

RELATIONSHIP BETWEEN THE MILNOR'S μ -INVARIANT AND HOMFLYPT POLYNOMIAL

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1. INTRODUCTION

For an ordered oriented link in the 3-sphere, J. Milnor [15, 16] defined a family of invariants, known as *Milnor's $\bar{\mu}$ -invariants*. For an n -component link L , Milnor invariant is determined by a sequence I of elements in $\{1, 2, \dots, n\}$ and denoted by $\bar{\mu}_L(I)$. It is known that Milnor invariants of length two are just linking numbers. In general, Milnor invariant $\bar{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\bar{\mu}_L(J)$ such that J is a subsequence of I obtained by removing at least one index or its cyclic permutation. If the sequence is of distinct numbers, then this invariant is also a link-homotopy invariant and we call it *Milnor's link-homotopy invariant*. Here, the *link-homotopy* is an equivalence relation generated by ambient isotopy and self-crossing changes.

In [3], N. Habegger and X. S. Lin showed that Milnor invariants are also invariants for string links, and these invariants are called Milnor's μ -invariants. For any string link σ , $\mu_\sigma(I)$ coincides with $\bar{\mu}_{\hat{\sigma}}(I)$ modulo $\Delta_{\hat{\sigma}}(I)$, where $\hat{\sigma}$ is a link obtained by the closure of σ . Milnor's μ -invariants of length k are finite type invariants of degree $k - 1$ for any natural integer k , as shown by D. Bar-Natan [1] and X. S. Lin [11].

In [17], M. Polyak gave a formula expressing Milnor's $\bar{\mu}$ -invariant of length 3 by the Conway polynomials of knots. His idea was derived from the following relation. Both Milnor's μ -invariant of length 3 for string link and the second coefficient of the Conway polynomial are finite type invariants of degree 2. He gave this relation by using Gauss diagram formulas.

Then, in [14], J-B. Meilhan and A. Yasuhara generalized it by using the clasper theory introduced by K. Habiro [4]. They showed that general Milnor's $\bar{\mu}$ -invariants can be represented by the HOMFLYPT polynomials of knots under some assumption. Moreover the author and A. Yasuhara improved it in [9].

In [8], we give a formula expressing Milnor's μ -invariant by the HOMFLYPT polynomials of knots under some assumption (Theorem 3.1) by using the clasper theory in [4]. The course of proof is similar to that in [14] and [9]. Moreover, Milnor's μ -invariants of length 3 for any string link are given by the HOMFLYPT polynomial, which is a finite type invariant of degree 2, and the linking number. Because a finite type knot invariant of degree 2 is only the second coefficient of the Conway polynomial essentially, Milnor's μ -invariants of length 3 are given by the second coefficient of the Conway polynomial and the linking number (Theorem 3.3). It is a string version of Polyak's result, and by taking modulo $\Delta(I)$, our result coincides with Polyak's result.

2. MILNOR'S μ -INVARIANT AND HOMFLYPT POLYNOMIAL

2.1. String link. Let n be a positive integer and $D^2 \subset \mathbb{R}^2$ the unit disk equipped with n marked points $x_1, x_2, \dots, x_n$ in its interior, lying in the diameter on the x -axis of $\mathbb{R}^2$ as in Figure 1. Let $I = [0, 1]$. An n -string link σ is the image of a proper embedding $\sqcup_{i=1}^n I_i \rightarrow D^2 \times I$ of the disjoint union of n copies of I in $D^2 \times I$, such that $\sigma|_{I_i}(0) = (x_i, 0)$ and $\sigma|_{I_i}(1) = (x_i, 1)$ for each i as in Figure 1. Each string of a string link inherits an orientation from the usual orientation of I . The n -string link $\{x_1, x_2, \dots, x_n\} \times I$ in $D^2 \times I$ is called the *trivial n -string link* and denoted by $\mathbf{1}_n$ or $\mathbf{1}$ simply.

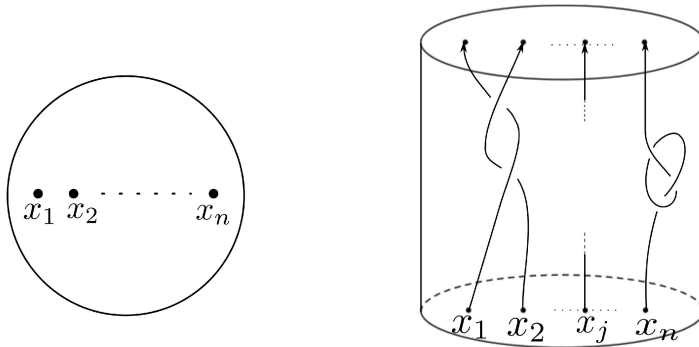


FIGURE 1. An n -string link

Given two n -string links σ and σ' , we denote their product by $\sigma \cdot \sigma'$, which is given by stacking σ' on the top of σ and reparametrizing the ambient cylinder $D^2 \times I$. By this product, the set of isotopy classes of n -string links has a monoid structure with unit given by the trivial string link $\mathbf{1}_n$. Moreover, the set of link-homotopy classes of n -string links is a group under this product.

2.2. Milnor's μ -invariant for string links. Let $\sigma = \cup_{i=1}^n \sigma_i$ in $D^2 \times I$ be an n -string link. We consider the fundamental group $\pi_1(D^2 \times I \setminus \sigma)$ of the complement of σ in $D^2 \times I$, where we choose a point b as a base point and curves $\alpha_1, \dots, \alpha_n$ as meridians in Figure 2.

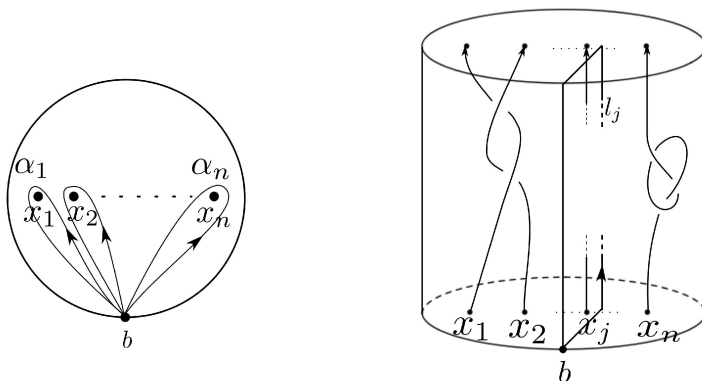


FIGURE 2. Longitude of string link

By Stallings' theorem [18], for any positive integer q , the inclusion map

$$\iota : D^2 \times \{0\} \setminus \{x_1, \dots, x_n\} \longrightarrow D^2 \times I \setminus \sigma$$

induce an isomorphism of the lower central series quotients of the fundamental groups

$$\iota_* : \frac{\pi_1(D^2 \times \{0\} \setminus \{x_1, \dots, x_n\})}{(\pi_1(D^2 \times \{0\} \setminus \{x_1, \dots, x_n\}))_q} \longrightarrow \frac{\pi_1(D^2 \times I \setminus \sigma)}{\pi_1(D^2 \times I \setminus \sigma)_q},$$

where given a group G , G_q means the q -th lower central subgroup of G . The fundamental group $\pi_1(D^2 \times \{0\} \setminus \{x_1, \dots, x_n\})$ is a free group generated by $\alpha_1, \dots, \alpha_n$. We then consider the j -th longitude l_j of σ in $D^2 \times I$, where l_j is the closure of the preferred parallel curve of σ_j , whose endpoints lie on the x -axis in $D^2 \times \{0, 1\}$ as in Figure 2. We then consider the image of the longitude $\iota_*^{-1}(l_j)$ by the Magnus expansion and denote $\mu(i_1, \dots, i_k, j)$ the coefficient of $X_{i_1}X_{i_2} \cdots X_{i_k}$ in the Magnus expansion.

Theorem 2.1 ([3]). *For any positive integer q , if $k < q$, then $\mu(i_1, \dots, i_k, j)$ is invariant under isotopy. Moreover, if the sequence $i_1, \dots, i_k, j$ is of distinct numbers, then $\mu(i_1, \dots, i_k, j)$ is also link-homotopy invariant.*

We call this invariant Milnor's μ -invariant.

2.3. HOMFLYPT polynomial. Recall the definition of the HOMFLYPT polynomial.

The *HOMFLYPT polynomial* $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link L is defined by the following two formulas:

- (1) $P(U; t, z) = 1$, and
- (2) $t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z)$,

where U denotes the trivial knot and L_+ , L_- and L_0 are link diagrams which are identical everywhere except near one crossing, where they look as follows:

$$L_+ = \begin{array}{c} \nearrow \\ \searrow \end{array} ; L_- = \begin{array}{c} \nwarrow \\ \swarrow \end{array} ; L_0 = \begin{array}{c} \nearrow \\ \nearrow \end{array} \quad \begin{array}{c} \nwarrow \\ \nwarrow \end{array}.$$

Recall that the HOMFLYPT polynomial of a knot K is of the form $P(K; t, z) = \sum_{k=0}^N P_{2k}(K; t)z^{2k}$, where $P_{2k}(K; t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $2k$ -th coefficient polynomial of K .

3. MAIN THEOREM

Given a sequence I of elements of $\{1, 2, \dots, n\}$, $J < I$ will be used for any subsequence J of I , possibly I itself, and $|J|$ will denote the length of the sequence J .

Let σ be an n -string link. Given a sequence $I = i_1 i_2 \cdots i_m$ obtained from $12 \cdots n$ by deleting some elements, and a subsequence $J = j_1 j_2 \cdots j_k$ of I , we define a knot $\overline{\sigma}_{I,J}$ as the closure of the product $b_I \cdot \sigma_J$. Here σ_J is the m -string link obtained from σ by deleting the i -th string, for all $i \in \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_m\}$ and replacing the i -th string with a trivial string underpassing all other components, for all $i \in \{i_1, i_2, \dots, i_m\} \setminus \{j_1, j_2, \dots, j_k\}$, and b_I is the m -braid associated with the permutation $b = \begin{pmatrix} i_1 & i_2 & \cdots & i_{m-1} & i_m \\ i_2 & i_3 & \cdots & i_m & i_1 \end{pmatrix}$ and such that the arc with connecting $(b^k(i_1), 0)$ with $(b^{k+1}(i_1), 1)$ underpasses all arcs with connecting $(b^k(i_1), 0)$ with $(b^{k'+1}(i_1), 1)$ in $[0, 1] \times [0, 1]$ of braid diagram for $k < k' < n$. See Figure 3 for an example. We then have the following Theorem.

Theorem 3.1. *Let σ be an n -string link ($n \geq 4$) with vanishing Milnor's link-homotopy invariants of length $\leq m - 2$. Then for any sequence I obtained from $12 \cdots n$ by deleting $n - m$ elements, we have*

$$\mu_\sigma(I) = \frac{(-1)^{m-1}}{(m-1)!2^{m-1}} \sum_{J < I} (-1)^{|J|} P_0^{(m-1)}(\overline{\sigma_{I,J}}; 1),$$

where $P_0^{(m-1)}(\cdot; 1)$ is the $(m-1)$ -th derivative of the 0-th coefficient $P_0(\cdot; t)$ of the HOM-FLYPT polynomial evaluated at $t = 1$.

Note that the above vanishing assumption for string link is equivalent to that any $(m-2)$ -substring link is link-homotopic to the trivial string link.

Remark 3.2. Theorem 1.1 remains valid if we use one of the following two alternative definitions of b_I . One is that we use “overpasses” instead of “underpasses”. The other is that we use “any $i \in \{i_1, i_2, \dots, i_m\}$ ” instead of “ i_1 ”.

We also give the case of μ -invariants of length 3 without the assumption.

Theorem 3.3. *Let σ be an n -string link and $I = i_1 i_2 i_3$ be a length 3 sequence with distinct numbers in $\{1, 2, \dots, n\}$. We then have*

$$\mu_\sigma(I) = - \sum_{J < I} (-1)^{|J|} a_2(\overline{\sigma_{I,J}}) - lk_\sigma(i_1 i_2) lk_\sigma(i_2 i_3) + A_I,$$

where a_2 is the second coefficient of the Conway polynomial, $lk_\sigma(ij)$ is the linking number of the i -th component and j -th component of σ , and

$$A_I = \begin{cases} lk_\sigma(i_1 i_2) & (i_2 < i_3 < i_1) \\ -lk_\sigma(i_1 i_2) & (i_1 < i_3 < i_2) \\ 0 & (\text{otherwise}). \end{cases}$$

Remark 3.4. This operation from a string link to a knot corresponds to Y -graph sum of links defined by M. Polyak. By taking this formula modulo $\Delta_{\overline{\sigma_{I,J}}}(I)$, we get Polyak's relation between Milnor's $\bar{\mu}$ -invariants and Conway polynomials [17].

Remark 3.5. In [19], K. Taniyama gave a formula expressing Milnor's $\bar{\mu}$ -invariants of length 3 for links by the second coefficient of the Conway polynomial assuming that all linking numbers vanish.

Remark 3.6. In [12], J.B. Meilhan showed that all finite type invariants of degree 2 for string link was given a formula by some invariants (Theorem 2.8). So the formula in Theorem 3.3 could also be derived from [12].

4. EXAMPLES

Example 4.1. Let σ be a 3-string link showed by Figure 3. Then $\mu_{123}(\sigma) = -1$, $\mu_{132}(\sigma) = \mu_{213}(\sigma) = 1$ and $\mu_{231}(\sigma) = \mu_{312}(\sigma) = \mu_{321}(\sigma) = 0$. And $lk_\sigma(12) = lk_\sigma(23) = 1$ and $lk_\sigma(13) = 0$.

On the other hand, $\overline{\sigma_{123,123}}$ and $\overline{\sigma_{123,23}}$ are the figure-eight knot, and $\overline{\sigma_{123,J}}$ ($J \neq 123, 23$) is the trivial knot. Therefore we obtain

$$- \sum_{J < 123} (-1)^{|J|} a_2(\overline{\sigma_{123,J}}) - lk_\sigma(12) lk_\sigma(23) = a_2(4_1) - a_2(4_1) - 1 \cdot 1 = -1.$$

Similarly, we have

$$\begin{aligned}
 & - \sum_{J < 231} (-1)^{|J|} a_2(\overline{\sigma_{231,J}}) - lk_{\sigma}(23)lk_{\sigma}(31) = a_2(3_1 \# 4_1) - a_2(3_1) - a_2(4_1) - 1 \cdot 0 = 0, \\
 & - \sum_{J < 312} (-1)^{|J|} a_2(\overline{\sigma_{312,J}}) - lk_{\sigma}(31)lk_{\sigma}(12) + lk_{\sigma}(13) = a_2(3_1) - a_2(3_1) - 0 \cdot 1 + 0 = 0.
 \end{aligned}$$

Moreover, $\overline{\sigma_{132,32}}$ is the figure-eight knot and $\overline{\sigma_{132,J}}$ ($J \neq 32$) is the trivial knot. Therefore we obtain

$$- \sum_{J < 132} (-1)^{|J|} a_2(\overline{\sigma_{132,J}}) - lk_{\sigma}(13)lk_{\sigma}(32) - lk_{\sigma}(13) = -a_2(4_1) - 0 \cdot 1 - 0 = 1.$$

Similarly, we have

$$\begin{aligned}
 & - \sum_{J < 213} (-1)^{|J|} a_2(\overline{\sigma_{213,J}}) - lk_{\sigma}(21)lk_{\sigma}(13) = a_2(7_6) - a_2(3_1) - a_2(4_1) - 1 \cdot 0 = 1, \\
 & - \sum_{J < 321} (-1)^{|J|} a_2(\overline{\sigma_{321,J}}) - lk_{\sigma}(32)lk_{\sigma}(21) = a_2(5_2) - a_2(3_1) - 1 \cdot 1 = 0.
 \end{aligned}$$

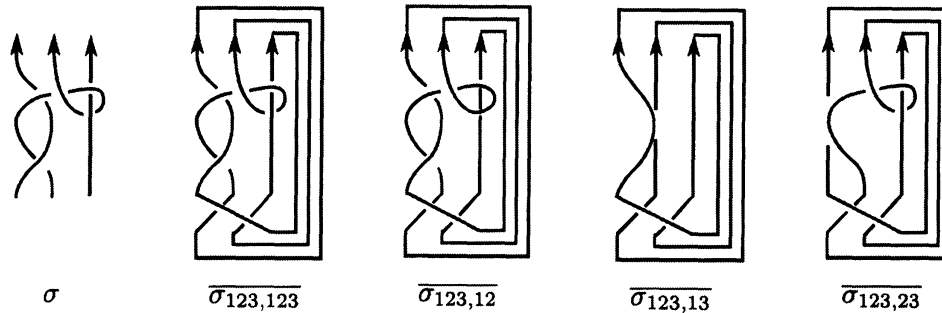


FIGURE 3

REFERENCES

- [1] D. Bar-Natan, *Vassiliev homotopy string link invariants*, J. Knot Theory Ram. **4**, no. 1 (1995), 13–32.
- [2] T. Fleming, A. Yasuhara, *Milnor's invariants and self C_k -equivalence*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 761–770.
- [3] N. Habegger and X.S. Lin, *The classification of links up to link-homotopy*, J. Amer. Math. Soc. **3** (1990), 389–419.
- [4] K. Habiro, *Claspers and finite type invariants of links*, Geom. Topol. **4** (2000), 1–83.
- [5] K. Habiro, J.B. Meilhan, *Finite type invariants and Milnor invariants for Brunnian links*, Int. J. Math. **19**, no. 6 (2008), 747–766.
- [6] T. Kanenobu, *C_n -moves and the HOMFLY polynomials of links*, Bol. Soc. Mat. Mexicana (3) **10** (2004), 263–277.
- [7] T. Kanenobu, Y. Miyazawa, *HOMFLY polynomials as Vassiliev link invariants*, in *Knot theory*, Banach Center Publ. **42**, Polish Acad. Sci., Warsaw (1998), 165–185.
- [8] Y. Kotorii, *A relation between Minor's μ -invariants and HOMFLYPT polynomials*, arXiv:math/1503.08026.
- [9] Y. Kotorii, A. Yasuhara, *Milnor invariants of length $2k+2$ for links with vanishing Milnor invariants of length $\leq k$* , Topology and its Applications, Vol 184, 87–100 (2015).
- [10] W. B. R. Lickorish, K. C. Millett, *A polynomial invariant of oriented links*, Topology **26** (1987), 107–141.

- [11] X.S. Lin, *Power series expansions and invariants of links*, in “Geometric topology”, AMS/IP Stud. Adv. Math. 2.1, Amer. Math. Soc. Providence, RI (1997) 184–202.
- [12] J.B. Meilhan, *On Vassiliev invariants of order two for string links*, J. Knot Theory Ram. **14** (2005), No. 5, 665–687.
- [13] J.B. Meilhan, A. Yasuhara, *On Cn -moves for links*, Pacific J. Math. **238** (2008), 119–143.
- [14] J.B. Meilhan, A. Yasuhara, *Milnor invariants and the HOMFLYPT polynomial*, Geom. Topol. **16** (2012), 889–917.
- [15] J. Milnor, *Link groups*, Ann. of Math. (2) **59** (1954), 177–195.
- [16] J. Milnor, *Isotopy of links*, Algebraic geometry and topology, A symposium in honor of S. Lefschetz, pp. 280–306, Princeton University Press, Princeton, N. J., 1957.
- [17] M. Polyak, *On Milnor’s triple linking number*, C. R. Acad. Sci. Paris Sé. I Math. **325** (1997), no. 1, 77–82.
- [18] J. Stallings, *Homology and central series of groups*, J. Algebra, **2** (1965), 170–181.
- [19] K. Taniyama, *Link homotopy invariants of graphs in $\mathbf{R}^3$* , Rev. Mat. Univ. Complut. Madrid **7** (1994), no. 1, 129–144.
- [20] A. Yasuhara, *Self Delta-equivalence for Links Whose Milnor’s Isotopy Invariants Vanish*, Trans. Amer. Math. Soc. **361** (2009), 4721–4749.

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