dv \\ &= 2^{-i-j} c_{0,0} \sum_{i \geq \nu_1 \geq 0, j \geq \nu_2 + \nu_3} (-1)^{j+\nu_1-\nu_2-\nu_3} \\ &\quad \frac{(\alpha+\gamma+\frac{3}{2})_{\nu_1+\nu_2} (\beta+\gamma+\frac{3}{2})_{i-\nu_1+\nu_3} (\alpha+1)_{\nu_1+\nu_2} (\beta+1)_{i-\nu_1+\nu_3}}{(\alpha+\beta+2\gamma+3)_{i+\nu_2+\nu_3} (\alpha+\beta+\gamma+\frac{5}{2})_{i+\nu_2+\nu_3}} \end{aligned}$$

The norm $(\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma}, \tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma})_{\gamma}$ has been evaluated by Heckman-Opdam in the form of a product of Gamma functions. The following expression is given by van Diejen (see [8]).

$$(\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma}, \tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma})_{\gamma} = 2^{2\alpha+2\beta+2\gamma+2+2(\lambda_1+\lambda_2)+3} \Delta_+(\alpha, \beta, \gamma) \Delta_-(\alpha, \beta, \gamma) \quad (8.28)$$

where

$$\begin{aligned} &\Delta_+(\alpha, \beta, \gamma) \\ &= \frac{\Gamma(\alpha+\beta+\gamma+\lambda_1+\frac{3}{2})\Gamma(\alpha+\gamma+\lambda_1+\frac{3}{2})\Gamma(\alpha+\beta+\lambda_2+1)\Gamma(\alpha+\lambda_2+1)}{\Gamma(\alpha+\beta+2\gamma+2\lambda_1+2)\Gamma(\alpha+\beta+2\lambda_2+1)} \\ &\quad \frac{\Gamma(\alpha+\beta+2\gamma+\lambda_1+\lambda_2+2)\Gamma(2\gamma+\lambda_1-\lambda_2+1)}{\Gamma(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{3}{2})\Gamma(\gamma+\lambda_1-\lambda_2+\frac{1}{2})} \\ &\Delta_-(\alpha, \beta, \gamma) \\ &= \frac{\Gamma(\gamma+\lambda_1+\frac{3}{2})\Gamma(\beta+\gamma+\lambda_1+\frac{3}{2})\Gamma(\lambda_2+1)\Gamma(\beta+\lambda_2+1)}{\Gamma(\alpha+\beta+2\gamma+2\lambda_1+3)\Gamma(\alpha+\beta+2\lambda_2+2)} \\ &\quad \frac{\Gamma(\alpha+\beta+\lambda_1+\lambda_2+2)\Gamma(\lambda_1-\lambda_2+1)}{\Gamma(\alpha+\beta+\gamma+\lambda_1+\lambda_2+\frac{5}{2})\Gamma(\gamma+\lambda_1-\lambda_2+\frac{3}{2})} \end{aligned}$$

From (6.7), $D_{N(l)}$ is evaluated by the identities

$$D_{N(l)} = \prod_{m=1}^l (\tilde{p}_{N(m)}^{\alpha, \beta, \gamma}, \tilde{p}_{N(m)}^{\alpha, \beta, \gamma})_{\gamma}$$

The operators A_1 , A_2 and $B = A_1^2 - 4A_2$ are defined by

$$f(u, v) \rightarrow uf(u, v) \quad (8.29)$$

$$f(u, v) \rightarrow vf(u, v) \quad (8.30)$$

$$f(u, v) \rightarrow (u^2 - 4v)f(u, v) \quad (8.31)$$

respectively.

They can be expressed in recurrence form by the use of orthogonal polynomials as follows (see [9], [29] for explicit forms.)

$$\begin{aligned} & A_1 : u\tilde{p}_{\lambda_1, \lambda_2}(x) \\ &= a_{\lambda_1, \lambda_2; \lambda_1+2, \lambda_2}^{(1)} \tilde{p}_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2}(x) + a_{\lambda_1, \lambda_2; \lambda_1+2, \lambda_2}^{(1)} \tilde{p}_{\lambda_1+2, \lambda_2}(x) \\ &+ a_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2-1}^{(1)} \tilde{p}_{\lambda_1+1, \lambda_2-1}(x) + a_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2-1}^{(1)} \tilde{p}_{\lambda_1-1, \lambda_2-1}(x) \\ &+ a_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2+1}^{(1)} \tilde{p}_{\lambda_1+1, \lambda_2+1}(x) + a_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2}^{(1)} \tilde{p}_{\lambda_1-1, \lambda_2}(x) \\ &+ a_{\lambda_1, \lambda_2; \lambda_1-2, \lambda_2}^{(1)} \tilde{p}_{\lambda_1-2, \lambda_2}(x) + a_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2+1}^{(1)} \tilde{p}_{\lambda_1-1, \lambda_2+1}(x) \end{aligned} \quad (8.32)$$

$$\begin{aligned} & A_2 : v\tilde{p}_{\lambda_1, \lambda_2}(x) \\ &= a_{\lambda_1, \lambda_2; \lambda_1+2, \lambda_2}^{(2)} \tilde{p}_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2}(x) + a_{\lambda_1, \lambda_2; \lambda_1+2, \lambda_2}^{(2)} \tilde{p}_{\lambda_1+2, \lambda_2}(x) \\ &+ a_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2-1}^{(2)} \tilde{p}_{\lambda_1+1, \lambda_2-1}(x) + a_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2-1}^{(2)} \tilde{p}_{\lambda_1-1, \lambda_2-1}(x) \\ &+ a_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2+1}^{(2)} \tilde{p}_{\lambda_1+1, \lambda_2+1}(x) + a_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2}^{(2)} \tilde{p}_{\lambda_1-1, \lambda_2}(x) \\ &+ a_{\lambda_1, \lambda_2; \lambda_1-2, \lambda_2}^{(2)} \tilde{p}_{\lambda_1-2, \lambda_2}(x) + a_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2+1}^{(2)} \tilde{p}_{\lambda_1-1, \lambda_2+1}(x) \end{aligned} \quad (8.33)$$

$$\begin{aligned} & B : (u^2 - 4v)\tilde{p}_{\lambda_1, \lambda_2}(x) \\ &= b_{\lambda_1, \lambda_2; \lambda_1+2, \lambda_2} \tilde{p}_{\lambda_1, \lambda_2; \lambda_1+2, \lambda_2}(x) + b_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2} \tilde{p}_{\lambda_1+1, \lambda_2}(x) \\ &+ b_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2-1} \tilde{p}_{\lambda_1+1, \lambda_2-1}(x) + b_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2-1} \tilde{p}_{\lambda_1-1, \lambda_2-1}(x) \\ &+ b_{\lambda_1, \lambda_2; \lambda_1+1, \lambda_2+1} \tilde{p}_{\lambda_1+1, \lambda_2+1}(x) + b_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2} \tilde{p}_{\lambda_1-1, \lambda_2}(x) \\ &+ b_{\lambda_1, \lambda_2; \lambda_1-2, \lambda_2} \tilde{p}_{\lambda_1-2, \lambda_2}(x) + b_{\lambda_1, \lambda_2; \lambda_1-1, \lambda_2+1} \tilde{p}_{\lambda_1-1, \lambda_2+1}(x) \end{aligned} \quad (8.34)$$

where the matrices $(a_{\lambda_1, \lambda_2; \mu_1, \mu_2}^{(1)})$, $(a_{\lambda_1, \lambda_2; \mu_1, \mu_2}^{(2)})$, and $(b_{\lambda_1, \lambda_2; \mu_1, \mu_2})$ define the bounded self-adjoint operators A_1 , A_2 , B on $L^2(\mathbb{Z}_{\geq 0}^2)$ respectively.

We have the connection formulae between the $\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma}(x) = \tilde{p}_{\lambda_1, \lambda_2}^{\gamma}(x)$ and $\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma+1}(x) = \tilde{p}_{\lambda_1, \lambda_2}^{\gamma+1}(x)$ as follows.

$$\begin{aligned} \tilde{p}_{\lambda_1, \lambda_2}^{\gamma}(x) &= \tilde{p}_{\lambda_1, \lambda_2}^{\gamma+1}(x) + \tilde{R}_{\lambda_1, \lambda_2/\lambda_1-1, \lambda_2-1} \tilde{p}_{\lambda_1-1, \lambda_2-1}^{\gamma+1}(x) \\ &+ \tilde{R}_{\lambda_1, \lambda_2/\lambda_1-1, \lambda_2+1} \tilde{p}_{\lambda_1-1, \lambda_2+1}^{\gamma+1}(x) \\ &+ \tilde{R}_{\lambda_1, \lambda_2/\lambda_1-1, \lambda_2} \tilde{p}_{\lambda_1-1, \lambda_2}^{\gamma+1}(x) + \tilde{R}_{\lambda_1, \lambda_2/\lambda_1-2, \lambda_2} \tilde{p}_{\lambda_1-2, \lambda_2}^{\gamma+1}(x) \end{aligned} \quad (8.35)$$

We put $\tilde{R}_{\lambda_1, \lambda_2/\mu_1, \mu_2}$ to be 0 for $(\lambda_1, \lambda_2) < (\mu_1, \mu_2)$. We denote by $\tilde{R}$ the lower triangular matrix $\{\tilde{R}_{\lambda_1, \lambda_2/\mu_1, \mu_2}\}_{\lambda_1, \lambda_2/\mu_1, \mu_2}$ thus obtained.

The matrix R is then defined by (6.19).

Let the operators A'_1, A'_2, B' be the corresponding operators for $\gamma + 1$ in place of γ . Then the following formulae of LR-transforms similar to (6.19) and (6.20) hold.

$$B = R \cdot {}^t R \quad (8.36)$$

$$A'_1 = R^{-1} \cdot A_1 \cdot R \quad (8.37)$$

$$A'_2 = R^{-1} \cdot A_2 \cdot R \quad (8.38)$$

These are equivalent to the identities (8.18), (8.19) and (8.20) respectively.

The elements of $\tilde{R}$ can be evaluated in a remarkable way.

Proposition 8

$$\begin{aligned} &\tilde{R}_{\lambda_1, \lambda_2/\lambda_1-1, \lambda_2+1} \\ &= -\frac{(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2)}{(\gamma + \lambda_1 - \lambda_2 - \frac{1}{2})(\gamma + \lambda_1 - \lambda_2 + \frac{1}{2})} \\ &\tilde{R}_{\lambda_1, \lambda_2/\lambda_1-1, \lambda_2} \\ &= \frac{4(\lambda_1 - \lambda_2)(\alpha - \beta)(\alpha + \beta)(\alpha + \beta + \lambda_1 + \lambda_2 + 1)}{(\alpha + \beta + 2\lambda_2)(\alpha + \beta + 2\lambda_2 + 2)(\alpha + \beta + 2\gamma + 2\lambda_1 + 1)(\alpha + \beta + 2\gamma + 2\lambda_1 + 1)} \\ &\tilde{R}_{\lambda_1, \lambda_2/\lambda_1-1, \lambda_2-1} \\ &= -\frac{4\lambda_2(\alpha + \lambda_2)(\beta + \lambda_2)(\alpha + \beta + \lambda_2)}{(\alpha + \beta + \gamma + \lambda_1 + \lambda_2 + \frac{1}{2})(\alpha + \beta + \gamma + \lambda_1 + \lambda_2 + \frac{3}{2})} \\ &\quad \cdot \frac{(\alpha + \beta + \lambda_1 + \lambda_2)(\alpha + \beta + \lambda_1 + \lambda_2 + 1)}{(\alpha + \beta + 2\lambda_2 - 1)(\alpha + \beta + 2\lambda_2)^2(\alpha + \beta + 2\lambda_2 + 1)} \end{aligned}$$

$$\begin{aligned}
& R_{\lambda_1, \lambda_2 / \lambda_1 - 2, \lambda_2} \\
= & \frac{4(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2)(\alpha + \beta + \lambda_1 + \lambda_2)(\alpha + \beta + \lambda_1 + \lambda_2 + 1)}{(\gamma + \lambda_1 - \lambda_2 - \frac{1}{2})(\gamma + \lambda_1 - \lambda_2 + \frac{1}{2})(\alpha + \beta + \gamma + \lambda_1 + \lambda_2 + \frac{1}{2})(\alpha + \beta + \gamma + \lambda_1 + \lambda_2 + 1)} \\
& \cdot \frac{(\alpha + \gamma + \lambda_1 + \frac{1}{2})(\beta + \gamma + \lambda_1 + \frac{1}{2})(\gamma + \lambda_1 + \frac{1}{2})(\alpha + \beta + \gamma + \lambda_1 + \frac{1}{2})}{(\alpha + \beta + 2\gamma + 2\lambda_1)(\alpha + \beta + 2\gamma + 2\lambda_1 + 1)^2(\alpha + \beta + 2\gamma + 2\lambda_1 + 2)}
\end{aligned}$$

The operator $B_+ = 1 - A_1 + A_2$ corresponding to the shift $\alpha \rightarrow \alpha + 1$ is defined by

$$B_+ : \varphi(u, v) \rightarrow (1 - u + v)\varphi(u, v)$$

Its connection formula for $\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma}(x) = \tilde{p}_{\lambda_1, \lambda_2}^{\alpha}(x)$ and $\tilde{p}_{\lambda_1, \lambda_2}^{\alpha+1, \beta, \gamma}(x) = \tilde{p}_{\lambda_1, \lambda_2}^{\alpha+1}(x)$ is given by

$$\begin{aligned}
& \tilde{p}_{\lambda_1, \lambda_2}^{\alpha}(x) = \tilde{p}_{\lambda_1, \lambda_2}^{\alpha+1}(x) + \tilde{R}_{\lambda_1, \lambda_2 / \lambda_1, \lambda_2 - 1} \tilde{p}_{\lambda_1, \lambda_2 - 1}^{\alpha+1}(x) \\
& + \tilde{R}_{\lambda_1, \lambda_2 / \lambda_1 - 1, \lambda_2} \tilde{p}_{\lambda_1 - 1, \lambda_2}^{\alpha+1}(x) + \tilde{R}_{\lambda_1, \lambda_2 / \lambda_1 - 1, \lambda_2 - 1} \tilde{p}_{\lambda_1 - 1, \lambda_2 - 1}^{\alpha+1}(x)
\end{aligned}$$

where

$$\begin{aligned}
& \tilde{R}_{\lambda_1, \lambda_2 / \lambda_1, \lambda_2 - 1} = -2 \frac{\lambda_2(\beta + \lambda_2)}{(\alpha + \beta + 2\lambda_2)(\alpha + \beta + 2\lambda_2 + 1)} \\
& \tilde{R}_{\lambda_1, \lambda_2 / \lambda_1 - 1, \lambda_2 + 1} \\
= & -2 \frac{(\lambda_1 - \lambda_2)(2\gamma + \lambda_1 - \lambda_2)}{(\gamma + \lambda_1 - \lambda_2 + \frac{1}{2})(\gamma + \lambda_1 - \lambda_2 - \frac{1}{2})} \\
& \cdot \frac{(\gamma + \lambda_1 + \frac{1}{2})(\beta + \gamma + \lambda_1 + \frac{1}{2})}{(\alpha + \beta + 2\gamma + 2\lambda_1 + 1)(\alpha + \beta + 2\gamma + 2\lambda_1 + 2)} \\
& \tilde{R}_{\lambda_1, \lambda_2 / \lambda_1 - 1, \lambda_2 - 1} \\
= & 4 \frac{\lambda_2(\beta + \lambda_2)(\gamma + \frac{1}{2})}{(\alpha + \beta + 2\lambda_2)(\alpha + \beta + 2\lambda_2 + 1)(\alpha + \beta + 2\gamma + 2\lambda_1 + 1)} \\
& \cdot \frac{(\beta + \gamma + \lambda_1 + \frac{1}{2})(\alpha + \beta + 2\gamma + \lambda_1 + \lambda_2 + 1)(\alpha + \beta + \lambda_1 + \lambda_2 + 1)}{(\alpha + \beta + 2\gamma + 2\lambda_1 + 2)(\alpha + \beta + \gamma + \lambda_1 + \lambda_2 + \frac{1}{2})(\alpha + \beta + \gamma + \lambda_1 + \lambda_2 + \frac{3}{2})}
\end{aligned}$$

One can obtain a similar formula for the operator $B_- = 1 + A_1 + A_2$ induced by the shift $\beta \rightarrow \beta + 1$, seeing that $\tilde{p}_{\lambda_1, \lambda_2}^{\alpha, \beta, \gamma}(x) = (-1)^{\lambda_1 + \lambda_2} \tilde{p}_{\lambda_1, \lambda_2}^{\beta, \alpha, \gamma}(-x)$.

We have the corresponding LR -transforms

$$\begin{aligned} 1 \mp A_1 + A_2 &= R \cdot {}^tR, \\ A'_1 &= R^{-1} \cdot A_1 \cdot R \\ A'_2 &= R^{-1} \cdot A_2 \cdot R \end{aligned}$$

respectively.

Details of these formulae will be discussed later.

Remark 3 *It seems interesting to generalize our observations to more general case. For example, Koornwinder polynomials have been generalized to BC-type orthogonal polynomials by Heckman-Opdam. Prof. K.Kadell has discussed the connection relations (8.15)-(8.16) in the case of Selberg-Jack polynomials. One may ask if simple product formulae like in Proposition 8 will be given in these cases.*

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