

Semigroup semantics for orthomodular logic

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Abstract

Quantum logic is usually considered as a logic which is based on orthomodular lattices. Here we introduce a different type of semantics, in which we use particular semigroups, and show that these two ways of interpretation of formulas are equivalent.

0 Basic notions

First we will give here some basic notions.

The language of our logics consists of :

- (i) a countable collection $\{ p_i \mid i < \omega \}$ of propositional variables,
- (ii) the connectives $\neg$ and $\wedge$ of negation and conjunction,
- (iii) parentheses (and).

The set Φ of formulas is defined in the usual way. That is, Φ is the minimum set which satisfies the following three conditions:

- (i) for every $i < \omega$, $p_i \in \Phi$,
- (ii) if $\alpha \in \Phi$, then $(\neg\alpha) \in \Phi$,
- (iii) if $\alpha, \beta \in \Phi$, then $(\alpha \wedge \beta) \in \Phi$.

The letters α, β , etc. are used as metavariables ranging over Φ . Parentheses may be omitted by the convention that $\neg$ binds strongly than $\wedge$. The disjunction $\alpha \vee \beta$ of α and β can be introduced as the abbreviation of $\neg(\neg\alpha \wedge \neg\beta)$.

Definition 0.1 (Orthomodular lattice) An *orthomodular lattice* $\mathcal{A}$ is a structure $\langle A, \leq, \sqcap, \sqcup, ^\perp, 1, 0 \rangle$, which satisfies the following conditions:

- (i) $\langle A, \leq, \sqcap, \sqcup, 1, 0 \rangle$ is a lattice with 1(maximum) and 0(minimum). We denote, for any $x, y \in A$, $x \sqcap y := \inf \{x, y\}$, $x \sqcup y := \sup \{x, y\}$.
- (ii) The unary operation $^\perp$ (*orthocomplement*) satisfies the following conditions, (a), (b) and (c): for any $x, y \in A$,
 - (a) $x \sqcap x^\perp = 0$
 - (b) $x^{\perp\perp} = x$
 - (c) $x \leq y$ implies $y^\perp \leq x^\perp$

$$(d) \quad x \leq y \quad \text{implies} \quad y = x \sqcup (x^\perp \sqcap y)$$

It is easy to see that $x \sqcup y = (x^\perp \sqcap y^\perp)^\perp$ holds in any orthomodular lattice. ■

Definition 0.2 (Valuation) A valuation is a function v , which associates with any formula $\alpha \in \Phi$ an element $v(\alpha)$ in an orthomodular lattice $\mathcal{A}$, and satisfies the following conditions:

for any formula α, β ,

- (i) $v(\neg\alpha) = (v(\alpha))^\perp$
- (ii) $v(\alpha \wedge \beta) = v(\alpha) \sqcap v(\beta)$

We call this v an *orthomodular valuation*. ■

It is easy to see that for any valuation v and for any formula α , the value $v(\alpha)$ is uniquely determined by the values $v(p_i)$ for propositional variables p_i appearing in α .

Definition 0.3 (Orthomodular logic) The *orthomodular logic* L is the set of pairs of formulas (α, β) satisfying the following conditions: for any orthomodular lattice $\mathcal{A}$ and for any orthomodular valuation v from Φ to $\mathcal{A}$, $v(\alpha) \leq v(\beta)$. We denote $\alpha \vdash_L \beta$ in place of $(\alpha, \beta) \in L$. ■

R.I Goldblatt proposed his “quantum model” for orthomodular logic in 1974[1].

Definition 0.4 (Quantum frame and quantum model) $\mathcal{F} = \langle X, \perp, \xi \rangle$ is a *quantum frame* if it satisfies the following conditions (i),(ii) and (iii).

- (i) X is a nonempty set.
- (ii) $\perp$ is an irreflexive and symmetric binary relation. (*orthogonality relation*)

- For $P \subseteq X$, $x \perp P$ means that $x \perp y$ for all $y \in P$.

- $P (\subseteq X)$ is $\perp$ -closed iff the following condition holds:

$$\forall x \in X (x \notin P), \exists y \in X [y \perp P \text{ and not}(y \perp x)]$$

- $P (\subseteq X)$ is $\perp$ -closed in Q ($Q \subseteq X$) iff the following condition holds:

$$\forall x \in Q (x \notin P), \exists y \in Q [y \perp P \text{ and not}(y \perp x)]$$

- (iii) ξ is a nonempty collection of $\perp$ -closed subsets of X , such that

- (a) ξ is closed under set-inclusion and the following operation † .

$$P^\dagger = \{x \in X | x \perp P\}$$

- (b) For any P, Q in ξ , if $P \subseteq Q$ then P is $\perp$ -closed in Q .

$\mathcal{Q} = \langle X, \perp, \xi, V \rangle$ is a *quantum model* if it satisfies the following:

- (i) $\mathcal{F} = \langle X, \perp, \xi \rangle$ is a quantum frame.
- (ii) V is a function assigning to each propositional variables p_i a member $V(p_i)$ of ξ .

The notion of truth in quantum models is defined inductively as follows: the symbol ' $\mathcal{Q} \models_x \alpha$ ' is read as "formula α is true at x in $\mathcal{Q}$ ".

- (i) $\mathcal{Q} \models_x p_i$ iff $p_i \in V(p_i)$,
- (ii) $\mathcal{Q} \models_x \alpha \wedge \beta$ iff $\mathcal{Q} \models_x \alpha$ and $\mathcal{Q} \models_x \beta$,
- (iii) $\mathcal{Q} \models_x \neg \alpha$ iff for any $y \in X$, $(\mathcal{Q} \models_y \alpha \Rightarrow x \perp y)$.

- α implies β in a model $\mathcal{Q}$ iff for all x in the model $\mathcal{Q}$, either $\mathcal{Q} \models_x \alpha$ does not hold, or $\mathcal{Q} \models_x \beta$ holds.

Using his quantum models, Goldblatt showed the following completeness theorem.

Theorem 0.5 (Completeness Theorem) For given formulas α and β , the statements (P) and (Q) are mutually equivalent, that is

- (P): for any orthomodular lattice $\mathcal{A}$ and any valuation $v : \Phi \rightarrow \mathcal{A}$, $v(\alpha) \leq v(\beta)$ holds.
- (Q): for any quantum model $\mathcal{Q}$, $\mathcal{Q} : \alpha \models \beta$ holds.

□

In study of orthomodular lattice, D.J.Foulis [2] found in 1960 the following representation theorem for orthomodular lattices with a particular kind of semigroups.

Theorem 0.6 (Foulis's representation theorem) Let $\mathcal{A}$ be an orthomodular lattice. Then $\mathcal{G}(\mathcal{A}) = \langle G(\mathcal{A}), \cdot, * \rangle$ is a Rickart $*$ semigroup and $\mathcal{A}$ is isomorphic to $P_c(G(\mathcal{A}))$. □

We will give another type of models for orthomodular logic using this representation theorem.

1 Rickart $*$ semigroups

Now we introduce a special type of semigroups called Rickart $*$ semigroups and lead some properties of them.

Definition 1.1 (Rickart $*$ semigroups) A Rickart $*$ semigroup is a structure $\mathcal{G} = \langle G, \cdot, * \rangle$ which satisfies the following conditions (i), (ii), (iii) and (iv).

- (i) $\langle G, \cdot \rangle$ is a semigroup, that is,
 - (a) $\cdot$ is a binary operation on G .
 - (b) For any $x, y, z \in G$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
- (ii) There exists the unique element 0 (zero element) in G such that $0 \cdot x = x \cdot 0 = 0$ holds for any $x \in G$.
- (iii) $*$ is a unary operation on G , which satisfies the following:
 - For any $x, y \in G$, (a): $(x^*)^* = x$. (b): $(x \cdot y)^* = y^* \cdot x^*$.

Before introducing the condition (iv), it is necessary to introduce some other notions.

- An element $e \in G$ is called a *projection* iff it satisfies $e^* = e \cdot e = e$.
We denote the set of all projections in G by $P(G)$.
- For an element $x \in G$, the set $\{x\}^{(r)} := \{y \in G \mid x \cdot y = 0\}$ is called the *right annihilator* for x .

By using these two notions, we formulate the condition (iv) as follows:

- (iv) For any $x \in G$, there exists a projection e such that the right annihilator for x can be expressed as: $\{x\}^{(r)} = e \cdot G = \{e \cdot y \mid y \in G\}$. We call this e a *right annihilating projection* for x .

Lemma 1.2 (Properties of $P(G)$) Let $\mathcal{G} = \langle G, \cdot, * \rangle$ be a Rickart $*$ semigroup.

- (i) For any $x \in G$, the right annihilating projection for x is uniquely determined. Hereafter, this will be written as x^r .
- (ii) There is the unit element in G , that is, an element 1 satisfying that for any $x \in G$, $x \cdot 1 = 1 \cdot x = x$.
- (iii) Both 0 and 1 are projections.
- (iv) For any $e, f \in P(G)$, the following three conditions are equivalent.
 - (a) $e \cdot f = e$.
 - (b) $f \cdot e = e$.
 - (c) $e \cdot G \subseteq f \cdot G$.

Proof :

- (i) Using the properties of the operation $*$.
- (ii) We can show that 0^r is the unit element 1 .
- (iii) By operating $*$ to both sides of the equation $0 = 0 \cdot 0^*$, we get that $0^* = 0$. Similarly we can show that $1^* = 1$.
- (iv) Not so hard.

□

The above Lemma 1.2 (iv) assures us the possibility of introducing a partial order on $P(G)$.

Definition 1.3 (Order on $P(G)$) Let $\mathcal{G} = \langle G, \cdot, * \rangle$ be a Rickart $*$ semigroup. Define a partial order $\leq$ on $P(G)$ as follows: for $e, f \in P(G)$, $e \leq f$ iff $e \cdot f = e$.

■

It is obvious that 1 is the maximum and that 0 is the minimum with respect to this order. Hence $P(G)$ can be regarded as a bounded partial ordered set.

In the proof of Lemma 1.2, we have defined the unary operation r from G to $P(G)$. Here we will see some of the basic properties of the operation r in detail, which will be used in the later discussion.

Lemma 1.4 (Properties of the operation r) Let $\mathcal{G} = \langle G, \cdot, * \rangle$ be a Rickart $*$ semigroup. For any $x, y \in G$ and for any $e, f \in P(G)$, the following statements can be verified.

- (i) $0^r = 1$, and $1^r = 0$. (v) If $e \leq f$, then $f^r \leq e^r$.
- (ii) $x \cdot x^r = 0$, and $x^r \cdot x^* = 0$. (vi) $x = x \cdot x^{rr}$, and $e \leq e^{rr}$.
- (iii) If $x \cdot e = 0$, then $e \leq x^r$. (vii) $x^r = x^{rrr}$.
- (iv) $x^r \leq (y \cdot x)^r$. (viii) If $e \cdot x = x \cdot e$, then $e^r \cdot x = x \cdot e^r$.

Proof : Here we prove only (vi) and (viii). Rest is not so hard.

- (vi) By (ii), $x^* \in \{x^r\}^{(r)} = x^{rr} \cdot G$. Then there exists some $s \in G$, such that $x^* = x^{rr} \cdot s$. By operating $*$ to this equation, we have that $x = x^{**} = s^* \cdot x^{rr*} = s^* \cdot x^{rr}$. Further operating x^{rr} from the right to the equation $x = s^* \cdot x^{rr}$, we can derive that $x \cdot x^{rr} = (s^* \cdot x^{rr}) \cdot x^{rr} = s^* \cdot x^{rr} = x$. In particular, when x is equal to a projection e , we have that $e \cdot e^{rr} = e$, that is, $e \leq e^{rr}$.

- (viii) Suppose that $e \cdot x = x \cdot e$. Then we have $e \cdot x \cdot e^r = x \cdot e \cdot e^r = 0$, since $e \cdot e^r = 0$. So $x \cdot e^r \in \{e\}^{(r)} = e^r \cdot G$, and there exists some $s \in G$ satisfying that $x \cdot e^r = e^r \cdot s$. By multiplying e^r from the left to both sides of this equation, we have that

$$e^r \cdot x \cdot e^r = e^r \cdot e^r \cdot s = e^r \cdot s = x \cdot e^r \quad \dots (1)$$

On the other hand, by operating $*$ to the supposition $e \cdot x = x \cdot e$, so we have that $x^* \cdot e = e \cdot x^*$. Then $e \cdot x^* \cdot e^r = x^* \cdot e \cdot e^r = 0$, which means that $x^* \cdot e^r \in \{e\}^{(r)} = e^r \cdot G$. So there exists some $t \in G$ such that $x^* \cdot e^r = e^r \cdot t$. By multiplying e^r from the left to both sides of this equation, we have that $e^r \cdot x^* \cdot e^r = e^r \cdot e^r \cdot t = e^r \cdot t = x^* \cdot e^r$. Further operating $*$ again, we get that

$$e^r \cdot x \cdot e^r = e^r \cdot x \quad \dots (2).$$

From (1) and (2), we can conclude that $x \cdot e^r = e^r \cdot x$.

□

Now we will consider a particular class of projections, called *closed projections*.

Definition 1.5 (Closed projection) A projection $f \in P(G)$ is called *closed* iff there exists an element $x \in G$ such that f is the right annihilating projection for x . This means that a closed projection f can be written as $f = x^r$ for some element $x \in G$. We denote the set of all closed projections in G by $P_c(G)$. ■

In other words, the set $P_c(G)$ is the range of the function r from G to $P(G)$. We give here a necessary and sufficient condition on a projection to be closed.

Proposition 1.6 For any $e \in P(G)$, $e \in P_c(G)$ if and only if $e^{rr} = e$. □

We will show that in $P_c(G)$ we can always find the supremum and the infimum of any two elements of it and hence this partially ordered set forms a lattice. Moreover we can show that $P_c(G)$ is an orthomodular lattice.

Lemma 1.7 (Existence of meet in $P_c(G)$)

- (i) For any closed projections e and f such that $e \cdot f = f \cdot e$, $e \cdot f \in P_c(G)$ holds, and there exists the infimum $(e \sqcap f)$ of e, f , which satisfies the equation $e \sqcap f = e \cdot f$.
- (ii) In general, for any closed projections e and f , there exists the infimum $(e \sqcap f)$ of e, f and the equation $e \sqcap f = e \cdot (f^r \cdot e)^r = (f^r \cdot e)^r \cdot e = e \sqcap (f^r \cdot e)^r$ holds.

Proof :

- (i) Suppose that $e \cdot f = f \cdot e$. We show that $e \cdot f \in P_c(G)$. Since $e, f \in P(G)$ and $e \cdot f = f \cdot e$, we can derive:

$$(e \cdot f)^* = f^* \cdot e^* = f \cdot e = e \cdot f, \quad \text{and} \quad (e \cdot f) \cdot (e \cdot f) = e \cdot e \cdot f \cdot f = e \cdot f.$$

Thus, $e \cdot f \in P(G)$. To prove that $e \cdot f \in P_c(G)$, by Proposition 1.5, it is enough to show that $(e \cdot f)^{rr} = e \cdot f$. Then we have only to show that $(e \cdot f)^{rr} \leq e \cdot f$ as the converse inequality holds always by Lemma 1.4 (vi). Considering the Lemma 1.4 (iv), we have that $e^r \leq (e \cdot f)^r$. Then by the Lemma 1.4 (v), we can derive that $(e \cdot f)^{rr} \leq e^{rr} = e$, which means $e \cdot (e \cdot f)^{rr} = (e \cdot f)^{rr}$. Similarly we can derive that $f \cdot (e \cdot f)^{rr} = (e \cdot f)^{rr}$. Therefore $e \cdot f \cdot (e \cdot f)^{rr} = e \cdot (e \cdot f)^{rr} = (e \cdot f)^{rr}$. Thus $(e \cdot f)^{rr} \leq e \cdot f$.

It is easy to see that $e \cdot f$ is the infimum of e and f .

- (ii) We put $u := f^r \cdot e$. By Lemma 1.4 (iv), we have that $e^r \leq (f^r \cdot e)^r = u^r$. This means that $e^r \cdot u^r = e^r = u^r \cdot e^r$. By applying Lemma 1.4 (viii), we have that $e \cdot u^r = u^r \cdot e$. Then by (i) of the present lemma, we can conclude that $e \cdot u^r \in P_c(G)$, and that $e \sqcap u^r = e \cdot u^r$. So it remains to show that $e \sqcap f = e \cdot u^r$.

- (a) Clearly, $e \cdot (e \cdot u^r) = e \cdot u^r$. So we have $e \cdot u^r \leq e$. On the other hand, $f^r \cdot e \cdot u^r = f^r \cdot e \cdot (f^r \cdot e)^r = 0$. So from Lemma 1.4 (iii), we derive that $e \cdot u^r \leq f^{rr} = f$. Thus $e \cdot u^r$ is a lower bound of $\{e, f\}$.
- (b) Take any $g \in P_c(G)$ such that $g \cdot e = g$ and $g \cdot f = g$. Then because $f \cdot f^r = 0$, we have that $g \cdot f \cdot f^r \cdot e = 0$. By our assumption on g , $g \cdot f^r \cdot e = 0$, which means that $g \cdot u = 0$. By Lemma 1.4 (iii), we can derive that $u \leq g^r$. So by Lemma 1.4 (v), $g = g^{rr} \leq u^r$. This is equivalent to $g \cdot u^r = g$. Again using the assumption on g , $g \cdot e \cdot u^r = g$. So we have derived that $g \leq e \cdot u^r$.

Thus we have shown that $e \sqcap f = e \cdot u^r$.

□

Therefore we have the following Proposition.

Proposition 1.8 For any $e, f \in P_c(G)$, the following equation holds:

$$e \cdot G \cap f \cdot G = (e \sqcap f) \cdot G.$$

□

Next we will see that $P_c(G)$ is an orthomodular lattice.

Theorem 1.9 $P_c(G)$ forms an orthomodular lattice, where the orthocomplement is the operation $^{\perp}$.

Proof : We can easily check the conditions in Definition 0.1 .

□

Next, in Section 2, we will introduce a semantics for orthomodular logic by using Rickart $*$ semigroups, and prove the soundness.

2 Semigroup semantics and soundness theorem

Definition 2.1 (Orthomodular model) $\mathcal{M} = \langle \mathcal{G}, u \rangle$ is a *orthomodular model* (OM model for short) iff $\mathcal{G} = \langle G, \cdot, * \rangle$ is a Rickart $*$ semigroup and u is a function assigning to each propositional variable p_i an element $u(p_i)$ of $P_c(G)$.

The notion of truth in OM models is defined inductively as follows: the symbol ' $(\mathcal{M}, x) \models \alpha$ ' is read as " a formula α is true at x in $\mathcal{M}$ ".

- (i) $(\mathcal{M}, x) \models p_i$ iff $x \in u(p_i) \cdot G$.
- (ii) $(\mathcal{M}, x) \models \alpha \wedge \beta$ iff $(\mathcal{M}, x) \models \alpha$ and $(\mathcal{M}, x) \models \beta$.
- (iii) $(\mathcal{M}, x) \models \neg \alpha$ iff $\forall y \in G, [(\mathcal{M}, y) \models \alpha \text{ only if } y^* \cdot x = 0]$.

■

For each formula α , define $\|\alpha\|^{\mathcal{M}} := \{x \in G \mid (\mathcal{M}, x) \models \alpha\}$. Then we can restate the above conditions in the following way:

- (i) $\|p_i\|^{\mathcal{M}} = u(p_i) \cdot G$.
- (ii) $\|\alpha \wedge \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$.
- (iii) $\|\neg \alpha\|^{\mathcal{M}} = \{x \in G \mid \forall y \in \|\alpha\|^{\mathcal{M}} (y^* \cdot x = 0) \}$.

Definition 2.2 Let α and β be formulas.

- (i) α *implies* β at x in an OM model $\mathcal{M}$ ($(\mathcal{M}, x) : \alpha \models \beta$) iff either $(\mathcal{M}, x) \models \alpha$ does not hold or $(\mathcal{M}, x) \models \beta$ holds.
- (ii) α *implies* β in an OM model $\mathcal{M}$ ($\mathcal{M} : \alpha \models \beta$) iff for all x in the model $\mathcal{M}$, $(\mathcal{M}, x) : \alpha \models \beta$ holds.

■

It is easy to see that $\mathcal{M} : \alpha \models \beta$ is equivalent to $\|\alpha\|^{\mathcal{M}} \subseteq \|\beta\|^{\mathcal{M}}$.

Lemma 2.3 Let $\mathcal{M} = \langle \mathcal{G}, u \rangle$ be an orthomodular model and e such an orthomodular valuation from Φ to $P_c(\mathcal{G})$ that $e(p_i) = u(p_i)$ holds for all propositional variables. Then for any formula α , $\|\alpha\|^{\mathcal{M}} = e(\alpha) \cdot G$ holds.

Proof : Induction on the construction of the formula α . □

Now we can prove the soundness theorem.

Theorem 2.4 (Soundness theorem) For given formulas α and β , let (S) and (T) be the statements as follows:

(S): for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \rightarrow \mathcal{A}$,
 $v(\alpha) \leq v(\beta)$.

(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

Then (S) implies (T). □

3 Monotone, residuated maps on an ordered set

Next, we will prove the Completeness Theorem. To show the direction $((S) \Leftarrow (T))$, we need to know how to build up an orthomodular model from a given orthomodular lattice. To do this, we need some preparations.

Definition 3.1 (Residuated, monotone maps on an ordered set) Let $\langle A, \leq \rangle$ be an ordered set.

- (i) A map φ from A to A is called *monotone* iff it satisfies the following condition: for any $x, y \in A$, if $x \leq y$, then $\varphi(x) \leq \varphi(y)$.

We denote the set of all monotone maps from A to A by $\overline{G}(A)$.

- (ii) A map $\varphi \in \overline{G}(A)$ is called *residuated* iff there exists a map $\varphi^\sharp \in \overline{G}(A)$ such that for any $x \in A$, $\varphi^\sharp(\varphi(x)) \geq x$ and $\varphi(\varphi^\sharp(x)) \leq x$.

We call this map $\varphi^\sharp$ a residual map for φ , and denote the set of all residuated, monotone maps on A by $G(A)$. ■

Lemma 3.2 (Properties of residual maps) Let $\langle A, \leq \rangle$ be an ordered set. Then the following holds.

- (i) For any $\varphi \in G(A)$, the residual map for φ is uniquely determined.
- (ii) For any $\varphi, \psi \in G(A)$, $(\varphi \cdot \psi)^\sharp = \psi^\sharp \cdot \varphi^\sharp$ holds, where $\cdot$ means the composition operator for maps. Therefore $G(A)$ is closed under this operation $\cdot$.

Proof : Using the monotonicity and the inequations which hold for $\varphi \in G(A)$ and its residual map $\varphi^\sharp$. □

It is guaranteed by (i) of Lemma 3.2 that we can write the residual map for φ as $\varphi^\sharp$. And (ii) of Lemma 3.2 means that $G(A)$ is a semigroup with respect to the operation $\cdot$.

Lemma 3.3 Let $\langle A, \leq, 0, 1 \rangle$ be an ordered set with the minimum element 0 and the maximum element 1 and let θ be a map defined by the condition: for all $x \in A$, $\theta(x) = 0$. Then θ is the zero element in the semigroup $G(A)$. $\square$

Lemma 3.4 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle$ be an ortholattice. Let $*$ be defined by the following: for any $\varphi \in G(A)$, $\varphi^*(x) := (\varphi^\sharp(x^\perp))^\perp$ for any $x \in A$. Then $\varphi^* \in G(A)$. Moreover the following conditions hold for every $\varphi, \psi \in G(A)$.

- (a) $\varphi^{**} = \varphi$.
- (b) $(\varphi \cdot \psi)^* = \psi^* \cdot \varphi^*$.

Proof : We put $\psi(x) := (\varphi(x^\perp))^\perp$ for any $x \in A$ and show that $\psi = \varphi^*$.

- (i) First we will show that ψ is monotone. Suppose that $x \leq y$ for $x, y \in A$. Then by the properties of the operation $\perp$, we have $x^\perp \geq y^\perp$. Since φ is monotone, we have $\varphi(x^\perp) \geq \varphi(y^\perp)$. Again by the properties of $\perp$, we have $(\varphi(x^\perp))^\perp \leq (\varphi(y^\perp))^\perp$, which means $\psi(x) \leq \psi(y)$. Therefore ψ is monotone.
- (ii) Next we will show that ψ is the residual map for φ . By the properties of the operation $\perp$ and the properties of $\varphi^\sharp$, we can derive: $\psi \cdot \varphi^*(x) = \psi \cdot (\varphi^\sharp(x^\perp))^\perp = [\varphi(\varphi^\sharp(x^\perp))^{\perp\perp}]^\perp = [\varphi(\varphi^\sharp(x^\perp))]^\perp \geq x^{\perp\perp} = x$. So we have $\psi \cdot \varphi^*(x) \geq x$. Similarly we can derive: $\varphi^* \cdot \psi(x) = \varphi^* \cdot (\varphi(x^\perp))^\perp = [\varphi^\sharp(\varphi(x^\perp))^{\perp\perp}]^\perp = [\varphi^\sharp(\varphi(x^\perp))]^\perp \leq x^{\perp\perp} = x$. So we have $\varphi^* \cdot \psi(x) \leq x$.

Hence we can conclude that $\psi = \varphi^*$ since the residual map of φ^* is unique. By (i) and (ii) in the above, we have that $\varphi^* \in G(A)$. Thus $*$ is a unary operator on $G(A)$. Now we will check the conditions (a) and (b). By the properties of the operation $\perp$, and the definition of φ^* , we calculate as follows: for any φ, ψ , and for any $x \in A$,

- (a): $\varphi^{**}(x) = [\varphi^\sharp(x^\perp)]^\perp = [(\varphi(x^{\perp\perp}))^\perp]^\perp = \varphi(x)$.
- (b): $\psi^* \cdot \varphi^*(x) = \psi^*(\varphi^\sharp(x^\perp))^\perp = [\psi^\sharp(\varphi^\sharp(x^\perp))^{\perp\perp}]^\perp = [\psi^\sharp \cdot \varphi^\sharp(x^\perp)]^\perp = [(\varphi \cdot \psi)^\sharp(x^\perp)]^\perp = (\varphi \cdot \psi)^*(x)$.

Consequently this $*$ satisfies conditions for the operator $*$ in Rickart $*$ semigroups. $\square$

From the above consideration, we can define the notions of projection, closed projection and right annihilator for an element in $G(A)$. In order to get a Rickart $*$ semigroup from $G(A)$, we must show that for any element $\varphi \in G(A)$, there exists some closed projection μ such that $\{\varphi\}^{(r)} := \{\psi \in G(A) \mid \varphi \cdot \psi = \theta\} = \mu \cdot G(A)$.

Lemma 3.5 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle$ be an orthomodular lattice. For each $a \in A$, define a map γ_a by $\gamma_a(x) := (x \sqcup a^\perp) \sqcap a$ for every $x \in A$.

- (i) γ_a is a projection in $G(A)$ for any $a \in A$.
- (ii) For any $\varphi \in G(A)$, if we put $a := \varphi^\sharp(0)$, then $\{\varphi\}^{(r)} = \gamma_a \cdot G(A)$ holds.

Proof : By our assumption, the following orthomodular law holds. For $a, b, c \in A$,
 (1) $a \leq b$ implies $b = (b \sqcap a^\perp) \sqcup a$. (2) $c \leq a$ implies $c = (c \sqcup a^\perp) \sqcap a$.
 It is easy to see that (2) follows from (1) and vice versa.

- (i) First we will show that $\gamma_a \in G(A)$. It is obvious that γ_a is monotone. We put $\psi(x) := (x \sqcap a) \sqcup a^\perp$ for any x in A . Clearly ψ is also monotone. Moreover, as shown below, it is the residual map for γ_a .

$$\begin{aligned}\gamma_a \cdot \psi(x) &= [((x \sqcap a) \sqcup a^\perp) \sqcup a^\perp] \sqcap a \\ &= [(x \sqcap a) \sqcup a^\perp] \sqcap a \\ &= x \sqcap a \leq x.\end{aligned}$$

In the last equation in the above, we used (2) since $x \sqcap a \leq a$.

$$\begin{aligned}\psi \cdot \gamma_a(x) &= [((x \sqcup a^\perp) \sqcap a) \sqcap a] \sqcup a^\perp \\ &= [(x \sqcup a^\perp) \sqcap a] \sqcup a^\perp \\ &= x \sqcup a^\perp \geq x\end{aligned}$$

Also, we used (1) since $x \sqcup a^\perp \geq a^\perp$.

Therefore $\gamma_a^\sharp(x) = \psi(x) = (x \sqcap a) \sqcup a^\perp$. So $\gamma_a \in G(A)$.

Next we will show that γ_a satisfies the conditions for projections.

$$\begin{aligned}\gamma_a^* &= (\gamma_a^\sharp(x^\perp))^\perp = [(x^\perp \sqcap a) \sqcup a^\perp]^\perp \\ &= (x^\perp \sqcap a)^\perp \sqcap a^{\perp\perp} \\ &= (x \sqcup a^\perp) \sqcap a = \gamma_a(x)\end{aligned}$$

$$\begin{aligned}\gamma_a \cdot \gamma_a(x) &= [\{(x \sqcup a^\perp) \sqcap a\} \sqcup a^\perp] \sqcap a \\ &= (x \sqcup a^\perp) \sqcap a = \gamma_a(x)\end{aligned}$$

Since $(x \sqcup a^\perp) \sqcap a \leq a$, we used (2) in the above calculation. Thus γ_a is a projection.

- (ii) First we will prove that $\gamma_a \cdot G(A) \subseteq \{\varphi\}^{(r)}$. Take any $\psi \in \gamma_a \cdot G(A)$. Then there exists some element $\lambda \in G(A)$ such that $\psi = \gamma_a \cdot \lambda$. For any $x \in A$, $\gamma_a(x) = (x \sqcup a^\perp) \sqcap a \leq a = \varphi^\sharp(0)$. So by the monotonicity of φ , we have that $\varphi \cdot \gamma_a(x) \leq \varphi \cdot \varphi^\sharp(0) \leq 0$. This means that $\varphi \cdot \gamma_a = \theta$. Then $\varphi \cdot \psi = \varphi \cdot \gamma_a \cdot \lambda = \theta$, that is $\psi \in \{\varphi\}^{(r)}$. Thus we conclude that $\gamma_a \cdot G(A) \subseteq \{\varphi\}^{(r)}$. Next we will show that $\{\varphi\}^{(r)} \subseteq \gamma_a \cdot G(A)$. Take any $\psi \in \{\varphi\}^{(r)}$.

Then ψ satisfies that $\varphi \cdot \psi = \theta$, which means that for any $x \in A$, we have that $\varphi \cdot \psi(x) = 0$. Taking 1 for x , we have $\varphi \cdot \psi(1) = 0$, and hence $a = \varphi^\sharp(0) = \varphi^\sharp \cdot \varphi \cdot \psi(1) \geq \psi(1)$. Therefore we have that for any $x \in A$, $\psi(x) \leq \psi(1) \leq a$. By combining this result with the orthomodular law (2), we have that $\gamma_a \cdot \psi(x) = (\psi(x) \sqcup a^\perp) \sqcap a = \psi(x)$. Consequently $\psi = \gamma_a \cdot \psi \in \gamma_a \cdot G(A)$.

Thus we have proved $\{\varphi\}^{(r)} = \gamma_a \cdot G(A)$.

□

Moreover, we can show the following lemma on the set of maps γ_a .

Lemma 3.6 For any orthomodular lattice $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle$, the relation $P_c(G(A)) = \{\gamma_a \mid a \in A\}$ holds.

Proof : Take any $\lambda \in P_c(G(A))$. Then there exists some $\mu \in G(A)$ such that $\{\mu\}^{(r)} = \lambda \cdot G(A)$. Now putting $b := \mu^\sharp(0)$, we have $\{\mu\}^{(r)} = \gamma_b \cdot G(A)$ by Lemma 3.5 (ii). So the uniqueness of the right annihilating projection gives us that $\lambda = \gamma_b \in \{\gamma_a \mid a \in A\}$.

Conversely, consider γ_a for $a \in A$. Since γ_a is a projection, $\gamma_a = \gamma_a \cdot \gamma_a = \gamma_a^*$ holds. We have that $\gamma_a \cdot \gamma_a^r = \theta$. So by operating $*$ to this equation, we get $\gamma_a^r \cdot \gamma_a = \theta$. Then of course, $\gamma_a^r \cdot \gamma_a \cdot \lambda = \theta$ for any $\lambda \in G(A)$ holds. Therefore we get $\{\gamma_a^r\}^{(r)} = \gamma_a \cdot G(A)$. Thus $\gamma_a \in P_c(G(A))$.

Consequently we have proved that $P_c(G(A)) = \{\gamma_a \mid a \in A\}$. $\square$

By all the lemmas 3.2, 3.3, 3.4 and 3.5, we can prove the following theorem.

Theorem 3.7 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle$ be an orthomodular lattice. Then $\mathcal{G}(\mathcal{A}) = \langle G(A), \cdot, * \rangle$ is a Rickart $*$ semigroup, where $\cdot$ is a composition operator of maps and $*$ is a unary operator defined in Lemma 3.3.

4 Corresponding model and Completeness Theorem

Now we have prepared all the notions for constructing the corresponding model for orthomodular logic.

Definition 4.1 (Corresponding model) Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle$ be an orthomodular lattice, and $v : \Phi \rightarrow A$ an orthomodular valuation. The corresponding model to $\mathcal{A}$ and v is the structure $\mathcal{M}_{\mathcal{A}} = \langle G(A), \cdot, *, u_{\mathcal{A}} \rangle$, where

- (i) $G(A)$ is the set of all residuated monotone maps on A ,
- (ii) $\cdot$ is the composition operator of maps on A ,
- (iii) $*$ is the unary operator on $G(A)$ defined in Lemma 3.4, that is,
for any $\varphi \in G(A)$, $\varphi^*(x) := (\varphi^\sharp(x^\perp))^\perp$ for all $x \in A$,
- (iv) $u_{\mathcal{A}}$ is a function assigning to each propositional variable p_i an element of the set $\{\gamma_a \mid a \in A\}$, such that, $u_{\mathcal{A}}(p_i) := \gamma_{v(p_i)}$.

■

Lemma 4.2 Let $\mathcal{A}$ be an orthomodular lattice and v an orthomodular valuation. Then the corresponding model $\mathcal{M}_{\mathcal{A}} = \langle G(A), \cdot, *, u_{\mathcal{A}} \rangle$ is an orthomodular model.

Proof : This is obvious from Lemma 3.6 and Lemma 3.7. $\square$

Since $\mathcal{M}_{\mathcal{A}}$ is an orthomodular model, the notion of truth in $\mathcal{M}_{\mathcal{A}}$ can be defined similarly in Definition 2.1 as follows Let α, β be formulas, φ, ψ elements in $G(A)$. Then:

- (i) $(\mathcal{M}_{\mathcal{A}}, \varphi) \models p_i$ iff $p_i \in u(p_i) \cdot G(A)$.
- (ii) $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha \wedge \beta$ iff $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha$ and $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \beta$.
- (iii) $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \neg \alpha$ iff $\forall \psi \in G(A), [(\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha \text{ only if } \psi^* \cdot \varphi = 0]$.

By denoting $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} := \{\varphi \in G(A) \mid (\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha\}$, we can restate the above conditions in the following way.

- (i) $\|p_i\|^{\mathcal{M}_{\mathcal{A}}} = u(p_i) \cdot G(A)$.
- (ii) $\|\alpha \wedge \beta\|^{\mathcal{M}_{\mathcal{A}}} = \|\alpha\|^{\mathcal{M}_{\mathcal{A}}} \cap \|\beta\|^{\mathcal{M}_{\mathcal{A}}}$.
- (iii) $\|\neg \alpha\|^{\mathcal{M}_{\mathcal{A}}} = \{\varphi \in G(A) \mid \forall \psi \in \|\alpha\|^{\mathcal{M}_{\mathcal{A}}} (\psi^* \cdot \varphi = 0)\}$.

Here we will make a comment about the order on $P_c(G(A))$, where A is an orthomodular lattice. Because $\gamma_a \in P_c(G)$ is a projection, the order on the set $\{\gamma_a \mid a \in A\}$ is defined as in Definition 1.4, that is,

$$\text{For } a, b \in A, \quad \gamma_a \leq \gamma_b \quad \text{iff} \quad \gamma_a \cdot \gamma_b = \gamma_a$$

By Lemma 1.3, we have that $\gamma_a \leq \gamma_b$ is equivalent to $\gamma_a \cdot G(A) \subseteq \gamma_b \cdot G(A)$.

We can show the following lemma on this order relation.

Lemma 4.3 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, ^\perp, 1, 0 \rangle$ be an orthomodular lattice. Then the following two conditions are equivalent.

- (i) $a \leq b$ on A .
- (ii) $\gamma_a \leq \gamma_b$ on $P_c(G(A))$.

Proof : ((i) $\Rightarrow$ (ii)): Suppose that $a \leq b$. Then, for all $x \in A$ the following holds:

$$\begin{aligned} \gamma_b \cdot \gamma_a(x) &= [\{(x \sqcup a^\perp) \sqcap a\} \sqcup b^\perp] \sqcap b \\ &= (x \sqcup a^\perp) \sqcap a = \gamma_a(x) \end{aligned}$$

Since we have $(x \sqcup a^\perp) \sqcap a \leq a \leq b$, we used the orthomodular law (2) in the proof of Lemma 3.5. Thus we conclude that $\gamma_a \leq \gamma_b$.

((ii) $\Leftarrow$ (i)): Suppose that $\gamma_a \leq \gamma_b$. This means that $\gamma_a \cdot \gamma_b = \gamma_b \cdot \gamma_a = \gamma_a$. Since $\gamma_a(1) \leq 1$, $\gamma_a(1) = \gamma_b \cdot \gamma_a(1) = \gamma_b(\gamma_a(1)) \leq \gamma_b(1)$. Recall here that $\gamma_a(x) := (x \sqcup a^\perp) \sqcap a$ for any $x \in A$, then we have that $a = \gamma_a(1) \leq \gamma_b(1) = b$. $\square$

As in Lemma 3.3, we can also extend the domain of valuation function $u_{\mathcal{A}}$ from the set of propositional variables to the set of all formulas Φ .

Lemma 4.4 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle$ be an orthomodular lattice and v an orthomodular valuation. Let $\mathcal{M}_{\mathcal{A}}$ be the canonical orthomodular model corresponding to $\mathcal{A}$. Then for any formula α , $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\alpha)} \cdot G(A)$.

Proof : Induction on the construction of the formula α . □

We have now reached the following Completeness Theorem.

Theorem 4.5 (Completeness theorem) For given formulas α and β , let (S) and (T) be the same statements in Theorem 2.4 . That is,

(S): for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \rightarrow A$,
 $v(\alpha) \leq v(\beta)$.

(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

Then (T) implies (S). □

5 Relation between two types of models

Theorem 5.1 Let $\mathcal{M} = \langle \mathcal{G}, u \rangle = \langle G, \cdot, *, u \rangle$ be an orthomodular model. Then $\mathcal{Q} = \langle G', R, \zeta, V \rangle$ is a quantum model, where,

- $G' := G \setminus \{0\}$,
- $\zeta := \{e \cdot G' \mid e \in P_c(G')\}$,
- R is a binary relation on G' defined as the following:
 for $x, y \in G'$, $xRy \Leftrightarrow x^* \cdot y = 0$,
- V is a function assigning to each p_i an element $V(p_i)$ of ζ .

Proof : Check the conditions for quantum model in Definition 0.4 . □

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