

# Existence of Periodic Solutions for Periodic Linear Functional Differential Equations in Banach Spaces (II)

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## 1 Introduction

Let  $R$  be a real line and  $E$  a Banach space with a norm  $|\cdot|$ . If  $x : (-\infty, a) \rightarrow E$ , then a function  $x_t : (-\infty, 0] \rightarrow E, t \in (-\infty, a)$ , is defined by  $x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$ . We deal with the linear functional differential equation with infinite delay in the Banach space  $E$ :

$$(L) \quad \frac{dx(t)}{dt} = Ax(t) + B(t, x_t) + F(t).$$

Let  $\mathcal{B}$  be a Banach space, consisting of functions  $\psi : (-\infty, 0] \rightarrow E$ , which satisfies some axioms demonstrated in Section 2. We assume that Eq.(L) always satisfies the following hypothesis(H):

- (i)  $A : \mathcal{D}(A) \subset E \rightarrow E$  is the infinitesimal generator of a  $C_0$ -semigroup  $T(t), t \geq 0$ , on  $E$ ;
- (ii)  $B : R \times \mathcal{B} \rightarrow E$  is continuous and  $B(t, \cdot) : \mathcal{B} \rightarrow E$  is linear;
- (iii)  $F : R \rightarrow E$  is continuous.

If  $B(t, \psi)$  and  $F(t)$  in Eq.(L) are periodic functions with a period  $\omega > 0$ , we denote Eq.(L) by Eq.(P $_{\omega}$ L). If  $F \equiv 0$ , we denote Eq.(L) and Eq.(P $_{\omega}$ L) by Eq.(L $_0$ ) and Eq.(P $_{\omega}$ L $_0$ ), respectively.

Chow and Hale [1] obtained the following two fixed point theorems for a linear affine map on a Banach space. Let  $X$  be a Banach space, and  $T : X \rightarrow X$  a linear affine map  $Tx = Lx + z, x \in X$ , where  $z \in X$  is fixed.

*Theorem A.* *If the range  $R(I - L)$  is closed and if there is an  $x_0 \in X$  such that  $\{x_0, Tx_0, T^2x_0, \dots\}$  is bounded in  $X$ , then  $T$  has a fixed point in  $X$ .*

*Theorem B.* *If there is an  $x_0 \in X$  such that  $\{x_0, Tx_0, T^2x_0, \dots\}$  is relatively compact in  $X$ , then  $T$  has a fixed point in  $X$ .*

Using Theorem A, we showed a result [8, Corollary 4.9] on the existence of periodic solutions of Eq.(P<sub>ω</sub>L). Its proof is based on the fact that, if the point 1 is a normal point of  $L$ , then the range  $R(I - L)$  is closed. More recently, using Theorem B, Hino and Murakami extended our result. The property that  $C_0$ -semigroup  $T(t)$  is compact for  $t > 0$  on  $E$  plays an essential role in their proof given in [4]. In such a direction, Li, Lim and Li [5] have also considered the existence of periodic solutions of Eq.(P<sub>ω</sub>L) with advanced and delay for the case where  $A = 0$  and  $E = R^n$ . However, Theorem B cannot apply even to the case where  $B(t, \cdot)$  is a compact operator for each  $t \in R$ , but either  $A = 0$  in Eq.(P<sub>ω</sub>L), or  $C_0$ -semigroup  $T(t)$  is compact only for  $t \geq t_0$ , where  $t_0$  is a positive constant.

The aim of this paper is to show the existence of periodic solutions for Eq.(P<sub>ω</sub>L) in succession to [8]. In particular, we will discuss directly the closedness of the range  $R(I - L)$  in Theorem A in the manner applicable for the case where the point 1 belongs to the essential spectrum of  $L$ . To do so, indeed, we make use of the theory of semi-Fredholm operators. As a result, we have general statements, Theorem 3.7 and Corollary 3.9, for the case that the phase space  $\mathcal{B} = UC_g$  (see Section 2) is a fading memory space; that is, a uniform fading memory space.

## 2 Preliminaries

First, we will explain the phase space  $\mathcal{B}$ . Let  $\mathcal{B}$  be a normed linear space consisting of some functions mapping  $(-\infty, 0]$  into  $E$ ; the norm in  $\mathcal{B}$  is denoted by  $|\cdot|_{\mathcal{B}}$ . Throughout this paper we assume that  $\mathcal{B}$  satisfies the following axioms.

(B-1) If a function  $x : (-\infty, \sigma + a) \rightarrow E$  is continuous on  $[\sigma, \sigma + a)$  and  $x_{\sigma} \in \mathcal{B}$ , then

(i)  $x_t \in \mathcal{B}$  for all  $t \in [\sigma, \sigma + a)$  and  $x_t$  is continuous in  $t \in [\sigma, \sigma + a)$ ;

(ii)  $H^{-1}|x(t)| \leq |x_t|_{\mathcal{B}} \leq K(t - \sigma) \sup\{|x(s)| : \sigma \leq s \leq t\} + M(t - \sigma)|x_{\sigma}|_{\mathcal{B}}$  for all  $t \in [\sigma, \sigma + a)$ , where  $H > 0$  is constant,  $K : [0, \infty) \rightarrow [0, \infty)$  is

continuous,  $M : [0, \infty) \rightarrow [0, \infty)$  is locally bounded and they are independent of  $x$ .

(B-2) The space  $\mathcal{B}$  is complete.

Let  $BC$  be the set of bounded, continuous functions mapping  $(-\infty, 0]$  into  $E$ , and  $C_{00}$  its subset consisting of functions with compact support. The space  $C_{00}$  is automatically contained in the space  $\mathcal{B}$  due to (B-1)-(i). The space  $BC$  is contained in  $\mathcal{B}$  under the additional axiom (C).

(C) If a uniformly bounded sequence  $\{\phi^n(\theta)\}$  in  $C_{00}$  converges to a function  $\phi(\theta)$  uniformly on every compact set of  $(-\infty, 0]$ , then  $\phi \in \mathcal{B}$  and  $\lim_{n \rightarrow \infty} \|\phi^n - \phi\|_{\mathcal{B}} = 0$ .

In fact,  $BC$  is continuously imbedded into  $\mathcal{B}$ ; put

$$\|\phi\|_{\infty} = \sup\{|\phi(\theta)| : \theta \leq 0\} \quad \text{for } \phi \in BC.$$

**Lemma 2.1 ([3])** *If the phase space  $\mathcal{B}$  satisfies the axiom (C), then there is a constant  $J > 0$  such that  $\|\phi\|_{\mathcal{B}} \leq J\|\phi\|_{\infty}$  for all  $\phi \in BC$ .*

Define operators  $S(t) : \mathcal{B} \rightarrow \mathcal{B}$ ,  $t \geq 0$ , as

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) & -t \leq \theta \leq 0, \\ \phi(t + \theta) & \theta \leq -t, \end{cases}$$

and denote by  $S_0(t)$  be the restriction of  $S(t)$  to  $\mathcal{B}_0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}$ . The phase space  $\mathcal{B}$  is called a fading memory space [3] if the axiom (C) holds and  $S_0(t)\phi \rightarrow 0$  as  $t \rightarrow \infty$  for each  $\phi \in \mathcal{B}_0$ . If  $\mathcal{B}$  is such a space, then  $\|S_0(t)\|$  is bounded for  $t \geq 0$ . In addition, if  $\|S_0(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ , then  $\mathcal{B}$  is called a uniform fading memory space. If the phase space  $\mathcal{B}$  is a fading memory space, then

$$\|x_t\|_{\mathcal{B}} \leq J \sup\{|x(s)| : \sigma \leq s \leq t\} + M\|x_{\sigma}\|_{\mathcal{B}}, \quad (1)$$

where  $M = (1 + HJ) \sup_{t \geq 0} \|S_0(t)\|$ .

**Example.** Take the phase space as  $\mathcal{B} = UC_g$ , the set of continuous functions,  $\phi(\theta)$  such that  $\phi(\theta)/g(\theta)$  is bounded and uniformly continuous on  $(-\infty, 0]$  with the norm

$$\|\phi\| = \sup\{|\phi(\theta)|/g(\theta) : \theta \leq 0\},$$

where  $g(\theta)$  is a positive continuous function such that  $g(\theta) \rightarrow \infty$  as  $\theta \rightarrow -\infty$ . Then  $\|S_0(t)\| = \sup_{s \leq 0} g(s)/g(s-t)$ , and it is a uniform fading memory space if and only if it is a fading memory space (cf.[3, p.191]).

Next, we recall the definition of the semi-Fredholm operator on Banach space  $X$ . A bounded linear operator  $L$  on Banach space  $X$  is said to be semi-Fredholm if the range  $R(L)$  is closed and at least one of  $\text{nul } L := \dim N(L)$ ,  $N(L) = \{x \in X | Lx = 0\}$ , and  $\text{def } L := \dim X/R(L)$  is finite. The set of all semi-Fredholm operators with  $\text{nul } L < \infty$  will be denoted by  $\mathcal{F}_+(X)$ .

Denote by  $F_T$  the set of fixed points of a linear affine map  $T$  given in Introduction. The following fixed point theorem is derived from Theorem A and properties of semi-Fredholm operators.

**Proposition 2.2** *Assume that  $I - L \in \mathcal{F}_+(X)$ . If there is an  $x_0 \in X$  such that  $\{x_0, Tx_0, T^2x_0, \dots\}$  is bounded in  $X$ , then  $F_T \neq \emptyset$ ,  $F_T$  is an affine set and  $\dim F_T = \dim N(I - L) < \infty$ .*

### 3 The phase space $UC_g$ and the existence of periodic solutions

A solution operator  $U(t, 0)$  of Eq.  $(P_\omega L_0)$  endowed with the initial condition  $x_0 = \phi \in \mathcal{B}$  is decomposed as  $U(t, 0)\phi = \hat{T}(t)\phi + K(t, 0)\phi$ , where

$$[\hat{T}(t)\phi](\theta) = \begin{cases} T(t + \theta)\phi(0) & t + \theta \geq 0, \\ \phi(t + \theta) & t + \theta \leq 0. \end{cases}$$

$$[K(t, 0)\phi](\theta) = \begin{cases} \int_0^{t+\theta} T(t + \theta - s)B(s, x_s(\sigma, \phi)) ds & t + \theta \geq 0, \\ 0 & t + \theta \leq 0. \end{cases}$$

In this section, we will show the closedness of the range  $R(I - U(\omega, 0))$  by using the theory of semi-Fredholm operators, where  $U(\omega, 0)$  is the solution operator for Eq.  $(P_\omega L_0)$ . Throughout this section we assume, in addition to the axioms (B-1) and (B-2), the following axiom :

(B-3)  $|\phi^1 - \phi^2|_{\mathcal{B}} = 0$  for  $\phi^1, \phi^2$  in  $\mathcal{B}$  if and only if  $\phi^1(\theta) = \phi^2(\theta)$  for  $\theta \in (-\infty, 0]$ .

**Lemma 3.1** *If the phase space  $\mathcal{B}$  satisfies the axiom (C) and  $T(t)$  is a  $C_0$ -semigroup on  $E$ , then a function  $\phi$  of  $N(I - \hat{T}(\omega))$  is an  $\omega$ -periodic continuous function given by  $\phi(\theta) = T(\theta + n\omega)\phi(0)$ ,  $\theta \in [-n\omega, 0]$ ,  $n = 1, 2, \dots$ , where  $\phi(0) \in N(I - T(\omega))$ , and*

$$\dim N(I - \hat{T}(\omega)) = \dim N(I - T(\omega)).$$

**Proof.** Suppose that  $\hat{T}(\omega)\phi = \phi$ . Since  $[\hat{T}(\omega)\phi](\theta) = \phi(\omega + \theta)$  for  $\omega + \theta \leq 0$ , it follows that  $\phi(\omega + \theta) = \phi(\theta)$  for  $\theta \leq -\omega$ ; that is,  $\phi(\theta)$  is  $\omega$ -periodic on  $(-\infty, 0]$ . Since  $\hat{T}(n\omega) = \hat{T}(\omega)^n$ ,  $n = 0, 1, 2, \dots$ , we have that  $\hat{T}(n\omega)\phi = \phi$ . On the other hand, if  $-n\omega \leq \theta \leq 0$ , then  $[\hat{T}(n\omega)\phi](\theta) = T(n\omega + \theta)\phi(0)$ ; hence,  $T(n\omega + \theta)\phi(0) = \phi(\theta)$  for  $-n\omega \leq \theta \leq 0$  and  $\phi(\theta)$  is continuous on  $[-n\omega, 0]$ . Set  $a = \phi(0)$  and  $x(t) = T(t)a$ ,  $t \geq 0$ . Then  $x(t) = \phi(t - n\omega)$  as long as  $0 \leq t \leq n\omega$ . Since  $n$  may be arbitrary, we can regard that  $x(t)$  is  $\omega$ -periodic and continuous in  $(-\infty, \infty)$ , and  $\phi = x_0$ . Since  $x(\omega) = x(0)$ , it follows that  $T(\omega)a = a$ ; that is,  $a \in N(I - T(\omega))$ .

Conversely, if  $a \in N(I - T(\omega))$ , then  $T(t+\omega)a = T(t)T(\omega)a = T(t)a$ ,  $t \geq 0$ ; that is,  $T(t)a$  is  $\omega$ -periodic in  $[0, \infty)$ . Suppose that  $x(t)$  is the  $\omega$ -periodic extension of  $T(t)a$  to  $(-\infty, \infty)$ , and set  $\phi = x_0$ . From the axiom (C) we see that  $\phi$  belongs to  $\mathcal{B}$ . Then it is obvious that  $\hat{T}(\omega)\phi = \phi$ . Moreover, the space  $N(I - T(\omega))$  is mapped bijectively onto the space  $N(I - \hat{T}(\omega))$ . Therefore, the proof is complete.

Let the null space  $N(I - T(\omega))$  be of finite dimension. Then there exists a closed subspace  $M$  of  $E$  such that  $E = M \oplus N$ , where  $N = N(I - T(\omega))$ , and let  $S_M$  be the restriction of  $I - T(\omega)$  to  $M$ . Then  $S_M : M \rightarrow R(I - T(\omega))$  is a continuous, bijective, linear operator. Thus there is the inverse operator  $S_M^{-1}$  of  $S_M$ . Of course, if  $R(I - T(\omega))$  is closed, then  $S_M^{-1}$  is continuous.

To prove that the range  $R(I - \hat{T}(\omega))$  is closed, we will solve the equation  $(I - \hat{T}(\omega))\phi = \psi$  and use the above notations.

**Proposition 3.2** *Suppose that the phase space  $\mathcal{B}$  satisfies the axiom (C), and that  $\dim N(I - T(\omega)) < \infty$ . Then  $\psi \in R(I - \hat{T}(\omega))$  if and only if  $\psi(0) \in R(I - T(\omega))$  and  $U\psi \in \mathcal{B}$ ,  $\psi \in \mathcal{B}$ , where  $U$  is defined as*

$$[U\psi](\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)S_M^{-1}\psi(0), \quad \theta \in [-k\omega, -(k-1)\omega], \quad (2)$$

for  $k = 1, 2, \dots$ .

**Proof.** First, we formally solve the equation  $(I - \hat{T}(\omega))\phi = \psi$ . The definition of  $\hat{T}(\omega)$  implies that

$$\phi(\theta) - T(\theta + \omega)\phi(0) = \psi(\theta), \quad -\omega \leq \theta \quad \text{and} \quad \phi(\theta) - \phi(\theta + \omega) = \psi(\theta), \quad \theta \leq -\omega.$$

From the first equation it follows that  $(I - T(\omega))\phi(0) = \psi(0)$ , and  $\phi(\theta) = \psi(\theta) + T(\theta + \omega)\phi(0)$  for  $-\omega \leq \theta \leq 0$ . From the second equation, it follows

that, for  $k = 2, 3, \dots$ ,  $\phi(\theta) = \psi(\theta) + \phi(\theta + \omega)$  for  $\theta \in [-k\omega, -(k-1)\omega]$ . Hence the solution  $\phi$  is determined as

$$\phi(\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)\phi(0), \quad \theta \in [-k\omega, -(k-1)\omega], \quad (3)$$

$k = 1, 2, \dots$ , uniquely for  $\phi(0)$ .

Assume that  $\psi \in R(I - \hat{T}(\omega))$ . Then  $\psi(0) \in R(I - T(\omega))$ , and there exists a function  $\hat{\phi} \in \mathcal{B}$  satisfying the equation  $(I - \hat{T}(\omega))\hat{\phi} = \psi$ . Obviously,  $\hat{\phi}(0)$  satisfies the equation  $(I - T(\omega))\hat{\phi}(0) = \psi(0)$ . Furthermore  $\hat{\phi}(0)$  is decomposed as  $\hat{\phi}(0) = S_M^{-1}\psi(0) + \phi_N(0)$ ,  $\phi_N(0) \in N$ ,  $S_M^{-1}\psi(0) \in M$ . Set  $\phi_N(\theta) = T(\theta + k\omega)\phi_N(0)$ ,  $\theta \in [-k\omega, 0]$ ,  $k = 0, 1, 2, \dots$ . Using Lemma 3.1 we see that  $\phi_N$  belongs to  $N(I - \hat{T}(\omega))$ . Hence  $\hat{\phi} = U\psi + \phi_N$ . Needless to say,  $U\psi$  belongs to  $\mathcal{B}$ .

Conversely, assume that  $\psi(0) \in R(I - T(\omega))$  and  $U\psi \in \mathcal{B}$ ,  $\psi \in \mathcal{B}$ . Then  $[(I - \hat{T}(\omega))U\psi](\theta) = \psi(\theta)$  for every  $\theta \in (-\infty, 0]$ ; that is,  $(I - \hat{T}(\omega))U\psi = \psi$ . The proof is complete.

**Theorem 3.3** *Suppose that the phase space  $\mathcal{B}$  satisfies the axiom (C), and that if  $|\phi^n - \phi|_{\mathcal{B}} \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\phi^n(\theta)$  converges to  $\phi(\theta)$  uniformly for  $\theta$  in any compact interval of  $(-\infty, 0]$ . Furthermore, Suppose that  $I - T(\omega) \in \mathcal{F}_+(E)$ . Then  $R(I - \hat{T}(\omega))$  is closed if and only if there exists a positive constant  $c$  such that  $|U\psi|_{\mathcal{B}} \leq c|\psi|_{\mathcal{B}}$ ,  $\psi \in \mathcal{B}$ , as long as  $\psi(0) \in R(I - T(\omega))$  and  $U\psi \in \mathcal{B}$ , where  $U$  is given by (2).*

**Proof.** Set  $D = \{U\psi : \psi \in R(I - \hat{T}(\omega))\}$ . Let  $F$  be the restriction of  $I - \hat{T}(\omega)$  to  $D$ . Then the operator  $F : D \rightarrow R(I - \hat{T}(\omega))$  have the following properties:  $N(F) = \{0\}$ ,  $FU\psi = \psi$  for  $\psi \in R(I - \hat{T}(\omega))$ ,  $R(F) = R(I - \hat{T}(\omega))$ , and  $F$  is a bounded linear operator. If  $F$  is a closed linear operator, then Theorem 3.3 follows from the well known theorem [8, Theorem 5.1, p.70] about the closed range property. If  $D$  is a closed subspace, then  $F$  is a closed operator. But it is difficult to see that  $D$  is closed. So, we show directly that  $F$  is a closed operator. To do so, suppose that a sequence  $\phi^n := U\psi^n$ ,  $n = 1, 2, \dots$  in  $D$  converges to a function  $\phi$  in  $\mathcal{B}$  and the sequence  $F\phi^n = \psi^n$  converges to a function  $\psi$  in  $\mathcal{B}$ . From the assumption in the theorem it follows that  $\phi^n(\theta) \rightarrow \phi(\theta)$ ,  $\psi^n(\theta) \rightarrow \psi(\theta)$  as  $n \rightarrow \infty$  uniformly for  $\theta$  in any compact interval of  $(-\infty, 0]$ . Since  $R(I - T(\omega))$  is closed, we have that  $\psi^n(0) \rightarrow \psi(0)$  as  $n \rightarrow \infty$  and  $\psi(0) \in R(I - T(\omega))$ . Then from the definition of the operator  $U$  it follows that  $U\psi^n(\theta) \rightarrow U\psi(\theta)$  as  $n \rightarrow \infty$  uniformly for  $\theta$  in any compact interval  $(-\infty, 0]$ . This implies that  $U\psi(\theta) = \phi(\theta)$  for all

$\theta \in (-\infty, 0]$ . Since  $\phi \in \mathcal{B}$ , it follows that  $\psi \in R(I - \hat{T}(\omega))$ ,  $\phi = U\psi \in D$  and  $F\phi = \psi$ .

From Theorem 5.1 in [7], Chapter III, it follows that  $R(I - \hat{T}(\omega))$  is closed if and only if there is a positive constant  $c$  such that  $|\phi|_{\mathcal{B}} \leq c|F\phi|_{\mathcal{B}}$  for all  $\phi \in D$ , which means that  $R(I - \hat{T}(\omega))$  is closed if and only if  $|U\psi|_{\mathcal{B}} \leq c|\psi|_{\mathcal{B}}$  for all  $\psi \in R(I - \hat{T}(\omega))$ . From Proposition 3.2 we have the conclusion of the theorem.

Let  $BUC$  be the set of all bounded and uniformly continuous functions from  $(-\infty, 0]$  into  $E$  with the supremum norm.

**Proposition 3.4** *Take the space  $BUC$  as the phase space of  $\hat{T}(\omega)$ . Then  $R(I - \hat{T}(\omega))$  is not closed in general.*

**Proof.** It suffices to show that there exists a sequence  $\{\phi^n\}$  in  $BUC$  such that  $|\phi^n|_{\mathcal{B}} \equiv 1$ , and  $\lim_{n \rightarrow \infty} |(I - \hat{T}(\omega))\phi^n|_{\mathcal{B}} = 0$ . Let  $e$  be a unit vector of  $E$ ; that is,  $|e|_{\mathcal{B}} = 1$ , and define  $x^n(t)$ ,  $n = 1, 2, \dots$ , as

$$x^n(t) = \begin{cases} e & t \leq -n\omega \\ (-t/n\omega)e & -n\omega \leq t \leq 0 \\ 0 & t \geq 0. \end{cases}$$

Set  $\phi^n = x^n$ ,  $n = 1, 2, \dots$ . Since  $\phi^n(0) = 0$ , we have  $[\hat{T}(\omega)(\phi^n)](\theta) = 0$  for  $\theta \in [-\omega, 0]$ ; in other words,  $\hat{T}(\omega)\phi^n = S_0(\omega)\phi^n$ . Thus it follows that  $(I - \hat{T}(\omega))\phi^n = \phi^n - S_0(\omega)\phi^n$ ; hence,  $|(I - \hat{T}(\omega))\phi^n|_{\mathcal{B}} = 1/n \rightarrow 0$  as  $n \rightarrow \infty$ . Clearly,  $|\phi^n|_{\mathcal{B}} \equiv 1$ . Thus this is a desired sequence.

**Theorem 3.5** *If  $\mathcal{B} = UC_g$  is a uniform fading memory space and if  $I - T(\omega) \in \mathcal{F}_+(E)$ , then the range  $R(I - \hat{T}(\omega))$  is closed; hence,  $I - \hat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$ .*

**Proof.** Since  $\mathcal{B} = UC_g$  is a uniform fading memory space, there are  $M_0 \geq 1$  and  $\epsilon_0 > 0$  such that  $\|S_0(t)\| \leq M_0 e^{-\epsilon_0 t}$  for  $t \geq 0$ . Namely,

$$\|S_0(t)\| = \sup_{s \leq 0} \frac{g(s)}{g(s-t)} = \sup_{s \leq -t} \frac{g(s+t)}{g(s)} \leq M_0 e^{-\epsilon_0 t}.$$

Suppose that  $\psi(0) \in R(I - T(\omega))$ ,  $U\psi \in \mathcal{B}$ ,  $\psi \in \mathcal{B}$ . Then we have that, for  $\theta \in [-k\omega, -(k-1)\omega]$ ,  $k \geq 1$ ,

$$\frac{1}{g(\theta)} \left| \sum_{j=0}^{k-1} \psi(\theta + j\omega) \right| \leq \sum_{j=0}^{k-1} \frac{g(\theta + j\omega)}{g(\theta)} \frac{|\psi(\theta + j\omega)|}{g(\theta + j\omega)}.$$

$$\begin{aligned}
&\leq \sum_{j=0}^{k-1} \|S_0(j\omega)\| \|\psi\| \\
&\leq \sum_{j=0}^{k-1} M_0 e^{-\epsilon_0 j\omega} \|\psi\| \leq \frac{M_0 \|\psi\|}{1 - e^{-\epsilon_0 \omega}}.
\end{aligned}$$

On the other hand, since  $S_M^{-1}$  is continuous, we have that

$$\frac{1}{g(\theta)} |T(\theta + k\omega) S_M^{-1} \psi(0)| \leq \sup\{\|T(t)\| : 0 \leq t \leq \omega\} \|S_M^{-1}\| \|\psi\|.$$

Summarizing these inequalities, (2) is estimated as

$$\|U\psi\| \leq \left( \frac{M_0}{1 - e^{-\epsilon_0 \omega}} + \sup\{\|T(t)\| : 0 \leq t \leq \omega\} \|S_M^{-1}\| \right) \|\psi\|, \quad (4)$$

which implies that the range  $R(I - \hat{T}(\omega))$  is closed, because of Theorem 3.3. Since  $I - T(\omega) \in \mathcal{F}_+(E)$ . From this fact and Lemma 3.1 it follows that  $I - \hat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$ , which proves the theorem.

The following result is well known in the theory of semi-Fredholm operators (refer to [2, Theorems 3.21, 3.22, pp.35-37], or [7, Theorems 6.3, 6.4, p.128]).

**Lemma 3.6** *Let  $L \in \mathcal{F}_+(X)$ .*

- 1) *If  $S$  is a compact operator on  $X$ , then  $L \pm S \in \mathcal{F}_+(X)$ .*
- 2) *There is a positive number  $\eta$  such that if  $S$  is a bounded linear operator on  $X$  satisfying  $\|S\| < \eta$ , then  $L \pm S \in \mathcal{F}_+(X)$  and  $\text{nul}(L \pm S) \leq \text{nul} L$ .*

Summarizing these results we can obtain one of main theorems of this paper.

**Theorem 3.7** *Assume that  $\mathcal{B} = UC_g$  is a uniform fading memory space and at least one of the following conditions is satisfied :*

- (i)  *$T(t)$  is a  $C_0$ -compact semigroup on  $E$ .*
- (ii) *For each  $t \in \mathbb{R}$ ,  $B(t, \cdot)$  is a compact operator and  $I - T(\omega) \in \mathcal{F}_+(E)$ . If  $\text{Eq.}(P_\omega L)$  has an  $E$ -bounded solution, then it has an  $\omega$ -periodic solution.*

**Proof.** The proof easily follows from Theorem 3.5, the assertion 1) in Lemma 3.6 and Proposition 2.2.

Finally, we consider the case where the both of  $T(t)$  and  $B(t, \cdot)$  are not compact in general. Set  $\|B\|_\infty := \sup\{\|B(t)\| : 0 \leq t < \infty\}$ , where



$\|B(t)\|$  is the operator norm of  $B(t, \cdot)$ . If  $\mathcal{B}$  is a fading memory space, if  $\|T(t)\| \leq M_w e^{wt}$ ,  $t \geq 0$ , and if  $\|B\|_\infty < \infty$ , then the Gronwall inequality implies that the solution  $x(t, \phi)$  of  $\text{Eq.}(P_\omega L_0)$  such that  $x_0 = \phi$  satisfies  $|x_t(\phi)|_{\mathcal{B}} \leq |\phi|_{\mathcal{B}} N(t; \|B\|_\infty)$  for  $t > 0$ , where

$$N(t; \|B\|_\infty) = (HJM_w + M) \exp\{t(M_w \|B\|_\infty J + \max\{w, 0\})\},$$

and  $M$  is the constant in the inequality (1). We denote by  $\mathcal{S}(\omega)$  the set of  $\omega$ -periodic solutions for  $\text{Eq.}(P_\omega L)$ .

**Theorem 3.8** *Let  $T(t)$  be a  $C_0$ -semigroup on  $E$  such that  $\|T(t)\| \leq M_w e^{wt}$ , and assume that  $I - \hat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$ . Let  $\eta$  be given as in Lemma 3.6 (2) for  $I - \hat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$ , and assume that  $\|B\|_\infty$  is so small as to satisfy the condition*

$$JM_w \|B\|_\infty N(\omega; \|B\|_\infty) \int_0^\omega e^{ws} ds < \eta.$$

*If  $\text{Eq.}(P_\omega L)$  has an  $E$ -bounded solution, then  $\mathcal{S}(\omega)$  is nonempty and*

$$\dim \mathcal{S}(\omega) \leq \dim N(I - \hat{T}(\omega)) < \infty.$$

*Proof.* Recall that the solution operator  $U(t, 0) : \mathcal{B} \rightarrow \mathcal{B}$  for  $\text{Eq.}(P_\omega L_0)$  is decomposed as  $U(t, 0) = \hat{T}(t) + K(t, 0)$ . Since

$$\begin{aligned} |K(\omega, 0)\phi|_{\mathcal{B}} &\leq J \sup_{0 \leq \tau \leq \omega} \int_0^\tau \|T(\tau - s)\| \|B(s)\|_\infty |x_s(\phi)|_{\mathcal{B}} ds \\ &\leq JM_w \|B\|_\infty N(\omega; \|B\|_\infty) |\phi|_{\mathcal{B}} \sup_{0 \leq \tau \leq \omega} \int_0^\tau e^{w(\tau-s)} ds, \end{aligned}$$

we have that

$$\|K(\omega, 0)\| \leq JM_w \|B\|_\infty N(\omega; \|B\|_\infty) |\phi|_{\mathcal{B}} \int_0^\omega e^{ws} ds.$$

Thus, if the right side of this inequality is less than  $\eta$ , then  $I - (\hat{T}(\omega) + K(\omega, 0)) \in \mathcal{F}_+(\mathcal{B})$ ; that is,  $I - U(\omega, 0) \in \mathcal{F}_+(\mathcal{B})$ . From Lemma 3.6 we have  $\dim N(I - (\hat{T}(\omega) + K(\omega, 0))) \leq \dim N(I - \hat{T}(\omega))$ . This proves the theorem.

**Corollary 3.9** *Assume that  $\mathcal{B} = UC_g$  is a uniform fading memory space,  $I - T(\omega) \in \mathcal{F}_+(E)$ , and that  $\|B\|_\infty$  satisfies the same condition as in Theorem 3.8. If  $\text{Eq.}(P_\omega L)$  has an  $E$ -bounded solution, then  $\mathcal{S}(\omega)$  is nonempty and*

$$\dim \mathcal{S}(\omega) \leq \dim N(I - T(\omega)) < \infty.$$

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