

Spaces of equivariant maps to toric varieties

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Abstract

The main purpose of this note is to announce the recent result in [13] concerning the homotopy type of spaces of algebraic maps from a real projective space to a compact smooth real toric variety. This note is also based on the joint work with Andrzej Kozłowski and Masahiro Ohno [8].

Toric varieties. An irreducible normal algebraic variety X (over $\mathbb{C}$) is called a *toric variety* if it has an algebraic action of the complex algebraic torus $\mathbb{T}_{\mathbb{C}}^r = (\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}_{\mathbb{C}}^r \cdot *$ of some point $*$ in X is dense in X and isomorphic to $\mathbb{T}_{\mathbb{C}}^r$.

A *strong convex rational polyhedral cone* σ in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $\sigma = \{\sum_{k=1}^s a_k \mathbf{n}_k \mid a_k \geq 0\}$, such that the set $\{\mathbf{n}_k\}_{k=1}^s \subset \mathbb{Z}^n$ does not contain any line.

A finite collection Σ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called a *fan* if every face of element of Σ is belongs to Σ and the intersection of any two elements of Σ is a face of each other. It is well-known that a toric variety X is completely characterized up to isomorphism by the fan Σ . We denote by X_{Σ} the corresponding toric variety associated to Σ . A cone σ in $\mathbb{R}^n$ is called *smooth* (reps. *simplicial*) if it is generated by a subset of a basis of $\mathbb{Z}^n$ (resp. a subset of a basis of $\mathbb{R}^n$). A fan Σ is called *complete* if $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$. It is known that X_{Σ} is compact iff Σ is complete, and that X_{Σ} is smooth iff every $\sigma \in \Sigma$ is smooth [4, Theorem 1.3.12]. It is also known that $\pi_1(X_{\Sigma})$ is isomorphic to the quotient of $\mathbb{Z}^n$ by the subgroup generated by $\bigcup_{\sigma \in \Sigma} \sigma \cap \mathbb{Z}^n$. [4, Theorem 12.1.10]. In particular, X_{Σ} is simply connected if it is compact.

Real toric varieties. For a fan Σ , let $X_{\Sigma, \mathbb{R}}$ denote the subspace of X_{Σ} consisting of all real points of X_{Σ} . Alternatively the space $X_{\Sigma, \mathbb{R}}$ is given by replacing the complex numbers $\mathbb{C}$ by the real numbers $\mathbb{R}$ everywhere in the given definitions of a toric variety X_{Σ} [18, Def. 6.1], and it is called a *real toric variety*. Note that $X_{\Sigma, \mathbb{R}}$ with the intersection $X_{\Sigma, \mathbb{R}} = X_{\Sigma} \cap \mathbb{R}P^N$ when X_{Σ} is a toric variety embedded in $\mathbb{C}P^N$.

Homogenous coordinates of $X_{\Sigma, \mathbb{K}}$. We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials, and we assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For polynomials $f_1, \dots, f_s \in$

$\mathbb{K}[z_1, \dots, z_r]$, let $V_{\mathbb{K}}(f_1, \dots, f_s)$ denote the affine variety

$$(1.1) \quad V_{\mathbb{K}}(f_1, \dots, f_s) = \{x \in \mathbb{K}^r \mid f_k(x) = 0 \text{ for each } 1 \leq k \leq s\}.$$

Let $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ denote the set of all one dimensional cones in a fan Σ , and let $\mathbf{n}_k \in \mathbb{Z}^n$ denote the generator of $\rho_k \cap \mathbb{Z}^n$ such that $\rho_k \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k$ (called *the primitive element* of ρ_k) for each $1 \leq k \leq r$. Define the affine variety $Z_{\Sigma, \mathbb{K}} \subset \mathbb{K}^r$ by

$$(1.2) \quad Z_{\Sigma, \mathbb{K}} = V_{\mathbb{K}}(z^{\hat{\sigma}} \mid \sigma \in \Sigma),$$

where $z^{\hat{\sigma}}$ denotes the monomial $z^{\hat{\sigma}} = \prod_{1 \leq k \leq r, \mathbf{n}_k \notin \sigma} z_k \in \mathbb{Z}[z_1, \dots, z_r]$. Let $\mathbb{T}_{\mathbb{K}}^r = (\mathbb{K}^*)^r$ and define the subgroup $G_{\Sigma, \mathbb{K}} \subset \mathbb{T}_{\mathbb{K}}^r$ by

$$(1.3) \quad G_{\Sigma, \mathbb{K}} = \{(\mu_1, \dots, \mu_r) \in \mathbb{T}_{\mathbb{K}}^r \mid \prod_{k=1}^r \mu_k^{\langle \mathbf{m}, \mathbf{n}_k \rangle} = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^n\}.$$

It is known that there is an isomorphism $X_{\Sigma, \mathbb{K}} \cong (\mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}) / G_{\Sigma, \mathbb{K}}$ for $\mathbb{K} = \mathbb{C}$ if the set $\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$ spans $\mathbb{R}^n$, where the group $G_{\Sigma, \mathbb{K}}$ acts on the complement $\mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}$ by the coordinate-wise multiplications and the space $(\mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}) / G_{\Sigma, \mathbb{K}}$ denotes its orbit space.

It is known that $G_{\Sigma, \mathbb{K}}$ acts freely on the complement $\mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}$ if Σ is smooth and $\mathbb{K} = \mathbb{C}$. In this case, for $\mathbb{K} = \mathbb{C}$ there are isomorphisms

$$(1.4) \quad X_{\Sigma, \mathbb{K}} \cong (\mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}) / G_{\Sigma, \mathbb{K}} \quad \text{and} \quad G_{\Sigma, \mathbb{K}} \cong \mathbb{T}_{\mathbb{K}}^{r-n}.$$

Note that (1.4) also holds for $\mathbb{K} = \mathbb{R}$ if Σ is smooth and complete [19, Lemma 7.3].

We say that a set of primitive elements $\{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_k}\}$ is *primitive* if they do not lie in any cone in Σ but every proper subset does. It is known that

$$(1.5) \quad Z_{\Sigma, \mathbb{K}} = \bigcup_{\{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_k}\}: \text{primitive}} V_{\mathbb{K}}(z_{i_1}, \dots, z_{i_k}).$$

So $Z_{\Sigma, \mathbb{K}}$ is a closed variety with real dimension $(r - r_{\min}) \dim_{\mathbb{R}} \mathbb{K}$, where we set

$$(1.6) \quad r_{\min} = \min \{k \in \mathbb{Z}_{\geq 1} \mid \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_k}\} \text{ is primitive}\}.$$

Spaces of continuous maps. For connected spaces X and Y , let $\text{Map}(X, Y)$ be the space of all continuous maps $f : X \rightarrow Y$ and $\text{Map}^*(X, Y)$ the corresponding subspace of all based continuous maps. If $m \geq 2$ and $g \in \text{Map}^*(\mathbb{RP}^{m-1}, X)$, let $F(\mathbb{RP}^m, X; g)$ denote the subspace of $\text{Map}^*(\mathbb{RP}^m, X)$ given by

$$(1.7) \quad F(\mathbb{RP}^m, X; g) = \{f \in \text{Map}^*(\mathbb{RP}^m, X) : f|_{\mathbb{RP}^{m-1}} = g\},$$

where we identify $\mathbb{RP}^{m-1} \subset \mathbb{RP}^m$ by putting $x_m = 0$. It is known that there is a homotopy equivalence $F(\mathbb{RP}^m, X; g) \simeq \Omega^m X$ if it is not an empty set.

Assumptions. From now on, we assume that the following two conditions are satisfied:

(1.7.1) Let Σ be a complete smooth fan in $\mathbb{R}^n$, $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ be the set of all one-dimension cones in Σ , and all primitive elements $\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$ of the fan Σ spans $\mathbb{R}^n$, where $\mathbf{n}_k \in \mathbb{Z}^n$ denotes the primitive element of ρ_k for $1 \leq k \leq r$.

(1.7.2) Let $D = (d_1, \dots, d_r)$ be an r -tuple of positive integers such that $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}$.

Then we can identify $X_{\Sigma, \mathbb{K}} = (\mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}})/G_{\Sigma, \mathbb{K}}$ and we denote by $[a_1, \dots, a_r]$ the corresponding element of $X_{\Sigma, \mathbb{K}}$ for each $(a_1, \dots, a_r) \in \mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}$.

Spaces of polynomials representing algebraic maps. Let $\mathcal{H}_{d, m}^{\mathbb{K}} \subset \mathbb{K}[z_0, \dots, z_m]$ denote the $\mathbb{K}$ -vector subspace consisting of all homogeneous polynomials of degree d . Let $A_D(m)$ denote the space $A_D^{\mathbb{K}}(m) = \mathcal{H}_{d_1, m}^{\mathbb{K}} \times \mathcal{H}_{d_2, m}^{\mathbb{K}} \times \dots \times \mathcal{H}_{d_r, m}^{\mathbb{K}}$ and let $A_{D, \Sigma}^{\mathbb{K}}(m) \subset A_D^{\mathbb{K}}(m)$ denote the subspace

$$(1.8) \quad A_{D, \Sigma}^{\mathbb{K}} = \{(f_1, \dots, f_r) \in A_D^{\mathbb{K}}(m) \mid F(\mathbf{x}) \notin Z_{\Sigma, \mathbb{K}} \text{ for any } \mathbf{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}\},$$

where we set $F(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))$.

Because $(1, 1, \dots, 1) \in \mathbb{K}^r \setminus Z_{\Sigma, \mathbb{K}}$, we choose $x_0 = [1, \dots, 1] \in X_{\Sigma, \mathbb{K}}$ as the base-point of $X_{\Sigma, \mathbb{K}}$. Define the subspace $A_D(m, X_{\Sigma, \mathbb{K}}) \subset A_{D, \Sigma}^{\mathbb{K}}(m)$ by

$$(1.9) \quad A_D(m, X_{\Sigma, \mathbb{K}}) = \{(f_1, \dots, f_r) \in A_{D, \Sigma}^{\mathbb{K}}(m) \mid (f_1(\mathbf{e}_1), \dots, f_r(\mathbf{e}_1)) = (1, 1, \dots, 1)\},$$

where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{m+1}$, and let us choose $[e_1] = [1 : 0 : \dots : 0]$ as the base-point of $\mathbb{RP}^m$. Define the natural map $j'_{D, \mathbb{K}} : A_{D, \Sigma}^{\mathbb{K}}(m) \rightarrow \text{Map}(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}})$ by

$$(1.10) \quad j'_{D, \mathbb{K}}(f_1, \dots, f_r)([\mathbf{x}]) = [f_1(\mathbf{x}), \dots, f_r(\mathbf{x})] \quad \text{for } \mathbf{x} = (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}.$$

Since the space $A_{D, \Sigma}^{\mathbb{K}}(m)$ is connected, the image of $j'_{D, \mathbb{K}}$ lies in a connected component of $\text{Map}(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}})$, which is denoted by $\text{Map}_D(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}})$. This gives the natural map

$$(1.11) \quad j'_{D, \mathbb{K}} : A_{D, \Sigma}^{\mathbb{K}}(m) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}}).$$

Note that $j'_{D, \mathbb{K}}(f_1, \dots, f_r) \in \text{Map}^*(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}})$ if $(f_1, \dots, f_r) \in A_D^{\mathbb{K}}(m, X_{\Sigma})$. Hence, if we set $\text{Map}_D^*(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}}) = \text{Map}^*(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}}) \cap \text{Map}_D(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}})$, we have the natural map

$$(1.12) \quad i_{D, \mathbb{K}} = j'_{D, \mathbb{K}}|_{A_D(m, X_{\Sigma, \mathbb{K}})} : A_D(m, X_{\Sigma, \mathbb{K}}) \rightarrow \text{Map}_D^*(\mathbb{RP}^m, X_{\Sigma, \mathbb{K}}).$$

Suppose that $m \geq 2$ and let us choose a fixed element $(g_1, \dots, g_r) \in A_D(m-1, X_{\Sigma, \mathbb{K}})$. For each $1 \leq k \leq r$, let $B_k^{\mathbb{K}} = \{g_k + z_m h : h \in \mathcal{H}_{d_k-1, m}^{\mathbb{K}}\}$. Then define the subspace $A_D(m, X_{\Sigma, \mathbb{K}}; g) \subset A_D(m, X_{\Sigma, \mathbb{K}})$ by

$$(1.13) \quad A_D(m, X_{\Sigma, \mathbb{K}}; g) = A_D(m, X_{\Sigma, \mathbb{K}}) \cap (B_1^{\mathbb{K}} \times B_2^{\mathbb{K}} \times \dots \times B_r^{\mathbb{K}}).$$

It is easy to see that $i_{D,\mathbb{K}}(f_1, \dots, f_r)|_{\mathbb{RP}^{m-1}} = g$ if $(f_1, \dots, f_r) \in A_D(m, X_{\Sigma,\mathbb{K}}; g)$, where g denotes the map in $\text{Map}_D^*(\mathbb{RP}^{m-1}, X_{\Sigma,\mathbb{K}})$ given by

$$(1.14) \quad g([x_0 : \dots : x_{m-1}]) = [g_1(\mathbf{x}), \dots, g_r(\mathbf{x})] \quad \text{for } \mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbb{R}^m \setminus \{0\}.$$

Then, one can define the map $i'_{D,\mathbb{K}} : A_D(m, X_{\Sigma,\mathbb{K}}; g) \rightarrow F(\mathbb{RP}^m, X_{\Sigma,\mathbb{K}}; g) \simeq \Omega^m X_{\Sigma,\mathbb{K}}$ by

$$(1.15) \quad i'_{D,\mathbb{K}} = i_{D,\mathbb{K}}|_{A_D(m, X_{\Sigma,\mathbb{K}}; g)} : A_D(m, X_{\Sigma,\mathbb{K}}; g) \rightarrow F(\mathbb{RP}^m, X_{\Sigma,\mathbb{K}}; g) \simeq \Omega^m X_{\Sigma,\mathbb{K}}.$$

Now consider the action of $G_{\Sigma,\mathbb{K}}$ on the space $A_{D,\Sigma}^{\mathbb{K}}(m)$ given by the coordinate-wise multiplication and define the space $\widetilde{A}_D(m, X_{\Sigma,\mathbb{K}})$ by the quotient space

$$(1.16) \quad \widetilde{A}_D(m, X_{\Sigma,\mathbb{K}}) = A_{D,\Sigma}^{\mathbb{K}}(m)/G_{\Sigma,\mathbb{K}}.$$

It is easy to see that one can define the map $j_{D,\mathbb{K}} : \widetilde{A}_D(m, X_{\Sigma,\mathbb{K}}) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_{\Sigma,\mathbb{K}})$ by

$$(1.17) \quad j_{D,\mathbb{K}}([f_1, \dots, f_r])([x_0, \dots, x_r]) = [f_1(\mathbf{x}), \dots, f_r(\mathbf{x})] \quad \text{for } \mathbf{x} \in \mathbb{R}^{m+1} \setminus \{0\}.$$

Let $d_{\min}$ and $D_{\mathbb{R}}(d_1, \dots, d_r; m, r)$ be the positive integer defined by

$$(1.18) \quad d_{\min} = \min\{d_1, d_2, \dots, d_r\}, \quad D(d_1, \dots, d_r; m) = (r_{\min} - m - 1)d_{\min} - 2.$$

From now on we write $(X_{\Sigma,\mathbb{K}}, Z_{\Sigma,\mathbb{K}}, G_{\Sigma,\mathbb{K}}) = (X_{\Sigma}, Z_{\Sigma}, G_{\Sigma})$ if $\mathbb{K} = \mathbb{C}$.

The main results. The main results of this note are stated as follows.

Theorem 1.1 ([13]). *Let Σ be a complete smooth fan in $\mathbb{R}^n$, let $\{d_k : 1 \leq k \leq r\}$ be the set of positive integers satisfying the conditions (1.7.1), (1.7.2), and let $X_{\Sigma,\mathbb{R}}$ be a smooth compact real toric variety associated to the fan Σ . Then if $1 \leq m \leq r_{\min} - 2$ and $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, the map*

$$i'_{D,\mathbb{R}} : A_D(m, X_{\Sigma,\mathbb{R}}; g) \rightarrow F(\mathbb{RP}^m, X_{\Sigma,\mathbb{R}}; g) \simeq \Omega^m X_{\Sigma,\mathbb{R}}$$

is a homology equivalence through dimension $D(d_1, \dots, d_r; m)$. □

Theorem 1.2 ([13]). *Under the same assumptions as Theorem 1.1, if $1 \leq m \leq r_{\min} - 2$ and $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, the maps*

$$\begin{cases} j_{D,\mathbb{R}} : \widetilde{A}_D(m, X_{\Sigma,\mathbb{R}}) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_{\Sigma,\mathbb{R}}) \\ i_{D,\mathbb{R}} : A_D(m, X_{\Sigma,\mathbb{R}}) \rightarrow \text{Map}_D^*(\mathbb{RP}^m, X_{\Sigma,\mathbb{R}}) \end{cases}$$

are homology equivalences through dimension $D(d_1, \dots, d_r; m)$. □

Remark 1.3. (i) A map $f : X \rightarrow Y$ is called a *homology equivalence through dimension D* if $f_* : H_k(X, \mathbb{Z}) \xrightarrow{\cong} H_k(Y, \mathbb{Z})$ is an isomorphism for any $k \leq D$.

(ii) Let G be a finite group and let $f : X \rightarrow Y$ be a G -equivariant map between G -spaces X and Y . Then the map $f : X \rightarrow Y$ is called a *G -equivariant homology equivalence through dimension D* if $f_*^H : H_k(X^H, \mathbb{Z}) \xrightarrow{\cong} H_k(Y^H, \mathbb{Z})$ is an isomorphism for any $k \leq D$ and any subgroup $H \subset G$, where $W^H = \{x \in W \mid g \cdot x = x \text{ for any } g \in H\}$ for a G -space W and f^H denotes the restriction map $f^H = f|X^H$.

(iii) Note that the complex conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$ -action on the space X_Σ , and it is easy to see that $(X_\Sigma)^{\mathbb{Z}/2} = X_{\Sigma, \mathbb{R}}$. Similarly, it also induces the $\mathbb{Z}/2$ -actions on the space $A_{D, \mathbb{C}}(m, X_\Sigma)$, $\tilde{A}_{D, \mathbb{C}}(m, X_\Sigma)$, $A_{D, \mathbb{C}}(m, X_\Sigma; g)$. Moreover, if we consider the space $\mathbb{RP}^m$ as a $\mathbb{Z}/2$ -space of the trivial action, the $\mathbb{Z}/2$ -action X_Σ also induces the $\mathbb{Z}/2$ -actions on the spaces $\text{Map}_D^*(\mathbb{RP}^m, X_\Sigma)$, $\text{Map}_D(\mathbb{RP}^m, X_\Sigma)$, $F(\mathbb{RP}^m, X_\Sigma; g)$.

Corollary 1.4 ([8], [13]). *Under the same assumptions as Theorem 1.1, if $2 \leq m \leq r_{\min} - 2$ and $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, the maps*

$$\begin{cases} i'_{D, \mathbb{C}} : A_D(m, X_\Sigma; g) \rightarrow F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \\ j_{D, \mathbb{C}} : \tilde{A}_D(m, X_\Sigma) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_\Sigma) \\ i_{D, \mathbb{C}} : A_D(m, X_\Sigma) \rightarrow \text{Map}_D^*(\mathbb{RP}^m, X_\Sigma) \end{cases}$$

are $\mathbb{Z}/2$ -equivariant homology equivalences through dimension $D(d_1, \dots, d_r; m)$. $\square$

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