数理解析研究所講究録 1306 巻 2003 年 217-222

AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL GAIN IN THE FULL INFORMATION DURATION PROBLEM

Vladimir Mazalov Karelian Research Center
玉置 光司 (Mitsushi Tamaki) Faculty of Business Administration, Aichi University

1 Introduction and summary

We consider here a full-information model for the duration problem [Ferguson, Hardwik, Tamaki 1992] with horizon n tending to infinity. Our objective here is to determine the asymptotics for the optimal gain $V(n)$.

Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables, uniformly distributed on $[0,1]$, where X_n denotes the value of the object at the n-th stage from the end. We call an object relatively best if it possesses the largest value among those observed so far. The task is to select a relatively best object with the view of maximizing the duration it stays relatively best. Let $v(x,n)$ denote the optimal expected return when there are n objects yet to be observed and the present maximum of past observations is x. Notice that $V(n) = v(0,n)$.

The Optimality Equation for $v(x,n)$ has form

$$v(x,n) = xv(x,n-1) + \int_x^1 \max\{w(t,n), v(t,n-1)\}dt, \quad v(x,0) = 0, \tag{1}$$

where $w(x,n)$ denote the expected payoff given that the nth object from the last is a relatively best object of value $X_n = x$ and we select it.

$$w(x,n) = 1 + x + x^2 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}. \tag{2}$$

Denote the point of intersection of functions $v(x,n-1)$ and $w(x,n)$ as x_n. It exists and unique because $w(x,n)$ are increasing in x for every n and $v(x,n)$ are nonicreasing in x for every n, and $w(0,n) = 1 \le \int_0^1 w(t,n)dt \le v(0,n)$, and $w(1,n) = n > 0 = v(1,n-1)$.

So, we can rewrite (1) in the form

$$v(x,n) = xv(x,n-1) + \int_x^{x_n} v(t,n-1)dt + \int_{x_n}^1 w(t,n)dt, \quad 0 \le x \le x_n \tag{3}$$

and

$$v(x,n) = xv(x,n-1) + \int_x^1 w(t,n)dt, \quad x_n \le x \le 1. \tag{4}$$

If we stop the selection with a relatively best object $X_n = x$, we receive $w(x,n)$. If we continue and select the next relatively best object, we expect to receive

$$\begin{aligned} u(x,n) &= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t,n-k)dt \\ &= \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1-x^j)/j. \end{aligned} \tag{5}$$

The problem is monotone [Ferguson et al, 1992], so the one-stage look-ahead rule (OLA) is optimal here and prescribes stopping if $w(x,n) \geq u(x,n)$; that is, if

$$\sum_{k=1}^{n} x^{k-1} \left(1 - \sum_{j=1}^{n-k} (1-x^j)/j\right) \geq 0. \tag{6}$$

It is equivalent that we stop selection on the step n if the relatively best object has value $X_n \geq x_n$ (see the fig. 1) with x_n as the solution of the equation (6).

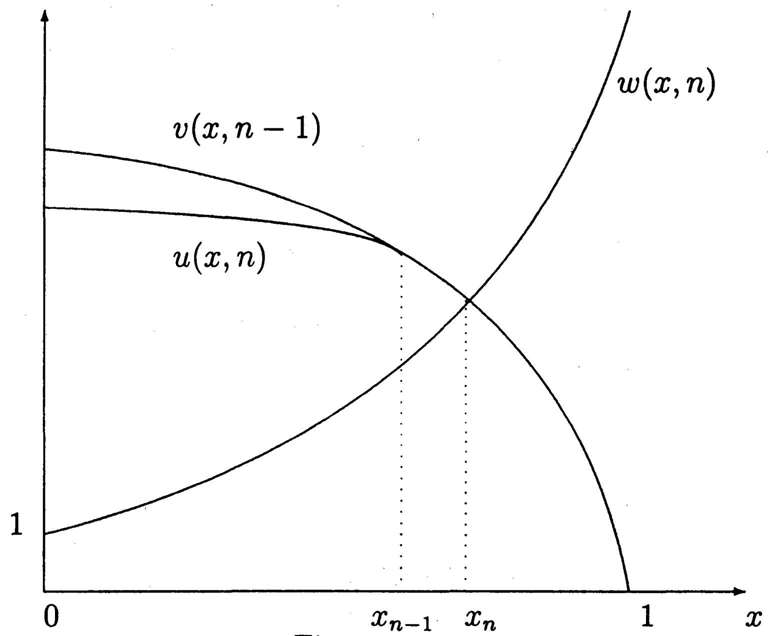

Fig. 1

According to [Porosinski, 1987], x_n written as $x_n = 1 - z_n/n$ satisfies the equation

$$\sum_{k=1}^{n} \left(1 - \frac{z_n}{n}\right)^{k-1} \left(1 - \sum_{j=1}^{n-k} (1 - (1 - \frac{z_n}{n})^j)/j\right) = 0,$$

and from here z_n must converge to a constant, $z_n \to z$, where $z \approx 2.11982$ satisfies the integral equation

$$\int_0^1 e^{-zv} \left[1 - \int_0^{1-v} \frac{1-e^{-zu}}{u} du\right] dv = 0.$$

Lemma 1. $V(n) \to \infty$ as $n \to \infty$.

Proof. It follows immediately from the inequality $v(x,n) \geq u(x,n+1)$ for every x,n and $u(0,n) = \sum_{j=1}^{n-1} 1/j \to \infty$ as $n \to \infty$.

Lemma 2. There exists a constant C such that $V(n) \leq Cn$ for all n.

Proof. Suppose that it is true for some $n-1$. Then from (3)-(4) it follows that for every x $v(x,n) \leq C(n-1) + \epsilon_n$ where $\epsilon_n = \int_{x_n}^1 w(t,n)dt$. Because $z_n \to z$, the sequence

$$\begin{aligned}
\epsilon_n &= \sum_{j=1}^{n} \frac{1-x_n^j}{j} \\
&= \sum_{j=1}^{n} \frac{1-(1-\frac{z_n}{n})^j}{j} \\
&= \sum_{j=1}^{n} \frac{1-(1-\frac{z_n}{n})^{\frac{j}{n}n}}{j/n} \frac{1}{n}
\end{aligned}$$

converges to $\int_0^1 (1-e^{-zx})/x dx \approx 1.3700$ and, consequently, it is bounded by some constant e, i.e. $\epsilon_n \leq e$. Hence, if we choose $C \geq e$ we obtain $v(x,n) \leq Cn$ for all x.

Remark. So, the function $V(n)$ tends to infinity not slower than $\log n$ and not faster than Cn.

Lemma 3. *Function $u(x,n)$ satisfies the equation*

$$u(x,n) = xu(x,n-1) + \int_x^1 w(t,n-1)dt, \quad u(x,1) = 0. \tag{7}$$

Proof. It follows from direct calculations

$$\begin{aligned}
u(x,n) - xu(x,n-1) &= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t,n-k)dt - \sum_{k=1}^{n-2} x^k \int_x^1 w(t,n-1-k)dt \\
&= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t,n-k)dx - \sum_{k=2}^{n-1} x^{k-1} \int_x^1 w(t,n-k)dx \\
&= \int_x^1 w(t,n-k)dx.
\end{aligned}$$

Let us introduce two new functions

$$y(x,n) = v(x,n) - u(x,n+1), \quad \Delta_n(x) = u(x,n) - w(x,n).$$

In the interval $[0, x_n]$ both functions are non-negative and $\Delta_n(x_n) = 0$. According to (3) and (7), $y(x, n)$ satisfies the equation

$$y(x,n) = xy(x,n-1) + \int_x^{x_n} \Big[y(t,n-1) + \Delta_n(t) \Big] dt, \quad 0 \le x \le x_n, \tag{8}$$

and $y(x, n) = 0$, for $x \ge x_n$. Also, notice that $y(x, 1) = y(x, 2) = 0$ (because $x_1 = x_2 = 0$) and $y(x, 3) = \int_x^{x_3} \Delta_3(t)dt$, where $\Delta_3(x) = 1/2 - x - 5/2x^2$.

Lemma 4. *Function $\Delta_n(x)$ satisfies the equations*

$$\Delta_n(x) = x\Delta_{n-1}(x) + \int_x^1 w(t, n-1)dt - 1, \quad \Delta_1(x) = -1. \tag{9}$$

$$\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} - x^n. \tag{10}$$

Proof. (9) follows from Lemma 3 and the simple identity

$$w(x, n) = xw(x, n-1) + 1.$$

From (9) we obtain

$$\Delta_{n+1}(x) - \Delta_n(x) = x \Big[\Delta_n(x) - \Delta_{n-1}(x) \Big] + \frac{1 - x^n}{n}.$$

After n iterations we have

$$\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} + x^n[\Delta_1(x) - \Delta_0(x)].$$

With $\Delta_1(x) - \Delta_0(x) = -1$ it proves (10).

Differentiating (8) in x we obtain

$$y'(x,n) = xy'(x,n-1) - \Delta_n(x), \quad 0 \le x \le x_n,$$

and, consequently, $y'(x, 3) = -\Delta_3(x)$,

$$y'(x,4) = \begin{cases} -x\Delta_3(x) - \Delta_4(x), & \text{if } 0 \le x \le x_3 \\ -\Delta_4(x), & \text{if } x_3 < x \le x_4 \end{cases}$$

and

$$y'(x,n) = -\sum_{j=i}^{n} x^{n-j} \Delta_j(x), \quad x_{i-1} \le x \le x_i, i = 3, 4, ..., n. \tag{11}$$

In integral form (11) is

$$y(x,n)=\sum_{j=i}^{n}\int_{x}^{x_j} t^{n-j}\Delta_j(t)dt, \quad x_{i-1}\le x\le x_i, i=3,4,...,n. \tag{12}$$

Letting in (12) $x=0$ we obtain

Lemma 5.

$$y_n=y(0,n)=\sum_{j=3}^{n}\int_{0}^{x_j} t^{n-j}\Delta_j(t)dt. \tag{13}$$

Consider now the difference $r_n=y_{n+1}-y_n$. Using (13) we can represent it as a sum of two expressions

$$\begin{aligned} r_n &= y_{n+1}-y_n \\ &= \sum_{j=3}^{n}\int_{0}^{x_j} t^{n-j}[\Delta_{j+1}(t)-\Delta_j(t)]dt+\sum_{j=3}^{n+1}\int_{x_{j-1}}^{x_j} t^{n-j+1}\Delta_j(t)dt \end{aligned} \tag{14}$$

Using (10) the first sum in (14) can be rewritten in the form

$$\sum_{j=3}^{n}\int_{0}^{x_j} t^{n-j}\left[\sum_{i=1}^{j}\frac{t^{j-i}-t^j}{i}-t^j\right]dt=\sum_{j=3}^{n}\left\{\sum_{i=1}^{j}\left[\frac{x_j^{n-i+1}}{n-i+1}-\frac{x_j^{n+1}}{n+1}\right]\frac{1}{i}-\frac{x_j^{n+1}}{n+1}\right\}. \tag{15}$$

As $n\to\infty$ letting $x_n=1-z_n/n$ we obtain that (15) converges to the integral

$$C_0=\int_0^1 e^{-\frac{z}{u}}\left[\int_0^u\left(\frac{e^{\frac{zv}{u}}-1}{v}+\frac{e^{\frac{zv}{u}}}{1-v}\right)dv-1\right]du. \tag{16}$$

Let us prove that the second sum in (14) tends to zero. Firstly notice that $\Delta_{j-1}(x)\le 0$ for $x\ge x_{j-1}$, consequently, from Lemma 4 it follows that

$$\Delta_j(x)\le\int_x^1 w(t,j-1)dt-1\le\sum_{i=1}^{j-1}\frac{1}{i}, \quad x\in[x_{j-1},x_j].$$

Hence,

$$\begin{aligned} 0 &\le \sum_{j=3}^{n+1}\int_{x_{j-1}}^{x_j} t^{n-j+1}\Delta_j(t)dt \\ &\le \sum_{j=3}^{n+1}\sum_{i=1}^{j-1}\int_{x_{j-1}}^{x_j}\frac{t^{n-j+1}}{i}dt \\ &= \sum_{i=1}^{n}\frac{1}{i}\sum_{j=i+1}^{n+1}\left[\frac{x_j^{n-j+2}-x_{j-1}^{n-j+2}}{n-j+2}\right] \end{aligned}$$

$$\leq \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} \left[x_j^{n-j+1}(x_j - x_{j-1}) \right]$$

$$= \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} \left[(1 - \frac{z_j}{j})^{n-j+1} \left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j} \right) \right]. \tag{17}$$

Using the inequality $1 - x \leq \exp(-x), \forall x$, (17) is not larger than

$$\sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} \left[\exp\left\{ -\frac{z_j}{j}(n-j+1) \right\} \left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j} \right) \right]$$

$$\leq \exp\{Z\} \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} \left[\exp\left\{ -\frac{z_j(n+1)}{j} \right\} \left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j} \right) \right], \tag{18}$$

where $Z = \sup z_j$. The last sum (18) converges as $n \to \infty$ to zero, because $z_n \to z$ and corresponding for internal sum integral $\int_{i+1}^{n+1} \exp\{-\frac{z(n+1)}{t}\} z/t^2 dt$ is not larger than $\exp\{-z\}/(n+1)$.

Summarizing all, we obtain the next result.

Theorem 1. *For large n*

$$\frac{V_n}{n} \to C_0,$$

where C_0 is the value of the integral (16).

Proof follows immediately from $\lim_{n\to\infty} V_n/n = \lim_{n\to\infty} v(0,n)/n = \lim_{n\to\infty}[y_n + u(0, n+1)]/n = \lim_{n\to\infty}[y_n + \sum_{j=1}^{n} 1/j]/n = C_0$. Package *Mathematica* calculates $C_0 \approx 0.4351708$.

REFERENCES

1. T.S.Ferguson, J.P.Hardwick, M.Tamaki, *Maximization of duration of owning a relatively best object*, Contemporary Mathematics, vol. 125, 1992, 37–57.

2. Z. Porosinski, *The Full-information best-choice problem with a random number of observations*, Stoch. Proc. and Appl. **24**, 1987, 293-307.