

Time regularity in Gevrey classes of solutions to general nonlinear partial differential equations

By

HIDETOSHI TAHARA*

Abstract

The paper considers general nonlinear partial differential equations

$$(E) \quad F\left(t, x, \{(\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j \leq m, |\alpha| \leq L}\right) = 0$$

(with $1 \leq m \leq L$) in Gevrey classes, and gives a sufficient condition for the following assertion to be valid: if a solution $u(t, x)$ is in C^∞ class with respect to the time variable t and in the Gevrey class $\mathcal{E}^{\{s\}}$ in the space variable x , then it is in the Gevrey class $\mathcal{E}^{\{s\}}$ also with respect to the time variable t for a suitable s . In [4] we have discussed this problem in a class of nonlinear partial differential equations; in this paper we will discuss the problem in the general case (E).

§ 1. Introduction

We denote by t the time variable in $\mathbb{R}_t$, and by $x = (x_1, \dots, x_n)$ the space variable in $\mathbb{R}_x^n$. We use the notations: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial_t = \partial/\partial t$, $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i$ ($i = 1, \dots, n$) and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$.

For $\sigma \geq 1$ and an open subset V of $\mathbb{R}_x^n$ we denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} |\partial_x^\alpha f(x)| \leq Ch^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

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*Department of Information and Communication Sciences, Sophia University, Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan. Email: h-tahara@hoffman.cc.sophia.ac.jp

A function in the class $\mathcal{E}^{\{\sigma\}}(V)$ is called a *function of the Gevrey class of order σ* .

If $\sigma = 1$, the class $\mathcal{E}^{\{1\}}(V)$ is nothing but the set of all analytic functions on V and usually it is denoted by $\mathcal{A}(V)$. If $1 < \sigma_1 < \sigma_2 < \infty$ we have

$$\mathcal{A}(V) \subset \mathcal{E}^{\{\sigma_1\}}(V) \subset \mathcal{E}^{\{\sigma_2\}}(V) \subset C^\infty(V).$$

Thus, functions in the class $\mathcal{E}^{\{\sigma_1\}}(V)$ are closer to analytic functions than those in $\mathcal{E}^{\{\sigma_2\}}(V)$; in this sense, we can say that functions in $\mathcal{E}^{\{\sigma_1\}}(V)$ are more regular than those in $\mathcal{E}^{\{\sigma_2\}}(V)$.

For an interval $[0, T] = \{t \in \mathbb{R}; 0 \leq t \leq T\}$ we denote by $C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V))$ the set of all infinitely differentiable functions $u(t, x)$ in $t \in [0, T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual locally convex topology (see [2]).

Similarly, for $s \geq 1$ and $\sigma \geq 1$ we denote by $\mathcal{E}^{\{s, \sigma\}}([0, T] \times V)$ the set of all functions $u(t, x) \in C^\infty([0, T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{(t, x) \in [0, T] \times K} |\partial_t^k \partial_x^\alpha u(t, x)| \leq Ch^{k+|\alpha|} k!^s |\alpha|!^\sigma, \quad \forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

Obviously, we have

$$\mathcal{E}^{\{s, \sigma\}}([0, T] \times V) \subset C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V)).$$

In the case $s = \sigma$ we write $\mathcal{E}^{\{\sigma\}}([0, T] \times V)$ instead of $\mathcal{E}^{\{\sigma, \sigma\}}([0, T] \times V)$.

Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$: the Gevrey class $\mathcal{E}^{\{s_1, \sigma, s_2\}}(\Omega)$ is defined in the same way. In this case, s_1 denotes the Gevrey order in t , σ denotes the Gevrey order in x , and s_2 denotes the Gevrey order in z .

In this paper, we will consider the following nonlinear partial differential equation

$$(1.1) \quad F\left(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j \leq m, |\alpha| \leq L}\right) = 0$$

where $1 \leq m \leq L$ are positive integers, and $F(t, x, \{z_{j, \alpha}\}_{j+|\alpha| \leq m})$ is a suitable function in a Gevrey class (for the precise assumptions, see §2). And, we will consider the following problem on Gevrey regularity in time:

Problem 1.1. Let $u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (1.1); can we have the result $u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, T] \times V)$ for a suitable $s \geq 1$? If this is true, determine the precise bound of the index s of the time regularity.

In the previous paper [4], we have studied this problem for the equation

$$t^\gamma \partial_t^m u = F\left(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j < m, |\alpha| \leq L}\right)$$

(with $\gamma \geq 0$ and $1 \leq m \leq L$). The purpose of this paper is to discuss the problem 1.1 for general equation (1.1).

§ 2. Formulation and main theorem

Let $m \in \mathbb{N}^*$ be fixed, Λ_0 be a finite subset of $\{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j < m\}$, and set $\Lambda = \Lambda_0 \cup \{(m, 0)\}$ and $d = \#\Lambda$. We denote by $z = \{z_{j,\alpha}\}_{(j,\alpha) \in \Lambda}$ the variable in $\mathbb{R}_z^d$. Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, let $F(t, x, z)$ be a C^∞ function on Ω , and let us consider the following nonlinear partial differential equation:

$$(2.1) \quad F(t, x, Du) = 0 \quad \text{with } Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha) \in \Lambda}.$$

Let $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$ be real numbers, V be an open subset of $\mathbb{R}_x^n$, and $T > 0$. Our basic assumptions are:

- $a_1)$ $m \geq 1, s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$;
- $a_2)$ $\Lambda = \Lambda_0 \cup \{(m, 0)\}$ and Λ_0 is a finite subset of $\{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j < m\}$,
- $a_3)$ $F(t, x, z) \in \mathcal{E}^{\{s_1, \sigma, s_2\}}(\Omega)$,
- $a_4)$ $u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (2.1) on $[0, T] \times V$; this includes the property: $(t, x) \in [0, T] \times V \implies (t, x, Du(t, x)) \in \Omega$.

Let us define

Definition 2.1. (1) Let $u(t)$ be a C^∞ -function in a neighborhood of $t = t_0$; we define the order of zero of $u(t)$ at $t = t_0$ (which we denote by $\text{ord}_{t=t_0}(u(t))$) by the following:

$$\text{ord}_{t=t_0}(u(t)) = \min\{k \in \mathbb{N}; u^{(k)}(t_0) \neq 0\}$$

(if $u^{(k)}(t_0) = 0$ for all $k \in \mathbb{N}$ we set $\text{ord}_{t=t_0}(u(t)) = \infty$).

(2) Let W be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n$, and let $f(t, x) \in C^\infty(W)$. We define a $\mathbb{N} \cup \{\infty\}$ -valued function $q(t_0, x_0; f)$ on W in the following way. Take any $(t_0, x_0) \in W$; then $f(t, x_0)$ is a C^∞ -function in a neighborhood of $t = t_0$ and so we can define the order $\text{ord}_{t=t_0}(f(t, x_0))$ of zero of $f(t, x_0)$ at $t = t_0$. We set $q(t_0, x_0; f) = \text{ord}_{t=t_0}(f(t, x_0))$.

Under the conditions $a_1)$, $a_2)$, $a_3)$ and $a_4)$ we set

$$(2.2) \quad \begin{aligned} k_{j,\alpha}(t_0, x_0) &= q(t_0, x_0; (\partial F / \partial z_{j,\alpha})(t, x, Du(t, x))) \\ &= \text{ord}_{t=t_0}((\partial F / \partial z_{j,\alpha})(t, x_0, (Du)(t, x_0))) \end{aligned}$$

(which is the order of zero of $(\partial F / \partial z_{j,\alpha})(t, x_0, (Du)(t, x_0))$ at $t = t_0$) and suppose:

(M) For any $(t_0, x_0) \in [0, T] \times V$ there are $\gamma \in \mathbb{N}$ and a neighborhood V_0 of $x_0 \in V$ which satisfy the following properties:

- 1) $k_{m,0}(t_0, x) = \gamma$ for any $x \in V_0$;
- 2) $k_{j,0}(t_0, x) \geq \gamma - m + j$ for any $x \in V_0$,
- 3) $k_{j,\alpha}(t_0, x) \geq \gamma - m + j + 1$ for any $x \in V_0$, if $|\alpha| > 0$.

In the condition (M), the constant γ may depends on (t_0, x_0) and so we may write $\gamma = \gamma(t_0, x_0)$: this function $\gamma(t, x)$ is locally constant with respect to x . Thus, if (M) is satisfied, we can take any connected neighborhood of x_0 as V_0 in the condition (M).

For any fixed $(t_0, x_0) \in [0, T] \times V$, by using $\gamma = \gamma(t_0, x_0)$ and a connected neighborhood V_0 of x_0 we set

$$k_{j,\alpha}(t_0, V_0) = \min_{x \in V_0} k_{j,\alpha}(t_0, x) \quad (\text{for } (j, \alpha) \in \Lambda_0), \quad \text{and}$$

$$s_0(t_0, V_0) = 1 + \max \left[0, \max_{(j,\alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \right) \right].$$

Note that these $k_{j,\alpha}(t_0, V_0)$ and $s_0(t_0, V_0)$ depend on $(t_0, x_0) \in [0, T] \times V$ and V_0 , and that by 2) and 3) we have the conditions: $k_{j,0}(t_0, V_0) \geq \gamma - m + j$, and $k_{j,\alpha}(t_0, V_0) \geq \gamma - m + j + 1$ if $|\alpha| > 0$. We also note that if $V_1 \subset V_0$ we have $k_{j,\alpha}(t_0, V_1) \geq k_{j,\alpha}(t_0, V_0)$ and so we have $s_0(t_0, V_1) \leq s_0(t_0, V_0)$.

By using these indices $s_0(t_0, V_0)$ (for $(t_0, x_0) \in [0, T] \times V$ and V_0) we define the index $s_0 \geq 1$ by the following:

$$(2.3) \quad s_0 = \sup_{(t_0, x_0) \in [0, T] \times V} \left(\inf_{V_0 \ni x_0} s_0(t_0, V_0) \right).$$

We note:

Lemma 2.2. *Under the above situation, for any $(t_0, x_0) \in [0, T] \times V$ and any sufficient small neighborhood V_0 of x_0 we have $s_0 \geq s(t_0, V_0)$.*

Proof. We set

$$K = \sup_{V_0 \ni x_0} k_{j,\alpha}(t_0, V_0).$$

If $K = \infty$, for any $N > 0$ we can take a V_0 such that $k_{j,\alpha}(t_0, V_0) > N$ and so we have $\min\{k_{j,\alpha}(t_0, V_0) - \gamma + m - j, m - j\} = m - j$ for any sufficiently small V_0 . If $K < \infty$, by the condition that $k_{j,\alpha}(t_0, V_0)$ is a $\mathbb{N}$ -valued function we see that $k_{j,\alpha}(t_0, V_0) = K$ holds for any sufficiently small V_0 . Thus, by the definition of $s_0(t_0, V_0)$ we can conclude that $s_0(t_0, V_0)$ is independent of V_0 if V_0 is sufficiently small. This proves Lemma 2.2. $\square$

The following result is the main theorem of this paper.

Theorem 2.3. *Suppose the conditions $a_1), a_2), a_3), a_4)$ and (M); then we have $u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, T] \times V)$ for any $s \geq \max\{s_0, s_1, s_2\}$.*

§ 3. Proof of Theorem 2.3

Take any $s \geq \max\{s_0, s_1, s_2\}$. To prove Theorem 2.3, it is enough to show the following assertion: *for any $(t_0, x_0) \in [0, T] \times V$ we can find a $\delta > 0$ and a small compact neighborhood V_1 of x_0 such that $u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([t_0, t_0 + \delta] \times V_1)$ (or $u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([t_0 - \delta, t_0] \times V_1)$) holds. By changing the variable $t \rightarrow t - t_0$, we have only to discuss the case $t_0 = 0$.*

Take any $(0, x_0)$ and fix it. Let $\gamma = \gamma(0, x_0)$ and take a sufficiently small connected neighborhood V_0 of x_0 : then by Lemma 2.2 we have $s_0 \geq s(t_0, V_0)$. For simplicity we set $k_{j, \alpha} = k_{j, \alpha}(0, V_0)$ (for $(j, \alpha) \in \Lambda_0$). Take a sufficiently small $T_0 > 0$, and we have $u(t, x) \in C^\infty([0, T_0], \mathcal{E}^{\{\sigma\}}(V_0))$. Since $k_{m, 0}(0, x) = \gamma$ holds on V_0 , we have

$$(3.1) \quad \left[t^{-\gamma} \frac{\partial F}{\partial z_{m, 0}}(t, x, Du) \right] \Big|_{t=0} \neq 0 \quad \text{on } V_0.$$

Moreover, we have $k_{j, 0} \geq \gamma - m + j$, and $k_{j, \alpha} \geq \gamma - m + j + 1$ if $|\alpha| > 0$. By the condition $s \geq \max\{s_0, s_1, s_2\}$ we have $s \geq \max\{s_1, s_2\}$ and

$$(3.2) \quad s \geq 1 + \max \left[0, \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j, \alpha} - \gamma + m - j, m - j\}} \right) \right].$$

We will consider the equation only on $[0, T_0] \times V_0$.

Let us reduce our equation (2.1) to an equation discussed in [4]. First, we take an integer q with

$$(3.3) \quad q \geq \gamma + m + 1$$

and set

$$u(t, x) = \varphi(t, x) + t^q w(t, x) \quad \text{with} \quad \varphi(t, x) = \sum_{k=0}^{q-1} \frac{(\partial_t^k u)(0, x)}{k!} t^k;$$

then we have $\varphi(t, x) \in \mathcal{E}^{\{1, \sigma\}}(\mathbb{R} \times V_0)$ and $w(t, x) \in C^\infty([0, T_0], \mathcal{E}^{\{\sigma\}}(V_0))$. Since

$$Du = D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j, \alpha) \in \Lambda}$$

(where $[\lambda]_0 = 1$ and $[\lambda]_p = \lambda(\lambda - 1) \cdots (\lambda - p + 1)$ for $p \geq 1$), and since $u(t, x)$ is a solution of (2.1) we have

$$(3.4) \quad F\left(t, x, D\varphi + \{t^{q-j}[t\partial_t + q]_j \partial_x^\alpha w\}_{(j, \alpha) \in \Lambda}\right) = 0$$

which is regarded as an equation with respect to $w(t, x)$. To see the result $u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, \delta] \times V_1)$, it is enough to show the condition: $w(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, \delta] \times V_1)$.

Let us do a further reduction. By the formula

$$[\lambda + q]_j = (\lambda + q)(\lambda + q - 1) \cdots (\lambda + q - j + 1) = \sum_{0 \leq i \leq j} c_{j,i} \lambda^i,$$

we define the constants $c_{j,i}$ ($0 \leq i \leq j \leq m$): we see that $c_{j,j} = 1$ holds. Set

$$Z(t, z) = \{Z_{j,\alpha}(t, z)\}_{(j,\alpha) \in \Lambda} \quad \text{with} \quad Z_{j,\alpha}(t, z) = t^{q-j} \sum_{0 \leq i \leq j} c_{j,i} z_{i,\alpha},$$

$$H(t, x, z) = \frac{1}{t^{q+\gamma-m}} \times F(t, x, D\varphi(t, x) + Z(t, z)),$$

$$\Theta^* w = \{(t\partial_t)^j \partial_x^\alpha w\}_{(j,\alpha) \in \Lambda} \quad \text{and} \quad z^* = (\Theta^* w)(0, x_0) \in \mathbb{R}^d.$$

By the definition we have $Z_{j,\alpha}(t, \Theta^* w) = t^{q-j} [t\partial_t + q]_j \partial_x^\alpha w$ for any $(j, \alpha) \in \Lambda$, and so we have $D\varphi(t, x) + Z(t, \Theta^* w) = Du(t, x)$. By the same argument as in [§3, [4]] we have

Lemma 3.1. *Under the above situation we have the following results:*

- (1) Set $\Omega_0 = \{(t, x, z) \in \mathbb{R}_t \times V_0 \times \mathbb{R}_z^d; (t, x, D\varphi(t, x) + Z(t, z)) \in \Omega\}$; then we have $(0, x_0, z^*) \in \Omega_0$ and $H(t, x, z) \in \mathcal{E}^{\{s^*, \sigma, s_2\}}(\Omega_0)$ for $s^* = \max\{s_1, s_2\}$.
- (2) $w(t, x) \in C^\infty([0, T_0], \mathcal{E}^{\{\sigma\}}(V_0))$ is a solution of

$$(3.5) \quad H(t, x, \Theta^* w) = 0, \quad \text{on } [0, T_0] \times V_0,$$

and we have $(t, x, \Theta^* w(t, x)) \in \Omega_0$ for any $(t, x) \in [0, T_0] \times V_0$.

- (3) $H(0, x_0, z^*) = 0$ and $(\partial H / \partial z_{m,0})(0, x_0, z^*) \neq 0$. In addition we have

$$(3.6) \quad \frac{\partial H}{\partial z_{m,0}}(t, x, \Theta^* w(t, x)) \Big|_{t=0} \neq 0 \quad \text{on } V_0.$$

- (4) We set

$$(3.7) \quad q_{j,\alpha} = \min_{x \in V_0} \left(\text{ord}_{t=0}((\partial H / \partial z_{j,\alpha})(t, x, \Theta^* w(t, x))) \right), \quad (j, \alpha) \in \Lambda_0.$$

Then we have $q_{j,\alpha} \geq 1$ if $|\alpha| > 0$. Moreover we see that the condition (3.2) implies

$$(3.8) \quad s \geq 1 + \max \left[0, \max_{(j,\alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{q_{j,\alpha}, m - j\}} \right) \right].$$

Proof. By the condition (M) and (3.3) we have

$$(\partial F / \partial z_{j,\alpha})(t, x, D\varphi(t, x)) = \begin{cases} O(t^{\gamma-m+j}) & \text{on } V_0, \quad \text{if } |\alpha| = 0, \\ O(t^{\gamma-m+j+1}) & \text{on } V_0, \quad \text{if } |\alpha| > 0 \end{cases}$$

(where $f(t, x) = O(t^a)$ means that $f(t, x) = O(t^a)$ uniformly in x (as $t \rightarrow +0$)). In particular, in the case $(j, \alpha) = (m, 0)$ we have $(\partial F / \partial z_{m,0})(t, x, D\varphi(t, x)) = O(t^\gamma)$ and by (3.1) we have

$$\left[t^{-\gamma} \frac{\partial F}{\partial z_{m,0}}(t, x, D\varphi) \right] \Big|_{t=0} \neq 0 \quad \text{on } V_0.$$

Therefore, by (3.4) and Taylor's formula we have

$$\begin{aligned} 0 &= F(t, x, D\varphi(t, x)) + \sum_{(j,\alpha) \in \Lambda} \frac{\partial F}{\partial z_{j,\alpha}}(t, x, D\varphi) O(t^{q-j}) + \sum_{(j,\alpha), (i,\beta) \in \Lambda} O(t^{q-j}) O(t^{q-i}) \\ &= F(t, x, D\varphi(t, x)) + \sum_{(j,\alpha) \in \Lambda} O(t^{\gamma-m+j}) \times O(t^{q-j}) + O(t^{2q-2m}) \\ &= F(t, x, D\varphi(t, x)) + O(t^{q+\gamma-m}) \quad \text{on } V_0 : \end{aligned}$$

this shows that $F(t, x, D\varphi(t, x)) = O(t^{q+\gamma-m})$ holds on V_0 . Hence, by the definition of $H(t, x, z)$ we have

$$\begin{aligned} H(t, x, z) &= \frac{1}{t^{q+\gamma-m}} \times F(t, x, D\varphi(t, x) + Z(t, z)) \\ &= \frac{1}{t^{q+\gamma-m}} \left[F(t, x, D\varphi(t, x)) + \sum_{(j,\alpha)} \frac{\partial F}{\partial z_{j,\alpha}}(t, x, D\varphi) O(t^{q-j}) + O(t^{2q-2m}) \right] \\ &= \frac{1}{t^{q+\gamma-m}} \left[O(t^{q+\gamma-m}) + \sum_{(j,\alpha)} O(t^{\gamma-m+j}) \times O(t^{q-j}) + O(t^{2q-2m}) \right] \end{aligned}$$

(where $f(t, x, z) = O(t^a)$ means that $f(t, x, z) = O(t^a)$ uniformly in (x, z) (as $t \rightarrow +0$)): this proves that $H(t, x, z)$ is well-defined as a C^∞ function on Ω_0 . By Proposition 5.1 (in Appendix) we have the condition: $H(t, x, z) \in \mathcal{E}^{\{s^*, \sigma, s_2\}}(\Omega_0)$ with $s^* = \max\{s_1, s_2\}$. This proves (1).

(2) is clear from the definition of $H(t, x, z)$ and Ω_0 . Since

$$\frac{\partial H}{\partial z_{m,0}}(t, x, \Theta^* w(t, x)) = \frac{1}{t^\gamma} \frac{\partial F}{\partial z_{m,0}}(t, x, Du(t, x))$$

holds, by (3.1) we have the result (3). Since

$$\begin{aligned} \frac{\partial H}{\partial z_{j,\alpha}}(t, x, \Theta^* w(t, x)) &= \frac{1}{t^{q+\gamma-m}} \sum_{l \geq j} \frac{\partial F}{\partial z_{l,\alpha}}(t, x, Du(t, x)) t^{q-l} c_{l,j} \\ &= \sum_{l \geq j} O(t^{k_{l,\alpha} - \gamma + m - l}) \quad \text{on } V_0, \end{aligned}$$

we have the result (4). $\square$

Now, let us apply the implicit function theorem [Theorem 5.2 (with $\sigma_1 = \sigma$ and $\sigma_2 = s_2$) in Appendix] to the functional equation

$$(3.9) \quad H(t, x, z) = 0 \quad \text{in a neighborhood of } (0, x_0, z^*).$$

We write $z^* = (z'^*, z_{m,0}^*)$ and $z = (z', z_{m,0})$. Since

$$H(0, x_0, z^*) = 0 \quad \text{and} \quad (\partial H / \partial z_{m,0})(0, x_0, z^*) \neq 0$$

hold, we can find an open neighborhood Ω_1 of $(0, x_0, z'^*) \in \mathbb{R}_t \times V_0 \times \mathbb{R}^{d-1}$ and a function $G(t, x, z')$ on Ω_1 which satisfy the following properties:

- 1) $G(t, x, z') \in \mathcal{E}^{s^*, \sigma, s_2}(\Omega_1)$;
- 2) $z_{m,0}^* = G(0, x_0, z'^*)$;
- 3) the functional relation (3.9) is equivalent to $z_{m,0} = G(t, x, z')$;
- 4) if we take $\delta > 0$ and an open neighborhood $V_1 \subset V_0$ of x_0 sufficiently small, we have $(t, x, \Theta w(t, x)) \in \Omega_1$ for any $(t, x) \in [0, \delta] \times V_1$ and the function $w(t, x) \in C^\infty([0, \delta], \mathcal{E}^{\{\sigma\}}(V_1))$ is a solution of the equation

$$(3.10) \quad (t\partial_t)^m w = G(t, x, \Theta w) \quad \text{with} \quad \Theta w = \{(t\partial_t)^j \partial^\alpha w\}_{(j,\alpha) \in \Lambda_0}.$$

Moreover, we have

Lemma 3.2. *We set*

$$q_{j,\alpha}(G) = \min_{x \in V_1} \left(\text{ord}_{t=0}((\partial G / \partial z_{j,\alpha})(t, x, \Theta w(t, x))) \right), \quad (j, \alpha) \in \Lambda_0.$$

Then we have $q_{j,\alpha}(G) \geq 1$ if $|\alpha| > 0$, and that the condition (3.2) implies

$$(3.11) \quad s \geq 1 + \max \left[0, \max_{(j,\alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{q_{j,\alpha}(G), m - j\}} \right) \right].$$

We denote by $s_0(G)$ the right-hand side of (3.11).

Proof. By the implicit function theorem we have

$$\frac{\partial G}{\partial z_{j,\alpha}}(t, x, \Theta w(t, x)) = - \frac{(\partial H / \partial z_{j,\alpha})(t, x, \Theta^* w(t, x))}{(\partial H / \partial z_{m,0})(t, x, \Theta^* w(t, x))}.$$

Since (3.6) holds, we have

$$\begin{aligned} & \text{ord}_{t=0}((\partial G / \partial z_{j,\alpha})(t, x, \Theta w(t, x))) \\ &= \text{ord}_{t=0}((\partial H / \partial z_{j,\alpha})(t, x, \Theta^* w(t, x))), \quad \forall x \in V_1 \end{aligned}$$

and by the condition $V_1 \subset V_0$ we have $q_{j,\alpha}(G) \geq q_{j,\alpha}$ for any $(j, \alpha) \in \Lambda_0$, where $q_{j,\alpha}$ ($(j, \alpha) \in \Lambda_0$) are the ones in (3.7). Thus, by (3.8) we have (3.11). $\square$

Thus, we have seen that $w(t, x)$ is a solution of (3.10). Since (3.10) is just an equation discussed in [4], by [Theorems 5.0.1 and 6.1 in [4]] we have the following result which proves Theorem 2.3.

Theorem 3.3. *Under the above situation, we have $w(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, \delta] \times V_1)$ for any $s \geq \max\{s_0(G), s^*\}$ with $s^* = \max\{s_1, s_2\}$.*

§ 4. Application

In this section, we will consider the equation (2.1) under the assumption

$$(4.1) \quad F(t, x, z) \in \mathcal{E}^{\{\sigma\}}(\Omega)$$

for some $\sigma \geq 1$, and we will consider the following problem:

Problem 4.1. Let $u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (2.1); can we have the result $u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times V)$?

By using Theorem 2.3, let us give a sufficient condition for this problem to be affirmative. We see: by Theorem 2.3 (with $s_1 = s_2 = \sigma$) we have the result $u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times V)$ if the condition $\sigma \geq s_0$ holds, that is, if for any $(t_0, x_0) \in [0, T] \times V$ and a sufficiently small neighborhood V_0 of x_0 we have

$$(4.2) \quad \sigma \geq 1 + \max \left[0, \max_{(j, \alpha) \in \Lambda_0, |\alpha| > 0} \left(\frac{j + \sigma|\alpha| - m}{\min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \right) \right]$$

which is equivalent to

$$(4.3) \quad \begin{aligned} m - j - \min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\} \\ \geq \sigma(|\alpha| - \min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\}) \\ \text{for any } (j, \alpha) \in \Lambda_0 \text{ with } |\alpha| > 0. \end{aligned}$$

If $j + |\alpha| > m$ holds for some $(j, \alpha) \in \Lambda_0$, we have

$$1 + \frac{j + \sigma|\alpha| - m}{\min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\}} \geq 1 + \frac{j + \sigma|\alpha| - m}{m - j} = \frac{\sigma|\alpha|}{m - j} > \sigma.$$

This shows that if $\Lambda_0 \not\subset \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m\}$ the condition (4.2) is not satisfied.

Let us consider the case:

$$(4.4) \quad \Lambda_0 \subset \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m\}.$$

If $|\alpha| \leq \min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\}$ holds, the condition (4.3) is clear from the fact that the right-hand side is nonpositive and the left-hand side is nonnegative. If $|\alpha| > \min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\}$ holds, by the condition $|\alpha| \leq m - j$ we have

$$\begin{aligned} m - j &\geq |\alpha| > \min\{k_{j, \alpha}(t_0, V_0) - \gamma + m - j, m - j\} \\ &= k_{j, \alpha}(t_0, V_0) - \gamma + m - j \end{aligned}$$

and so the inequality (4.3) is equivalent to

$$\sigma \leq \frac{m-j-(k_{j,\alpha}(t_0, V_0) - \gamma + m - j)}{|\alpha| - (k_{j,\alpha}(t_0, V_0) - \gamma + m - j)} = 1 + \frac{m-j-|\alpha|}{|\alpha| - (k_{j,\alpha}(t_0, V_0) - \gamma + m - j)}$$

Therefore, if we set $\Delta = \{(j, \alpha) \in \Lambda_0; k_{j,\alpha}(t_0, V_0) - \gamma + m - j < |\alpha|\}$, our condition (4.2) is equivalent to

$$(4.5) \quad 1 \leq \sigma \leq 1 + \min \left[\infty, \min_{(j,\alpha) \in \Delta} \left(\frac{m-j-|\alpha|}{|\alpha| - (k_{j,\alpha}(t_0, V_0) - \gamma + m - j)} \right) \right].$$

Thus, summinig up we have the following result. Set

$$(4.6) \quad \Delta(t_0, V_0) = \{(j, \alpha); k_{j,\alpha}(t_0, V_0) - \gamma + m - j < |\alpha|\},$$

$$(4.7) \quad \sigma_0(t_0, V_0) = 1 + \min \left[\infty, \min_{(j,\alpha) \in \Delta(t_0, V_0)} \left(\frac{m-j-|\alpha|}{|\alpha| - (k_{j,\alpha}(t_0, V_0) - \gamma + m - j)} \right) \right]$$

where $\gamma = \gamma(t_0, x_0)$. We have:

Theorem 4.2. *Let $\sigma \geq 1$. Suppose the conditions (4.1), (4.4), a_4), (M), and*

$$(4.8) \quad \sigma \leq \inf_{(t_0, x_0) \in [0, T] \times V} \left(\sup_{V_0 \ni x_0} \sigma_0(t_0, V_0) \right) :$$

then we have $u(t, x) \in \mathcal{E}^{\{\sigma\}}([0, T] \times V)$.

§ 5. Appendix

In §3, we have used two results: one is a result on the composition of Gevrey functions, and the other is the implicit function theorem in Gevrey classes. We present here precise formulations and their proofs.

First, let us show a result on the composition of Gevrey functions. We write $t \in \mathbb{R}_t$, $x = (x_1, \dots, x_n) \in \mathbb{R}_x^n$ and $z = (z_1, \dots, z_d) \in \mathbb{R}_z^d$. We have

Proposition 5.1. *Let $s \geq 1$, $s_1 \geq 1$, $s_2 \geq 1$, $\sigma \geq 1$, let Ω and W be open subsets of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$. If the conditions*

- 1) $F(t, x, z) \in \mathcal{E}^{\{s_1, \sigma, s_2\}}(\Omega)$,
- 2) $u_i(t, x, z) \in \mathcal{E}^{\{s, \sigma, s_2\}}(W)$ ($i = 1, \dots, d$),
- 3) $W \ni (t, x, z) \implies (t, x, u(t, x, z)) \in \Omega$, where $u = (u_1, \dots, u_d)$,
- 4) $\sigma \geq s_2$ and $s \geq \max\{s_1, s_2\}$

hold, we have $F(t, x, u(t, x, z)) \in \mathcal{E}^{\{s, \sigma, s_2\}}(W)$.

Proof. Take any compact subset Z of W ; then the image L of Z by the mapping $(t, x, z) \longrightarrow (t, x, u(t, x, z))$ is also a compact subset of Ω . We take constants $A_{p,q,\nu} \geq 0$ and $B_{i,k,\beta,\gamma} \geq 0$ so that

- $\max_{(t,x,z) \in L} \left| \frac{F^{(p,q,\nu)}(t, x, z)}{p!q!\nu!} \right| \leq A_{p,q,\nu} p!^{s_1-1} |q|!^{\sigma-1} |\nu|!^{s_2-1},$
- $\max_{(t,x,z) \in Z} \left| \frac{u_i^{(k,\beta,\gamma)}(t, x, z)}{k!\beta!\gamma!} \right| \leq B_{i,k,\beta,\gamma} (k-1)!^{s-1} |\beta|!^{\sigma-1} |\gamma|!^{s_2-1}, \text{ if } k \geq 1,$
- $\max_{(t,x,z) \in Z} \left| \frac{u_i^{(0,\beta,\gamma)}(t, x, z)}{\beta!\gamma!} \right| \leq B_{i,0,\beta,\gamma} (|\beta|-1)!^{\sigma-1} |\gamma|!^{s_2-1}, \text{ if } |\beta| \geq 1,$
- $\max_{(t,x,z) \in Z} \left| \frac{u_i^{(0,0,\gamma)}(t, x, z)}{\gamma!} \right| \leq B_{i,0,0,\gamma} (|\gamma|-1)!^{s_2-1}, \text{ if } |\gamma| \geq 1,$

where $p \in \mathbb{N}$, $q \in \mathbb{N}^n$, $\nu \in \mathbb{N}^d$, $k \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $\gamma \in \mathbb{N}^d$. We set also

$$G(t, x, z) = \sum_{p+|q|+|\nu| \geq 0} A_{p,q,\nu} t^p x^q z^\nu,$$

$$w_i(t, x, z) = \sum_{k+|\beta|+|\gamma| \geq 1} B_{i,k,\beta,\gamma} t^k x^\beta z^\gamma, \quad i = 1, \dots, d.$$

Then, $G(t, x, z) = G(t, x, z_1, \dots, z_d)$ and $w_i(t, x, z)$ ($i = 1, \dots, d$) are convergent in a neighborhood of $(t, x, z) = (0, 0, 0)$, and so the function

$$H(t, x, z) = G(t, x, w_1(t, x, z), \dots, w_d(t, x, z))$$

$$= \sum_{p+|q|+|\nu| \geq 0} A_{p,q,\nu} t^p x^q \left(\sum_{k_1+|\beta_1|+|\gamma_1| \geq 1} B_{1,k_1,\beta_1,\gamma_1} t^{k_1} x^{\beta_1} z^{\gamma_1} \right)^{\nu_1} \times$$

$$\times \dots \times \left(\sum_{k_d+|\beta_d|+|\gamma_d| \geq 1} B_{d,k_d,\beta_d,\gamma_d} t^{k_d} x^{\beta_d} z^{\gamma_d} \right)^{\nu_d}$$

(with $\nu = (\nu_1, \dots, \nu_d)$) is also convergent in a neighborhood of $(t, x, z) = (0, 0, 0)$. If we set

$$H(t, x, z) = \sum_{(m,\alpha,\mu) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^d} C_{m,\alpha,\mu} t^m x^\alpha z^\mu$$

we have $C_{0,0,0} = A_{0,0,0}$ and for $m + |\alpha| + |\mu| \geq 1$

$$(5.1) \quad C_{m,\alpha,\mu} = \sum_{\substack{1 \leq p+|q|+|\nu| \\ \leq m+|\alpha|+|\mu|}} A_{p,q,\nu} \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)| \geq 1}} \prod_{i=1}^d \prod_{j=1}^{\nu_i} (B_{i,k_i(j),\beta_i(j),\gamma_i(j)}),$$

where $|k^*| = \sum_{i=1}^d (k_i(1) + \cdots + k_i(\nu_i))$ and

$$s(\beta(\nu)) = \sum_{i=1}^d \sum_{j=1}^{\nu_i} \beta_i(j) \in \mathbb{N}^n, \quad s(\gamma(\nu)) = \sum_{i=1}^d \sum_{j=1}^{\nu_i} \gamma_i(j) \in \mathbb{N}^d.$$

Since $H(t, x, z)$ is a holomorphic function in a neighborhood of $(t, x, z) = (0, 0, 0)$, by Cauchy's inequality we have $C_{m, \alpha, \mu} \leq M \eta^{m+|\alpha|+|\mu|}$ ($m + |\alpha| + |\mu| = 0, 1, 2, \dots$) for some $M > 0$ and $\eta > 0$. Therefore, to see the condition $h(t, x, z) = F(t, x, u(t, x, z)) \in \mathcal{E}^{\{s, \sigma, s_2\}}(Z)$ it is sufficient to prove the following inequalities:

$$(5.2) \quad \frac{1}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} \max_{(t, x, z) \in Z} \left| \frac{1}{m! \alpha! \mu!} h^{(m, \alpha, \mu)}(t, x, z) \right| \leq 3^{(m+|\alpha|+|\mu|)(s_2-1)} C_{m, \alpha, \mu} \quad \text{for any } m + |\alpha| + |\mu| \geq 1.$$

Now, let us recall that by Faà di Bruno's formula (see [1]) or [[4], Lemma 4.3] we have

$$\begin{aligned} \frac{1}{m! \alpha! \mu!} h^{(m, \alpha, \mu)} &= \sum_{1 \leq p+|q|+|\nu| \leq m+|\alpha|+|\mu|} \frac{1}{p! q! \nu!} F^{(p, q, \nu)} \times \\ &\quad \times \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)| \geq 1}} \prod_{i=1}^d \prod_{j=1}^{\nu_i} \left(\frac{1}{k_i(j)! \beta_i(j)! \gamma_i(j)!} u_i^{(k_i(j), \beta_i(j), \gamma_i(j))} \right). \end{aligned}$$

Thereofre, by using estimates $(1/p!q!\nu!)|F^{(p, q, \nu)}| \leq A_{p, q, \nu} p!^{s_1-1} |q|!^{\sigma-1} |\nu|!^{s_2-1}$ on L and $(1/k!\beta!\gamma!)|u_i^{(k, \beta, \gamma)}| \leq B_{i, k, \beta, \gamma} (k - \delta_1)!^{s-1} (|\beta| - \delta_2)!^{\sigma-1} (|\gamma| - \delta_3)!^{s_2-1}$ (where $(\delta_1, \delta_2, \delta_3) = (1, 0, 0)$ if $k \geq 1$, $(\delta_1, \delta_2, \delta_3) = (0, 1, 0)$ if $k = 0$ and $|\beta| \geq 1$, and $(\delta_1, \delta_2, \delta_3) = (0, 0, 1)$ if $k = 0$, $\beta = 0$ and $|\gamma| \geq 1$) on Z , we have

$$(5.3) \quad \begin{aligned} &\frac{1}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} \left| \frac{1}{m! \alpha! \mu!} h^{(m, \alpha, \mu)} \right| \\ &\leq \frac{1}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} \sum_{1 \leq p+|q|+|\nu| \leq m+|\alpha|+|\mu|} A_{p, q, \nu} p!^{s_1-1} |q|!^{\sigma-1} |\nu|!^{s_2-1} \times \\ &\quad \times \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)| \geq 1}} \prod_{i=1}^d \prod_{j=1}^{\nu_i} \left(B_{i, k_i(j), \beta_i(j), \gamma_i(j)} (k_i(j) - \delta_1)!^{s-1} \times \right. \\ &\quad \left. \times (|\beta_i(j)| - \delta_2)!^{\sigma-1} (|\gamma_i(j)| - \delta_3)!^{s_2-1} \right) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{1 \leq p+|q|+|\nu| \leq m+|\alpha|+|\mu|} A_{p,q,\nu} \frac{p!^{s_1-1} |q|!^{\sigma-1} |\nu|!^{s_2-1}}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} \times \\
&\quad \times \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)| \geq 1}} (|k^*| - n_1)!^{s-1} (|\beta(\nu)^*| - n_2)!^{\sigma-1} (|\gamma(\nu)^*| - n_3)!^{s_2-1} \\
&\quad \times \prod_{i=1}^d \prod_{j=1}^{\nu_i} \left(B_{i,k_i(j),\beta_i(j),\gamma_i(j)} \right),
\end{aligned}$$

where $|\beta(\nu)^*| = \sum_{i=1}^d \sum_{j=1}^{\nu_i} |\beta_i(j)|$, $|\gamma(\nu)^*| = \sum_{i=1}^d \sum_{j=1}^{\nu_i} |\gamma_i(j)|$, $n_1 = \#\{(i,j); k_i(j) \geq 1\}$, $n_2 = \#\{(i,j); k_i(j) = 0, |\beta_i(j)| \geq 1\}$, and $n_3 = \#\{(i,j); k_i(j) = 0, \beta_i(j) = 0, |\gamma_i(j)| \geq 1\}$. Since $n_1 + n_2 + n_3 = |\nu|$, $s \geq s_2$ and $\sigma \geq s_2$ hold, we have

$$|\nu|!^{s_2-1} \leq (3^{|\nu|} n_1! n_2! n_3!)^{s_2-1} \leq 3^{|\nu|(s_2-1)} n_1!^{s-1} n_2!^{\sigma-1} n_3!^{s_2-1},$$

and so

$$\begin{aligned}
&\frac{p!^{s_1-1} |q|!^{\sigma-1} |\nu|!^{s_2-1}}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} (|k^*| - n_1)!^{s-1} (|\beta(\nu)^*| - n_2)!^{\sigma-1} (|\gamma(\nu)^*| - n_3)!^{s_2-1} \\
&\leq \frac{p!^{s-1} |q|!^{\sigma-1} \times 3^{|\nu|(s_2-1)} n_1!^{s-1} n_2!^{\sigma-1} n_3!^{s_2-1}}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} \times \\
&\quad \times (m - p - n_1)!^{s-1} (|\alpha| - |q| - n_2)!^{\sigma-1} (|\mu| - n_3)!^{s_2-1} \\
&= 3^{|\nu|(s_2-1)} \frac{p!^{s-1} n_1!^{s-1} (m - p - n_1)!^{s-1}}{m!^{s-1}} \\
&\quad \times \frac{|q|!^{\sigma-1} n_2!^{\sigma-1} (|\alpha| - |q| - n_2)!^{\sigma-1}}{|\alpha|!^{\sigma-1}} \frac{n_3!^{s_2-1} (|\mu| - n_3)!^{s_2-1}}{|\mu|!^{s_2-1}} \\
&\leq 3^{|\nu|(s_2-1)} \leq 3^{(m+|\alpha|+|\mu|)(s_2-1)}.
\end{aligned}$$

Thus, by applying this to (5.3) we obtain

$$\begin{aligned}
&\frac{1}{m!^{s-1} |\alpha|!^{\sigma-1} |\mu|!^{s_2-1}} \left| \frac{1}{m! \alpha! \mu!} h^{(m,\alpha,\mu)} \right| \\
&\leq 3^{(m+|\alpha|+|\mu|)(s_2-1)} \sum_{\substack{1 \leq p+|q|+|\nu| \\ \leq m+|\alpha|+|\mu|}} A_{p,q,\nu} \sum_{\substack{|k^*|=m-p \\ s(\beta(\nu))=\alpha-q \\ s(\gamma(\nu))=\mu \\ k_i(j)+|\beta_i(j)|+|\gamma_i(j)| \geq 1}} \prod_{i=1}^d \prod_{j=1}^{\nu_i} \left(B_{i,k_i(j),\beta_i(j),\gamma_i(j)} \right) \\
&= 3^{(m+|\alpha|+|\mu|)(s_2-1)} C_{m,\alpha,\mu}
\end{aligned}$$

on Z . This proves (5.2). $\square$

Some versions of the implicit function theorem in ultra-differentiable classes are given in Komatsu [3] and Yamanaka [5]. For the self-containedness, we will give here an implicit function theorem which is used in §3.

As before, we write $t \in \mathbb{R}_t$, $x = (x_1, \dots, x_n) \in \mathbb{R}_x^n$, $z = (z_1, \dots, z_d) \in \mathbb{R}_z^d$ and $w \in \mathbb{R}_w$. Let Ω be an open neighborhood of $(0, 0, 0, 0) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d \times \mathbb{R}_w$, let $F(t, x, z, w)$ be a C^∞ -function on Ω , let $s_1 \geq 1$, $s_2 \geq 1$, $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$, and suppose: $\sigma_i \geq s_2$ for $i = 1, 2$. We have

Theorem 5.2. *Suppose the following conditions: $F(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$, $F(0, 0, 0, 0) = 0$ and $(\partial F / \partial w)(0, 0, 0, 0) \neq 0$. Then, there are an open neighborhood W of $(0, 0, 0) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$ and a function $\phi(t, x, z) \in C^\infty(W)$ which satisfy $\phi(0, 0, 0) = 0$ and the following properties:*

$$(5.4) \quad W \ni (t, x, z) \implies (t, x, z, \phi(t, x, z)) \in \Omega,$$

$$(5.5) \quad F(t, x, z, \phi(t, x, z)) = 0 \quad \text{on } W.$$

Moreover, we have $\phi(t, x, z) \in \mathcal{E}^{\{s, \sigma_1, \sigma_2\}}(W)$ for any $s \geq \max\{s_1, s_2\}$.

Proof. The former half of the result is nothing but the result of the implicit function theorem in the C^∞ class, and so we know that there are an open neighborhood W of $(0, 0, 0) \in \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$ and a function $\phi(t, x, z) \in C^\infty(W)$ which satisfy $\phi(0, 0, 0) = 0$, (5.4) and (5.5).

Let us show that $\phi(t, x, z) \in \mathcal{E}^{\{s, \sigma_1, \sigma_2\}}(W)$ holds for any $s \geq \max\{s_1, s_2\}$. Since the problem is set in a local sense, without loss of generality we may suppose that $(\partial F / \partial w)(t, x, z, w) \neq 0$ holds on Ω . Then, by (5.5) we have the equality

$$\begin{aligned} \phi_t(t, x, z) &= -\frac{F_t(t, x, z, \phi)}{F_w(t, x, z, \phi)}, \\ \phi_{x_i}(t, x, z) &= -\frac{F_{x_i}(t, x, z, \phi)}{F_w(t, x, z, \phi)}, \quad i = 1, \dots, n, \\ \phi_{z_j}(t, x, z) &= -\frac{F_{z_j}(t, x, z, \phi)}{F_w(t, x, z, \phi)}, \quad j = 1, \dots, d, \end{aligned}$$

where $\phi_t = \partial \phi / \partial t$, $F_t = \partial F / \partial t$, $\phi_{x_i} = \partial \phi / \partial x_i$, $F_{x_i} = \partial F / \partial x_i$ and so on. Therefore, if we set $G(t, x, z) = -(F_t / F_w)(t, x, z, w)$, $H_i(t, x, z) = -(F_{x_i} / F_w)(t, x, z, w)$ ($i = 1, \dots, n$) and $K_j(t, x, z, w) = -(F_{z_j} / F_w)(t, x, z, w)$ ($j = 1, \dots, d$) we have the system of equations

$$(5.6) \quad \begin{cases} \phi_t = G(t, x, z, \phi), \\ \phi_{x_i} = H_i(t, x, z, \phi), \quad i = 1, \dots, n, \\ \phi_{z_j} = K_j(t, x, z, \phi), \quad j = 1, \dots, d. \end{cases}$$

Since $F_w(t, x, z, w) \neq 0$ on Ω , by the condition $F(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega)$ we have

$$(5.7) \quad \begin{aligned} &\bullet G(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega), \\ &\bullet H_i(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega), \quad i = 1, \dots, n, \\ &\bullet K_j(t, x, z, w) \in \mathcal{E}^{\{s_1, \sigma_1, \sigma_2, s_2\}}(\Omega), \quad j = 1, \dots, d. \end{aligned}$$

Thus, to complete the proof of Theorem 5.2 it is enough to show

Proposition 5.3. *Let $s_1 \geq 1$, $s_2 \geq 1$, $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$. Suppose the conditions (5.7) and $\sigma_i \geq s_2$ for $i = 1, 2$. If $\phi(t, x, z) \in C^\infty(W)$ is a solution of (5.6), we have $\phi(t, x, z) \in \mathcal{E}^{\{s, \sigma_1, \sigma_2\}}(W)$ for any $s \geq \max\{s_1, s_2\}$.*

Proof of Proposition 5.3. Let Z be a compact subset of W , and let L be the image of Z by the mapping: $(t, x, z) \longrightarrow (t, x, z, \phi(t, x, z)) \in \Omega$. We take $A_{p, q_1, q_2, k} \geq 0$ ($p \in \mathbb{N}$, $q_1 \in \mathbb{N}^n$, $q_2 \in \mathbb{N}^d$ and $k \in \mathbb{N}$), $B_{q_1, q_2, k}^{(i)} \geq 0$ ($i = 1, \dots, n$, $q_1 \in \mathbb{N}^n$, $q_2 \in \mathbb{N}^d$ and $k \in \mathbb{N}$) and $C_{q_2, k}^{(j)} \geq 0$ ($j = 1, \dots, d$, $q_2 \in \mathbb{N}^d$ and $k \in \mathbb{N}$) so that

$$\begin{aligned} & \bullet \max_{(t, x, z, w) \in L} \left| \frac{G^{(p, q_1, q_2, k)}(t, x, z, w)}{p! q_1! q_2! k!} \right| \leq A_{p, q_1, q_2, k} p!^{s_1-1} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1}, \\ & \bullet \max_{(t, x, z, w) \in L} \left| \frac{H_i^{(0, q_1, q_2, k)}(t, x, z, w)}{q_1! q_2! k!} \right| \leq B_{q_1, q_2, k}^{(i)} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1}, \\ & \bullet \max_{(t, x, z, w) \in L} \left| \frac{K_j^{(0, 0, q_2, k)}(t, x, z, w)}{q_2! k!} \right| \leq C_{q_2, k}^{(j)} |q_2|!^{\sigma_2-1} k!^{s_2-1}. \end{aligned}$$

Let us consider the following functional equation:

$$\begin{aligned} (5.8) \quad Y = & t \sum_{p+|q_1|+|q_2|+k \geq 0} A_{p, q_1, q_2, k} t^p x^{q_1} z^{q_2} (3^{s_2-1} Y)^k \\ & + \sum_{i=1}^n x_i \sum_{|q_1|+|q_2|+k \geq 0} B_{q_1, q_2, k}^{(i)} x^{q_1} z^{q_2} (2^{s_2-1} Y)^k + \sum_{j=1}^d z_j \sum_{|q_2|+k \geq 0} C_{q_2, k}^{(j)} z^{q_2} Y^k. \end{aligned}$$

Since this is an analytic functional equation, by the implicit function theorem in holomorphic category we see that (5.8) has a unique holomorphic solution $Y(t, x, z)$ in a neighborhood of $(0, 0, 0) \in \mathbb{C}_t^n \times \mathbb{C}_x^n \times \mathbb{C}_z^d$ satisfying $Y(0, 0, 0) = 0$. Let

$$Y(t, x, z) = \sum_{m+|\alpha_1|+|\alpha_2| \geq 1} Y_{m, \alpha_1, \alpha_2} t^m x^{\alpha_1} z^{\alpha_2},$$

be the Taylor expansion of $Y(t, x, z)$. We see that the coefficients $Y_{m, \alpha_1, \alpha_2}$ ($m + |\alpha_1| + |\alpha_2| \geq 1$) are determined by the following recurrent formulas:

$$Y_{1,0,0} = A_{0,0,0,0}, \quad Y_{0,e_i,0} = B_{0,0,0}^{(i)} \quad (i = 1, \dots, n), \quad Y_{0,0,\epsilon_j} = C_{0,0}^{(j)} \quad (j = 1, \dots, d)$$

(where $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1) \in \mathbb{N}^n$ and $\epsilon_1 = (1, 0, \dots, 0), \dots, \epsilon_d =$

$(0, \dots, 0, 1) \in \mathbb{N}^d$), and for $M = m + |\alpha_1| + |\alpha_2| \geq 2$ we have

$$\begin{aligned}
 (5.9) \quad Y_{m, \alpha_1, \alpha_2} = & \sum_{0 \leq p + |q_1| + |q_2| + k \leq M-1} A_{p, q_1, q_2, k} \sum_{\substack{|\mu^*| = m-p-1 \\ s(\beta_1^*) = \alpha_1 - q_1 \\ s(\beta_2^*) = \alpha_2 - q_2}} \prod_{l=1}^k 3^{s_2-1} Y_{\mu(l), \beta_1(l), \beta_2(l)} \\
 & + \sum_{i=1}^n \sum_{0 \leq |q_1| + |q_2| + k \leq M-1} B_{q_1, q_2, k}^{(i)} \sum_{\substack{|\mu^*| = m \\ s(\beta_1^*) = \alpha_1 - q_1 - e_i \\ s(\beta_2^*) = \alpha_2 - q_2}} \prod_{l=1}^k 2^{s_2-1} Y_{\mu(l), \beta_1(l), \beta_2(l)} \\
 & + \sum_{j=1}^d \sum_{0 \leq |q_2| + k \leq M-1} C_{q_2, k}^{(j)} \sum_{\substack{|\mu^*| = m \\ s(\beta_1^*) = \alpha_1 \\ s(\beta_2^*) = \alpha_2 - q_2 - e_j}} \prod_{l=1}^k Y_{\mu(l), \beta_1(l), \beta_2(l)},
 \end{aligned}$$

where $|\mu^*| = \mu(1) + \dots + \mu(k)$, $s(\beta_1^*) = \sum_{l=1}^k \beta_1(l)$ and $s(\beta_2^*) = \sum_{l=1}^k \beta_2(l)$. We have

Lemma 5.4. *Take any $s \geq \max\{s_1, s_2\}$. Then, we have:*

- (1) $\max_{(t, x, z) \in Z} \left| \frac{\phi^{(m, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} \right| \leq Y_{m, \alpha_1, \alpha_2} (m-1)!^{s-1} |\alpha_1|!^{\sigma_1-1} |\alpha_2|!^{\sigma_2-1}, \text{ if } m \geq 1,$
- (2) $\max_{(t, x, z) \in Z} \left| \frac{\phi^{(0, \alpha_1, \alpha_2)}(t, x, z)}{\alpha_1! \alpha_2!} \right| \leq Y_{0, \alpha_1, \alpha_2} (|\alpha_1| - 1)!^{\sigma_1-1} |\alpha_2|!^{\sigma_2-1}, \text{ if } |\alpha_1| \geq 1,$
- (3) $\max_{(t, x, z) \in Z} \left| \frac{\phi^{(0, 0, \alpha_2)}(t, x, z)}{\alpha_2!} \right| \leq Y_{0, 0, \alpha_2} (|\alpha_2| - 1)!^{\sigma_2-1}, \text{ if } |\alpha_2| \geq 1.$

Proof of Lemma 5.4. We will prove this by induction on $M = m + |\alpha_1| + |\alpha_2|$. By (5.6) we have $|\phi_t| = |G(t, x, z, \phi)| \leq A_{0,0,0,0} = Y_{1,0,0}$, $|\phi_{x_i}| = |H_i(t, x, z, \phi)| \leq B_{0,0,0}^{(i)} = Y_{0,e_i,0}$ ($i = 1, \dots, n$) and $|\phi_{z_j}| = |K_j(t, x, z, \phi)| \leq C_{0,0}^{(j)} = Y_{0,0,e_j}$ ($j = 1, \dots, d$). This proves the case $M = 1$.

Suppose that $M = m + |\alpha_1| + |\alpha_2| \geq 2$. If $m \geq 1$, by (5.6) we have

$$\begin{aligned}
 & \frac{\phi^{(m, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} \\
 &= \frac{\phi_t^{(m-1, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} = \frac{1}{m! \alpha_1! \alpha_2!} \partial_t^{m-1} \partial_x^{\alpha_1} \partial_z^{\alpha_2} G(t, x, z, \phi) \\
 &= \frac{1}{m} \sum_{0 \leq p + |q_1| + |q_2| + k \leq M-1} \frac{G^{(p, q_1, q_2, k)}}{p! q_1! q_2! k!} \sum_{\substack{|\mu^*| = m-p-1 \\ s(\beta_1^*) = \alpha_1 - q_1 \\ s(\beta_2^*) = \alpha_2 - q_2}} \prod_{l=1}^k \frac{\phi^{(\mu(l), \beta_1(l), \beta_2(l))}}{\mu(l)! \beta_1(l)! \beta_2(l)!}
 \end{aligned}$$

and therefore

(5.10)

$$\begin{aligned}
& \left| \frac{\phi^{(m, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} \right| \\
& \leq \frac{1}{m} \sum_{0 \leq p + |q_1| + |q_2| + k \leq M-1} A_{p, q_1, q_2, k} p!^{s_1-1} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} k!^{s_2-1} \times \\
& \times \sum_{\substack{|\mu^*| = m-p-1 \\ s(\beta_1^*) = \alpha_1 - q_1 \\ s(\beta_2^*) = \alpha_2 - q_2}} \prod_{l=1}^k Y_{\mu(l), \beta_1(l), \beta_2(l)} (\mu(l) - \delta_0)!^{s-1} (|\beta_1(l)| - \delta_1)!^{\sigma_1-1} (|\beta_2(l)| - \delta_2)!^{\sigma_2-1}
\end{aligned}$$

where $(\delta_0, \delta_1, \delta_2) = (1, 0, 0)$ if $\mu(l) \geq 1$, $(\delta_0, \delta_1, \delta_2) = (0, 1, 0)$ if $\mu(l) = 0$ and $|\beta_1(l)| \geq 1$, and $(\delta_0, \delta_1, \delta_2) = (0, 0, 1)$ if $\mu(l) = 0$, $\beta_1(l) = 0$ and $|\beta_2(l)| \geq 1$. If we set $n_0 = \#\{l; \mu(l) \geq 1\}$, $n_1 = \#\{l; \mu(l) = 0, |\beta_1(l)| \geq 1\}$ and $n_2 = \#\{l; \mu(l) = 0, \beta_1(l) = 0, |\beta_2(l)| \geq 1\}$, then we have $n_0 + n_1 + n_2 = k$ and

$$k!^{s_2-1} \leq (3^k n_0! n_1! n_2!)^{s_2-1} \leq 3^{k(s_2-1)} n_0!^{s-1} n_1!^{\sigma_1-1} n_2!^{\sigma_2-1}.$$

By applying this to (5.10) and by using $s \geq s_1$ we have

$$\begin{aligned}
& \left| \frac{\phi^{(m, \alpha_1, \alpha_2)}(t, x, z)}{m! \alpha_1! \alpha_2!} \right| \\
& \leq \frac{1}{m} \sum_{0 \leq p + |q_1| + |q_2| + k \leq M-1} A_{p, q_1, q_2, k} p!^{s-1} |q_1|!^{\sigma_1-1} |q_2|!^{\sigma_2-1} \times n_0!^{s-1} n_1!^{\sigma_1-1} n_2!^{\sigma_2-1} \times \\
& \times (m - p - 1 - n_0)!^{s-1} (|\alpha_1| - |q_1| - n_1)!^{\sigma_1-1} (|\alpha_2| - |q_2| - n_2)!^{\sigma_2-1} \\
& \times \sum_{\substack{|\mu^*| = m-p-1 \\ s(\beta_1^*) = \alpha_1 - q_1 \\ s(\beta_2^*) = \alpha_2 - q_2}} \prod_{l=1}^k 3^{s_2-1} Y_{\mu(l), \beta_1(l), \beta_2(l)} \\
& \leq (m-1)!^{s-1} |\alpha_1|!^{\sigma_1-1} |\alpha_2|!^{\sigma_2-1} \times \\
& \times \frac{1}{m} \sum_{0 \leq p + |q_1| + |q_2| + k \leq M-1} A_{p, q_1, q_2, k} \sum_{\substack{|\mu^*| = m-p-1 \\ s(\beta_1^*) = \alpha_1 - q_1 \\ s(\beta_2^*) = \alpha_2 - q_2}} \prod_{l=1}^k 3^{s_2-1} Y_{\mu(l), \beta_1(l), \beta_2(l)} \\
& \leq (m-1)!^{s-1} |\alpha_1|!^{\sigma_1-1} |\alpha_2|!^{\sigma_2-1} \times Y_{m, \alpha_1, \alpha_2}.
\end{aligned}$$

This proves (1).

By using $\phi_{x_i} = H_i(t, x, z, \phi)$ ($i = 1, \dots, n$) and $\phi_{z_j} = K_j(t, x, z, \phi)$ ($j = 1, \dots, d$), we can prove (2) and (3) in the same way. $\square$

Completion of the proof of Proposition 5.3. Since $Y(t, x, z)$ is a holomorphic function in a neighborhood of $(0, 0, 0) \in \mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_z^d$, by Cauchy's inequality we can take $C > 0$ and $h > 0$ so that

$$Y_{m, \alpha_1, \alpha_2} \leq Ch^{m+|\alpha_1|+|\alpha_2|}$$

holds for all $m \in \mathbb{N}$, $\alpha_1 \in \mathbb{N}^n$ and $\alpha_2 \in \mathbb{N}^d$. By combining this with Lemma 5.4 we have the result: $\phi(t, x, z) \in \mathcal{E}^{\{s, \sigma_1, \sigma_2\}}(Z)$. This proves Proposition 5.3. $\square$

This completes the proof of Theorem 5.2. $\square$

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