

APPLICATIONS OF NONCOMMUTATIVE DEFORMATIONS

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ABSTRACT. For a general class of contractions of a variety X to a base Y , I discuss recent joint work with M. Wemyss defining a noncommutative enhancement of the locus in Y over which the contraction is not an isomorphism, along with applications to the derived symmetries of X .

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Derived symmetry groups of algebraic varieties extend classical symmetry groups to include contributions from symplectic geometry via homological mirror symmetry, and from birational geometry. In a recent joint paper with M. Wemyss [9], for a general class of birational contractions $f: X \rightarrow Y$, we construct a sheaf of noncommutative algebras $\mathcal{A}$ on Y which, in an appropriate crepant setting, induces a derived symmetry of X . This short note explains key features of our results.

The sheaf $\mathcal{A}$ is supported on the locus of Y over which f is not an isomorphism. In previous joint work [6, 8] we considered contractions of 3-folds for which this locus is just a point. In this setting we studied an algebra of noncommutative deformations A which allowed new constructions of derived symmetries, and extended and unified known invariants of such contractions. I begin by reviewing this, as $\mathcal{A}$ may be viewed as a sheafy version of the algebra A .

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I also briefly discuss an example in which f is a Springer resolution (§3), and indicate recent work in which deformation algebras are used to recover the geometry of contractions (§4).

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Conventions. I work over the ground field $\mathbb{C}$, though this assumption can be weakened. Varieties X are assumed quasi-projective, with bounded derived category of coherent sheaves denoted by $D(X)$. The variety of hyperplanes in a vector space V is denoted by $\mathbb{P}V$.

1. DEFORMATION ALGEBRAS FOR 3-FOLDS

The theorem below applies noncommutative deformations to study derived symmetries of 3-folds. Given smooth 3-folds X and X' related by a flop, Bridgeland [3] constructs certain canonical Fourier–Mukai equivalences

$$D(X) \begin{array}{c} \xleftarrow{F} \\ \sim \\ \xrightarrow{F'} \end{array} D(X').$$

These equivalences are not mutually inverse: the theorem explains this using deformations of curves on X .

Consider a 3-fold Y with an isolated rational singular point p , and a resolution $f: X \rightarrow Y$ of this singularity, with one-dimensional exceptional locus. Write C^i for the components of the exceptional locus.

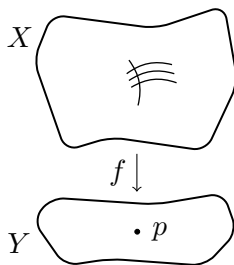


FIGURE A. Contraction f for Theorem A.

Theorem A. [6, 7, 8] *The subvarieties $C^1, \dots, C^n$ of the resolution X are projective lines, and:*

- (1) *there exists a $\mathbb{C}^n$ -algebra A which represents the functor of non-commutative deformations of the sheaves $\mathcal{O}_{C^i}(-1)$ on X .*

Write E for the corresponding universal sheaf on X . If the contraction f corresponds to a flop of X , then:

- (2) *there is a Fourier–Mukai autoequivalence T_E of $D(X)$, fitting into a distinguished triangle of functors*

$$\mathbb{R}\mathrm{Hom}_X(E, -) \overset{\mathbb{L}}{\otimes}_A E \longrightarrow \mathrm{Id}_{D(X)} \longrightarrow \mathsf{T}_E \longrightarrow;$$

- (3) *there is a natural isomorphism of functors*

$$\mathsf{T}_E \cong (\mathsf{F}' \circ \mathsf{F})^{-1}.$$

In the simplest flopping situation, where f contracts a $(-1, -1)$ -curve, the autoequivalence T_E is a spherical twist in the sense of Seidel–Thomas [18]. For a contraction of a $(-2, 0)$ -curve, it is a generalized spherical twist as first constructed by Toda [19], who furthermore established the conclusion of Theorem A(3) in this case.

Remark. The noncommutative deformation theory used here relies on work of Laudal [15], Eriksen [11], E. Segal [17], and Efimov–Lunts–Orlov [10].

Remark. The algebra A above, and similar noncommutative deformation algebras, have now been applied in a number of settings including: enumerative geometry of curves on 3-folds by Toda and Hua–Toda [20, 13]; flops of families of curves in higher dimensions by Bodzenta and Bondal [2]; construction of autoequivalences and exceptional objects by Kawamata [14]; and new braid-type groups of derived symmetries of 3-folds by the author and Wemyss [8].

Remark. The full statement of Theorem A does not require X to be smooth: I leave details to the references.

2. GENERAL RESULTS

The following theorem from [9] gives a sheafy analogue of the deformation algebra A , applicable in higher dimensions. For a birational contraction $f: X \rightarrow Y$ satisfying the assumption below, we define a sheaf of algebras $\mathcal{A}$ on Y which is supported on the locus over which f is not an isomorphism. We furthermore construct an associated autoequivalence of $D(X)$.

Assumption. Suppose that $f: X \rightarrow Y$ is a contraction with $\dim X \geq 2$, and that either:

- (a) the variety X has an f -relative tilting generator with summand $\mathcal{O}_X$, where f is crepant, and Y is Gorenstein;

or, alternatively,

- (b) the fibres of f have dimension at most one.

Remark. The tilting generator assumption from (a) is satisfied in a range of situations, including symplectic resolutions of quotient singularities as established by Bezrukavnikov and Kaledin [1], and contractions with fibres of dimension at most two under conditions of Toda and Uehara [21].

Write Z for the locus in Y over which f is not an isomorphism.

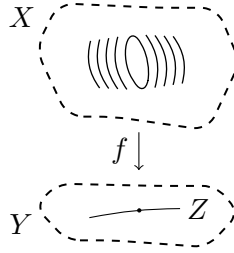


FIGURE B. Contraction f for Theorem B.

Theorem B. [9] Under the Assumption above, there is a sheaf of algebras $\mathcal{A}$ on Y , inducing an object $\mathcal{E}$ of $D(X)$, such that:

- (1) the support of $\mathcal{A}$ is Z ;

and, assuming furthermore that

- (i) the contraction f is crepant,
- (ii) the base Y is complete locally a hypersurface at each point of Z ,

and that either $\text{codim } Z \geq 3$ or, alternatively,

- (iii) the sheaf $\mathcal{A}$ is Cohen–Macaulay, and
- (iv) the object $\mathcal{E}$ is perfect,

then:

- (2) there is a Fourier–Mukai autoequivalence $\mathbb{T}_{\mathcal{E}}$ of $D(X)$, fitting into a distinguished triangle of functors

$$f^{-1}\mathbb{R}f_*\mathbb{R}\text{Hom}_X(\mathcal{E}, -) \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{A}} \mathcal{E} \longrightarrow \text{Id}_{D(X)} \longrightarrow \mathbb{T}_{\mathcal{E}} \longrightarrow .$$

Remark. When Z is a point, the autoequivalence $T_{\mathcal{E}}$ reduces to the autoequivalence T_E appearing in Theorem A(2).

I now indicate the construction of the sheaf of algebras $\mathcal{A}$. Under the Assumption above, we have an f -relative tilting generator $\mathcal{O}_X \oplus N$, either by assertion in case (a), or by a theorem of Van den Bergh [22] in case (b). Let $\mathcal{T}$ denote the relative endomorphism algebra of $\mathcal{O}_X \oplus N$, a sheaf of algebras on Y . We then establish that

$$\mathcal{T} = f_* \mathcal{E}nd_X(\mathcal{O}_X \oplus N) \cong \mathcal{E}nd_Y f_*(\mathcal{O}_X \oplus N).$$

This allows us to make the following definition, generalizing a construction of the algebra A from our previous work [6].

Definition. Let $\mathcal{A} = \mathcal{T}/\mathcal{I}$, a sheaf of algebras on Y , where $\mathcal{I}$ is the ideal of sections of $\mathcal{T}$ which factor, at each stalk, through a sum of copies of $\mathcal{O}_Y$.

The object $\mathcal{E}$ of $D(X)$ is then defined as the image of $\mathcal{A}$ under an appropriate tilting equivalence: I refer to [9, Section 3] for a precise statement.

Remark. Although the tilting generator $\mathcal{O}_X \oplus N$, and thence the sheaf of algebras $\mathcal{A}$, is not canonically defined (see for instance the construction of Van den Bergh in [22]) it seems that the autoequivalence $T_{\mathcal{E}}$ may be canonical, given a choice of contraction f .

Remark. For f a flopping contraction, it would be interesting to establish when $T_{\mathcal{E}}$ is related to a flop-flop functor, as in Theorem A(3).

Remark. It is tempting to speculate that the ‘tilting’ condition in the requirement for an f -relative tilting generator in (a) may be relaxed by upgrading $\mathcal{A}$ to an appropriate sheaf of differential graded algebras.

In the case of one-dimensional fibres, the construction above has the following deformation-theoretic interpretation.

Theorem B (continued). *For a point z of Z such that $f^{-1}(z)$ is one-dimensional with components C^i , then:*

- (3) *the completion $\mathcal{A}_z$ is an algebra which prorepresents the functor of noncommutative deformations of the sheaves $\mathcal{O}_{C^i}(-1)$ on X , up to Morita equivalence;*
- (4) *the restriction of $\mathcal{E}$ to the formal fibre of f over z is a sheaf, namely the universal family corresponding to (3), up to summands of finite sums of sheaves.*

I record the following 3-fold setting where the assumptions of Theorem B may be established.

Theorem C. [9] *With $\dim X = 3$, assume that*

- (i) *the contraction f is crepant,*
- (ii) *the base Y is complete locally a hypersurface at each point of Z ,*
- (iii) *the exceptional fibres of f are irreducible curves.*

Then the assumptions of Theorem B hold, and there exists an associated autoequivalence $\mathsf{T}_{\mathcal{E}}$ of $D(X)$.

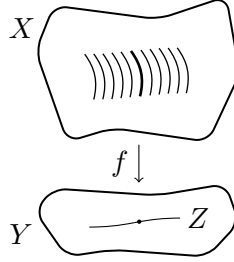


FIGURE C. Contraction f for Theorem C.

3. SPRINGER RESOLUTION EXAMPLE

For an example in which the theory of the previous section applies to a contraction with higher-dimensional fibres, consider the Springer resolution of the variety of singular d -by- d matrices. Namely, for a vector space V of dimension d with $d \geq 2$, take the singular cone

$$Y = \{M \in \operatorname{End} V \mid \det M = 0\},$$

which is a Gorenstein hypersurface. It has a resolution by

$$X = \{(M, H) \in \operatorname{End} V \times \mathbb{P}V \mid \operatorname{Im} M \subseteq H\}$$

whose natural projection f to $\operatorname{End} V$ surjects onto Y . This resolution f is crepant. Its exceptional fibres lie over points M in Y with $\operatorname{rk} M < d - 1$, and are projective spaces of dimension $d - 1 - \operatorname{rk} M$.

A tilting generator for X has been constructed by Buchweitz, Leuschke, and Van den Bergh [4], so that we are in the setting of Assumption (a). Conditions (i) and (ii) of Theorem B are noted above, and $\operatorname{codim} Z \geq 3$, so we can apply Theorem B(2) to obtain an autoequivalence $\mathsf{T}_{\mathcal{E}}$ of $D(X)$.

Remark. For $d = 2$, the variety X is just a 3-fold resolving a conifold Y with exceptional fibre a $(-1, -1)$ -curve, and we are in the setting of Theorem A.

Remark. In joint work with E. Segal [5], I studied the resolution f , along with more general resolutions where Y is the variety of d -by- d matrices of rank at most r for $0 < r < d$. We constructed certain ‘Grassmannian twist’ autoequivalences of the corresponding $D(X)$ by quite different methods: it would be interesting to compare these with $T_{\mathcal{E}}$.

Remark. The sheaf of algebras $\mathcal{A}$ for this example may be computed from the presentation of the endomorphism algebra $\mathcal{T}$ in [4].

4. CONJECTURES

In the setting of a 3-fold flopping contraction as in Theorem A, we made a conjecture [6, Conjecture 1.4] that the complete local neighbourhood of the 3-fold Y near the singularity p is determined, up to isomorphism, by the deformation algebra A . This conjecture is clear in the following simple cases, namely the two kinds of flopping curve for which A is commutative, but remains open more generally.

- (1) Contractions of $(-1, -1)$ -curves. In this case $A = \mathbb{C}$, and the completion of Y at p is necessarily a conifold singularity.
- (2) Contractions of $(-2, 0)$ -curves. Here $A \cong \mathbb{C}[\varepsilon]/\varepsilon^w$ with $w \geq 2$, where w is the width invariant of Reid [16], and the completion of Y at p is determined by w .

Hua and Toda subsequently proposed an A_{∞} version of the conjecture [13, Conjecture 5.3] in which A is endowed with the structure of an A_{∞} -algebra. They established their conjecture for contractions to weighted homogeneous hypersurface singularities [13, Theorem 5.5], and it has now been settled in general by Hua [12]. A key idea in these works is that the deformation algebra A may be viewed as a noncommutative analogue of the Milnor algebra, and that the A_{∞} structure on it allows recovery of the Milnor algebra along with enough information to apply a version of the Mather–Yau theorem.

Remark. It would be interesting to extend these results to higher dimensions, and to non-isolated singularities, for instance to establish assumptions under which the complete local neighbourhood of the variety Y near the non-isomorphism locus Z is determined by $(Z, \mathcal{A})$, potentially along with some appropriate A_{∞} structure.

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