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A free boundary problem for the calculus of variations

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1. ABSTRACT

In this paper, we treat a free boundary problem for Non-linear elliptic equations derived from a variational problem. This type of problem has been studied by H.W.Alt, L.A.Caffarelli and A.Friedmann.

We show in this paper Lipschitz continuity of the minimum if the domain is 2 dimensional, and it immediately follows that the free boundary has a finite 1 dimensional Hausdorff measure, that is, the free boundary is contably 1 rectifiable sets.

2. PROBLEM

We consider a following minimizing problem.

Problem A

Find a minimum of the functional below,

$$J(u) = \int_\Omega \left(a^{ij}(u)D_i u\, D_j u + Q^2(x)\, \chi(\{u>0\}) \right) dx ,$$

in the function class K, and show the regularity of $\partial\{x \mid u>0\}$ (the free boundary).

The notations are as follows.

Ω ; $(\subset \mathbf{R}^n)$ an open and connected domain (unbounded)

The boundary Ω, $\partial\Omega$, is a Lipschitz Graph.

$u(x)$; $\Omega \to \mathbf{R}$

$Q(x)$; a given measurable function with $0<c\leqq Q(x)\leqq C<\infty$.

K ; $K := \{ u\in L^2_{loc}(\Omega) \mid \nabla u \in L^2(\Omega),\ u=u^0 \text{ on } S \}$

u^0 is a given function, bounded and non-negative, and

$u^0 \in L^2_{loc}(\Omega)$, $\nabla u^0 \in L^2(\Omega)$ and $J(u^0)<\infty$.

S is the subset of $\partial\Omega$ with $H^{n-1}(S)>0$.

H^{n-1} is the n-1 dimensional Hausdorff measure.

χ ; the characteristic function

$$\chi(\{u>0\}) = \begin{cases} 1 & x \in \{x \mid u>0\} \\ 0 & x \notin \{x \mid u>0\} \end{cases}$$

$a^{ij}(z)$; $\in C^\infty(\mathbb{R})$

$$0<\lambda\,|\xi|^2 \leqq a^{ij}(z)\xi^i\xi^j \leqq \Lambda\,|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n-\{0\}$$

$$0 \leqq \dot{a}^{ij}(z)\xi^i\xi^j \leqq \tilde{\Lambda}\,|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n-\{0\}$$

($\dot{a}^{ij}(z)$: the derivatives of $a^{ij}(z)$ with respect to z)

For the problem A, we obtained a following result.

Conclusion A1 (S.Omata, 1989. See [Om].)

Assume that n=2, then the free boundary $\partial\{x \mid u>0\}$ has the finite 1 dimensional Hausdorff measure locally (that is, the free boundary is a countably n rectifiable set; see [S]).

This type of problem has been studied by H.W.Alt and L.A.Caffarelli and A.Friedmann. Their problems and conclusions are as follows.

Problem B

Find a minimum of the functional below,

$$J(u) = \int_\Omega \left(F\left(\left| \nabla u \right|^2 \right) + Q^2(x)\ \chi(\{u>0\}) \right) dx,$$

in the function class K, and show the regularity of $\partial\{x \mid u>0\}$ (the free boundary).

Conclusion B1 (H.W.Alt, L.A.Caffarelli, 1981. See [AC].)

Assume F(t)=t, and Q is Hölder continuous, and if n=2 then $\partial\{x \mid u>0\}$ is a $C^{1,\alpha}$-curve locally, and if $n\geqq 3$, then the reduced free boundary $\partial^*\{x \mid u>0\}$ is a $C^{1,\alpha}$-curve locally.

Conclusion B2 (H.W.Alt, L.A.Caffarelli, A.Friedmann, 1984)

Assume $F(t) \in C^{2,1}[0,\infty)$ and $F(0)=0$ with $c \leqq F'(t) \leqq C$ and $0 \leqq \frac{F''(t)}{1+t} \leqq C$, and Q is Hölder continuous, and if $n=2$ then $\partial\{x \mid u>0\}$ is a $C^{1,\alpha}$-curve locally, and if $n \geqq 3$, then the reduced free boundary $\partial^*\{x \mid u>0\}$ is a $C^{1,\alpha}$-curve locally.

3. FIRST VARIATION

It is easy to see that the minimum is Hölder continuous inside Ω by using the methods in the book [LU]. So, $\Omega \cap \{u>0\}$ becomes open set. Then we can choose $\zeta \in C_0^\infty(\Omega \cap \{u>0\})$ with $\zeta \geqq 0$ arbitrarily. Since $\chi(\{u>0\}) = \chi(\{u+\varepsilon\zeta>0\})$ for $\varepsilon>0$, we have;

$$0 = \lim_{\varepsilon \to 0} \frac{J(u+\varepsilon\zeta)-J(u)}{\varepsilon}$$

$$= \int_{\Omega \cap \{u>0\}} \left(- a^{ij}(u)\, D_i u\, D_j \zeta - \frac{1}{2}\, \dot{a}^{ij}(u)\, D_i u\, D_j u\, \zeta \right) dx$$

Denote this equation $Lu = 0$.

For whole domain Ω, we have the following inequality;

$$0 \leqq \lim_{\varepsilon \to 0} \frac{J(u-\varepsilon\zeta)-J(u)}{\varepsilon}$$

$$= \int_{\Omega} \left(- a^{ij}(u)\, D_i u\, D_j \zeta - \frac{1}{2}\, \dot{a}^{ij}(u)\, D_i u\, D_j u\, \zeta \right) dx$$

for all $\zeta \in C_0^\infty(\Omega)$ with $\zeta \geqq 0$.

This means that u is the L-subsolution globally in Ω.

$$\begin{cases} Lu = 0 & \text{in} \quad \Omega \cap \{u>0\} \\ Lu \geqq 0 & \text{in} \quad \Omega \end{cases}$$

Since $Lu=0$ in $\Omega \cap \{u>0\}$, u is smooth inside $\Omega \cap \{u>0\}$. Here we are interested in the global Lipschitz continuity which is essential for the estimates of the length of free boundary.

4. THE LIPSCHITZ CONTINUITY OF THE MINIMUM

We apply the method of [ACF] to nonlinear problem, and we have the Lipschitz estimates sufficiently near the free boundary.

Theorem 4.1

Let u be the minimum of J and choose $x_0 \in \Omega$, arbitrary with $R := \mathrm{dist}(x_0,\{u>0\}) < \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)$. And let y be a point in $\partial B_R(x_0) \cap \{u=0\}$. Choose $x_r \in \overline{x_0 y}$ with $r = d(x_r, y) \leqq R$, then there exists a positive number $M<\infty$ such that

$$\limsup_{r\to 0} \frac{u(x_r)}{r} \leqq M$$

Proof (by contradiction)

Assume that $M=\infty$. For convenience, let us assume that y=0.

Let v be the minimizer of the functional below,

$$I_{B_r(0)}(v) := \int_{B_r(0)} a^{ij}(u)\, D_i v\, D_j v\, dx ,$$

with v=u on $\partial B_r(0)$. It is easy to see that v satisfies

$$\int_{B_r(0)} - a^{ij}(u)\, D_i v\, D_j \xi = 0 , \qquad \forall \xi \in C_0^\infty (B_r(0)) .$$

(Denote this the weak form of the linear equation $\tilde{L} v = 0$.)

Minimality of u tells us

$$\int_{B_r(0)} \left(a^{ij}(u) D_i u\, D_j u + Q^2 \chi(\{u>0\}) \right) \leqq \int_{B_r(0)} \left(a^{ij}(v) D_i v\, D_j v + Q^2 \chi(\{v>0\}) \right) .$$

By the regularity assumption on $a^{ij}(\cdot)$, and ellipticity of $[a^{ij}(\cdot)]$,

$$(4.1) \quad \lambda \int_{B_r(0)} |\nabla (u-v)|^2 \leqq c \sup_{B_r(0)} |v-u| \int_{B_r(0)} |\nabla v|^2 + \int_{B_r(0)} Q^2 \chi(\{u=0\}) .$$

Since v satisfies a good elliptic equation, then we have,

$$v(x) \geqq \tilde{C}\, M(\, r - |x|\,) \quad \text{in} \quad B_r(0)$$

Take two disjoint balls $B_{\frac{1}{8}r}(y_i)$ $(i=1,2)$ in $B_{\frac{1}{2}r}(0)$. Let ξ vary on $\partial B_r(0)$ and draw a line to y_i, and choose a point $\eta_i(\xi)$ on the line such that length of the $\overline{\xi\,\eta_i(\xi)}$ become largest with $\eta_i(\xi) \notin B_{\frac{1}{8}r}(y_i)$ and $u(\eta_i(\xi)) = 0$. Here, $\eta_i(\xi) = \xi$ if $u(x) \neq 0$ for all $x \in \ell_i(\xi)$.

First we fix $\xi \in \partial B_r(0)$, and by using the fundamental theorem of calculus in the direction $\overline{\xi\,\eta_i(\xi)}$, we can write (5.10) as follows;

$$\int_{\xi}^{\eta_i(\xi)} \frac{d}{d\ell_i(\xi)} \Big(\tilde{C}M(r-|x|) \Big)\, d\ell_i(\xi) \leqq \int_{\xi}^{\eta_i(\xi)} \frac{d}{d\ell_i(\xi)} \Big(v(x)-u(x) \Big)\, d\ell_i(\xi).$$

For the sake of simplicity denote $\eta_i(\xi) = \eta$ and $\ell_i(\xi) = \ell$

Immediately we have;

$$\int_{\xi}^{\eta} c\, \tilde{C}\, M \;\, d\ell \leqq \int_{\xi}^{\eta} \Big| \nabla(u(x) - v(x)) \Big| \;\, d\ell\, .$$

Denote by S_i the union of all the segments $\ell_i(\xi)$, and let $S=S_1 \cup S_2$.

By integrating over $\xi \in \partial B_r(0)$, and taking the union of S_i, we obtain;

$$C(n)\, M \int_S dx \leqq \int_S \Big| \nabla(u(x) - v(x)) \Big| \; dx\, .$$

And by using the Schwartz inequality,

$$c^2\, M^2\, |S| \leqq \int_S \Big| \nabla(u(x) - v(x)) \Big|^2 dx.$$

Here $|S|$ describes the Lebesgue measure of S. Using (4.1)

$$(4.2)\quad CM^2 \Big| S \Big| \leqq c \sup_{B_r(0)} \Big| v - u \Big| \int_{B_r(0)} \Big| \nabla v \Big|^2 + \int_{B_r(0)} Q^2 \chi(\{u=0\})$$

We have the following estimate,

$$cM^2 r^n \leqq c\, r^{\alpha} r^{n-2+2\alpha} + Q_{max}\, r^n.$$

Thus we obtained the following inequality;

$$\tilde{c}\, M^2 \leqq r^{3\alpha-2} + Q^2_{max}, \qquad (5.3)$$

where $\tilde{c}$ depends only on λ, Λ.

Letting M large enough, which contradicts (5.3). Thus we have the theorem. (α >2/3 is easily obtained)

5. THE NONDEGENERACY THEOREM

In section 4, we have the Lipschitz estimates for the minimum. This enables us to prove the following theorem.

Theorem 5.1

For any $p>1$ and for any $0<\kappa<1$, there is a constant $C_\kappa = C(n,\kappa)$, such that for the minimum u and for any balls $B_r \subset \Omega$ the following conclusion holds:

$$\frac{1}{r}\left(\fint_{B_r} u^p \right)^{\frac{1}{p}} \leqq C_\kappa \quad \text{implies} \quad u = 0 \quad \text{in } B_{\kappa r}.$$

6. ESTIMATES FOR THE FREE BOUNDARY

We will define the Radon measure λ as follows.

For the open set A;

$$\lambda(A) = \sup_{\substack{|\xi| \leqq 1 \\ \operatorname{supp}\xi \subset A}} \left(\int_\Omega - a^{ij}(u)\, D_i u\, D_j \xi - \frac{1}{2}\, \dot{a}^{ij}(u)\, D_i u\, D_j u\, \xi \right).$$

And for the arbitrary subset B, consider the open covering A_i, and take infimum as below

$$\lambda(B) = \inf_{B \subset \cup A_i} \lambda(A_i), \quad \text{where } A_i \text{ is the open set.}$$

First, we will estimate free boundary from above by the Hausdorff measure; We take ξ suitable function, i.e.

$$\xi = \begin{cases} 1 & \text{in } x \in B_{r-\varepsilon} \\ 0 & \text{on } x \in \partial B_r. \end{cases}$$

Letting $\varepsilon \to 0$ then we have;

$$\int_{B_r(x)} \zeta \, d\lambda \leqq Cr^{n-1} + cr^n \leqq Cr^{n-1}$$

Next, we will esimate from below.

Here we will introduce the positive Green's function of the principal term of the Euler equation with the pole at y, and y is not on the support of λ. Thus;

$$(6.1) \quad \int_{B_r(x)} G_y \, d\lambda = \int_{B_r(x)\setminus B_{c(\kappa)r}(y)} G_y \, d\lambda$$

$$\leqq \sup_{B_r(x)\setminus B_{c(\kappa)r}(y)} G_y \int_{B_r(x)} d\lambda$$

$$\leqq C(\kappa)\, r^{2-n} \int_{B_r(x)} d\lambda .$$

Next, by the nature of the Green's function and some easy adjustments for boundary condition, we have the following estimates;

$$(6.2) \quad \int_{B_r(x)} G_y \, d\lambda \geqq C\, u(y) \geqq C\, c(\kappa) r$$

From (6.1) and (6.2), we have the estimates from below;

$$C \cdot r^{n-1} \leqq \int_{B_r(x)} d\lambda .$$

Then we have the following theorem.

Theorem 6.1

The Radon measure λ is absolutely continuous with the n-1 dimensional Hausdorff measure, i.e. for $x \in \partial\{u>0\}$ there are constants $0<c<C<\infty$ independent of x and r such that

$c H^{n-1}(\partial\{u>0\}\cap B_r(x)) \leqq \lambda(\partial\{u>0\}\cap B_r(x)) \leqq C H^{n-1}(\partial\{u>0\}\cap B_r(x)).$

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