

Iterated convolution product of index $1/k$ and endlessly continuable functions

By

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Abstract

In [6], [8], [13], analytic continuation of iterated convolution products of endlessly continuable functions was discussed and a certain uniform estimate for them was given. It is in principle for analytic functions given as the Borel transform of formal series of Gevrey order 1. In [1], it is generalized to the case of Gevrey order $1/k$ ($k > 0$) and applied to nonlinear ordinary differential equations. In this paper, we report on the results in the forthcoming paper [1].

§ 1. Introduction

In [6], [8], [13], analytic continuation of iterated convolution products of endlessly continuable functions was discussed and a certain uniform estimate for them was given. The results enable us to analyze singularities of convolution products of endlessly continuable functions, verify the resurgence of formal series solutions of nonlinear ordinary differential equations and so on. The target of these works is endlessly continuable functions arose from formal series of Gevrey order 1 as their Borel transform. As a generalization of the works, we consider the case of Gevrey order $1/k$ in this paper with a view to applications to differential equations discussed in Section 4. About the Borel summability (and the multisummability also) for such cases, plenty of results has already been obtained. (See e.g. [2], [3] and [9] for the details.) However, we have few

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literatures about the resurgence though it is reduced to the case $k = 1$ in principle. In the forthcoming paper [1], such cases are handled by generalizing the results in [6]. In this paper, we report on the results in [1].

§ 2. Endlessly continuable functions

In this section, we introduce several definitions to discuss analytic continuation of Borel transformed formal series of Gevrey order $s(> 0)$.

§ 2.1. The Borel transform

Let $\varphi(x) = \sum_{j=0}^{\infty} \varphi_j x^j \in \mathbb{C}[[x]]$ be a formal series. We say $\varphi(x)$ is of Gevrey order s if it satisfies the following: there exist positive constants $C, A > 0$ such that

$$(2.1) \quad |\varphi_j| \leq CA^j \Gamma(sj)$$

holds for any $j \geq 0$. The set of formal series of Gevrey order s is denoted by $\mathbb{C}[[x]]_s$. When $\varphi_0 = 0$, we define the Borel transform $\mathfrak{B}_s$ of index s by

$$\mathfrak{B}_s : \varphi(x) = \sum_{j=1}^{\infty} \varphi_j x^j \mapsto \hat{\varphi}(\xi) = \sum_{j=1}^{\infty} \frac{\varphi_j}{\Gamma(sj)} \xi^{j-1/s}.$$

We see $\xi^{1/s} \hat{\varphi}(\xi)$ converges at $\xi = 0$ and defines an analytic function when $\varphi(x) \in \mathbb{C}[[x]]_s$. Similarly, we define the Borel transform $\mathfrak{B}_1$ of index 1 for $\varphi|_s(y) := \varphi(y^s)$ by

$$\mathfrak{B}_1 : \varphi|_s(y) = \sum_{j=1}^{\infty} \varphi_j y^{sj} \mapsto \hat{\varphi}|_s(\eta) = \sum_{j=1}^{\infty} \frac{\varphi_j}{\Gamma(sj)} \eta^{sj-1}.$$

By the definition, we see $\hat{\varphi}|_s(\eta) = \hat{\varphi}(\eta^s)$ holds.

Let $\varphi(x), \psi(x) \in \mathbb{C}[[x]]_s$ satisfy $\varphi_0, \psi_0 = 0$. Then, the convolution product of index s of $\hat{\varphi}(\xi)$ and $\hat{\psi}(\xi)$ is defined by

$$\hat{\varphi} *_s \hat{\psi}(\xi) := \int_0^\eta \hat{\varphi}|_s(\eta - \eta') \hat{\psi}|_s(\eta') d\eta' \Big|_{\eta=\xi^{1/s}}.$$

By the definition, we have

$$\mathfrak{B}_s[\varphi\psi](\xi) = \hat{\varphi} *_s \hat{\psi}(\xi).$$

See [2] and [9] for further details about the Borel transform.

§ 2.2. Endless Riemann surfaces

We first introduce “discrete filtered set”, which is useful to describe singular locus of endlessly continuable functions:

Definition 2.1. A discrete filtered set is a family $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ ¹ of subsets in

¹We use the notation $\mathbb{R}_{\geq c}$ for $c \in \mathbb{R}$ to denote the set $\{\lambda \in \mathbb{R} \mid \lambda \geq c\}$.

$\mathbb{C}$ such that

- i) Ω_L is a finite set,
- ii) $\Omega_{L_1} \subset \Omega_{L_2}$ if $L_1 \leq L_2$,
- iii) there exists $R > 0$ such that $\Omega_R = \emptyset$.

A discrete filtered set defined by $\Omega_L = \emptyset$ for all $L \in \mathbb{R}_{\geq 0}$ is called trivial and denoted by $\emptyset$.

We introduce the following operations on discrete filtered sets:

Definition 2.2. Let Ω and Ω' be discrete filtered sets. Then, we define a discrete filtered set $\Omega * \Omega'$ by

$$(\Omega * \Omega')_L := \{ \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1}, \omega_2 \in \Omega'_{L_2}, L_1 + L_2 = L \} \cup \Omega_L \cup \Omega'_L \quad (L \geq 0).$$

We set $\Omega^{*n} := \underbrace{\Omega * \cdots * \Omega}_{n \text{ times}}$ for $n \geq 1$ and define $\Omega^{*\infty}$ by

$$\Omega^{*\infty} := \varinjlim_n \Omega^{*n}.$$

We define $R(\Omega)$ and $\mathcal{S}_\Omega$ by

$$R(\Omega) := \sup \{ L \in \mathbb{R}_{\geq 0} \mid \Omega_L = \emptyset \},$$

$$\mathcal{S}_\Omega := \{ (\lambda, \omega) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \omega \in \Omega_\lambda \}.$$

We denote the set of Lipschitz continuous paths $\gamma : [0, 1] \rightarrow \mathbb{C}$ satisfying $\gamma(0) = 0$ by Π and the length of $\gamma \in \Pi$ by $L(\gamma)$. We then introduce two classes of paths as follows:

Definition 2.3. Let Ω be a discrete filtered set. We say a path $\gamma \in \Pi$ is Ω -allowed if it satisfies $\tilde{\gamma}(t) := (L(\gamma|_{[0,t]}), \gamma(t)) \notin \overline{\mathcal{S}_\Omega}$ for any $t \in [0, 1]$, where $\overline{\mathcal{S}_\Omega}$ denotes the closure of $\mathcal{S}_\Omega$ in $\mathbb{R}_{\geq 0} \times \mathbb{C}$. The set of Ω -allowed paths is denoted by Π_Ω .

We set

$$\Pi_\Omega^\circ = \{ \gamma \in \Pi_\Omega \mid \gamma(t) \neq 0 \ (t \in (0, 1]) \}.$$

Definition 2.4. An Ω -endless Riemann surface is a triple $(X, \mathbf{p}, \underline{0})$ such that X is a connected and simply connected Riemann surface, $\mathbf{p} : X \rightarrow \mathbb{C}$ is a local biholomorphism, $\mathbf{p}(\underline{0}) = 0$, and any path $\gamma : [0, 1] \rightarrow \mathbb{C}$ of Π_Ω has a lift $\underline{\gamma} : [0, 1] \rightarrow X$ such that

$\gamma(0) = \underline{0}$. A morphism $\mathfrak{q} : (X, \mathfrak{p}, \underline{0}) \rightarrow (X', \mathfrak{p}', \underline{0}')$ of Ω -endless Riemann surfaces is given by a local biholomorphism $\mathfrak{q} : X \rightarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccc} (X, \underline{0}) & \xrightarrow{\mathfrak{q}} & (X', \underline{0}') \\ & \searrow \mathfrak{p} \quad \swarrow \mathfrak{p}' & \\ & (\mathbb{C}, 0) & \end{array}$$

Then, we have the following

Theorem 2.5 ([8]). *There exists an Ω -endless Riemann surface $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ such that, for any Ω -endless Riemann surface $(X, \mathfrak{p}, \underline{0})$, there is a unique morphism*

$$\mathfrak{q} : (X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega) \rightarrow (X, \mathfrak{p}, \underline{0}).$$

We define a discrete filtered set $\Omega^\circ = (\Omega_L^\circ)_{L \in \mathbb{R}_{\geq 0}}$ associated with a discrete filtered set Ω by

$$(2.2) \quad \Omega_L^\circ = \begin{cases} \Omega_L & (0 \leq L < 2R(\Omega)), \\ \Omega_L \cup \{0\} & (L \geq 2R(\Omega)). \end{cases}$$

By the definition, we see

$$(\Omega^\circ)^\circ = \Omega^\circ.$$

§ 2.3. Endlessly continuable functions

In what follows, we consider the case $s = 1/k$ for a positive integer k . We fix a discrete filtered set Ω satisfying $\Omega = \Omega^\circ$. We define $X_\Omega^{(k)}$ by

$$\begin{aligned} X_\Omega^{(k)} &:= \mathbb{C}_\xi \times_{\mathbb{C}_\eta} X_\Omega \\ &= \{(\xi, \underline{\eta}) \in \mathbb{C}_\xi \times X_\Omega \mid \xi^k = \mathfrak{p}_\Omega(\underline{\eta})\}, \end{aligned}$$

where $\mathbb{C}_\xi$ (resp. $\mathbb{C}_\eta$) denotes the complex plane with the variable ξ (resp. η). We see $X_\Omega^{(k)}$ defines a Riemann surface satisfying the following commutative diagram

$$\begin{array}{ccc} (X_\Omega^{(k)}, \underline{0}_\Omega^{(k)}) & \xrightarrow{\mathfrak{r}_\Omega^{(k)}} & (X_\Omega, \underline{0}_\Omega) \\ \mathfrak{p}_\Omega^{(k)} \downarrow & & \downarrow \mathfrak{p}_\Omega \\ (\mathbb{C}_\xi, 0) & \xrightarrow{\eta = \xi^k} & (\mathbb{C}_\eta, 0) \end{array} ,$$

where $\underline{0}_\Omega^{(k)} := (0, \underline{0}_\Omega)$ and $\mathfrak{p}_\Omega^{(k)}$ (resp. $\mathfrak{r}_\Omega^{(k)}$) is a morphism defined by $(\xi, \underline{\eta}) \mapsto \xi$ (resp. $(\xi, \underline{\eta}) \mapsto \underline{\eta}$). Then, $\mathfrak{r}_\Omega^{(k)} : X_\Omega^{(k)} \rightarrow X_\Omega$ gives a branched covering with branching points only at $(\mathfrak{p}_\Omega^{(k)})^{-1}(0)$. Therefore, restricting it to $(X_\Omega^\circ)^{(k)} := (\mathfrak{p}_\Omega^{(k)})^{-1}(\mathbb{C}_\xi^\circ)$ and $X_\Omega^\circ := (\mathfrak{p}_\Omega)^{-1}(\mathbb{C}_\eta^\circ)$, where $\mathbb{C}^\circ := \mathbb{C} \setminus \{0\}$, we see $\mathfrak{r}_\Omega^{(k)} : (X_\Omega^\circ)^{(k)} \rightarrow X_\Omega^\circ$ gives an unbranched covering. Let $\gamma \in \Pi_\Omega^\circ$ be a path on $\mathbb{C}_\eta$. Then, the restriction of γ to $(0, 1]$ has a lift $\underline{\gamma}^{(k)}$ to $(X_\Omega^\circ)^{(k)}$ such that $\mathfrak{p}_\Omega \circ \mathfrak{r}_\Omega^{(k)} \circ \underline{\gamma}^{(k)} = \gamma|_{(0, 1]}$. We see we have k choices for the lift.

Definition 2.6. We say an analytic germ $\hat{\varphi} \in \mathbb{C}\{\xi\}$ is $\Omega^{(k)}$ -continuable if $\hat{\varphi}$ is analytically continuable along any lift $\underline{\gamma}^{(k)}$ of any path $\gamma \in \Pi_\Omega^\circ$. The space of $\Omega^{(k)}$ -continuable functions is denoted by $\hat{\mathcal{R}}_\Omega^{(k)}$.

Then, we have the following isomorphism:

$$(2.3) \quad (\mathfrak{p}_\Omega^{(k)})^* : \hat{\mathcal{R}}_\Omega^{(k)} \xrightarrow{\sim} \Gamma(X_\Omega^{(k)}, \mathcal{O}_{X_\Omega^{(k)}}),$$

where $\mathcal{O}_{X_\Omega^{(k)}}$ is the sheaf of holomorphic functions on $X_\Omega^{(k)}$.

Using the representation (2.3), we can naturally introduce a structure of Fréchet space to $\hat{\mathcal{R}}_\Omega^{(k)}$ as follows: we set for $L, \delta > 0$

$$\mathcal{M}_\Omega^{\delta, L} := \{ (\lambda, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \text{dist}((\lambda, \xi), \overline{\mathcal{S}}_\Omega) \geq \delta, \lambda \leq L \},$$

$$\Pi_\Omega^{\delta, L} := \{ \gamma \in \Pi_\Omega \mid (L(\gamma|_t), \gamma(t)) \in \mathcal{M}_\Omega^{\delta, L} \text{ for all } t \in [0, 1] \},$$

where $\text{dist}(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$, and define compact subsets $K_\Omega^{\delta, L}$ of X_Ω by

$$K_\Omega^{\delta, L} := \{ \underline{\gamma}(1) \in X_\Omega \mid \gamma \in \Pi_\Omega^{\delta, L} \}.$$

Since X_Ω is exhausted by $(K_\Omega^{\delta, L})_{\delta, L > 0}$, and hence, $X_\Omega^{(k)}$ is exhausted by $((\mathfrak{r}_\Omega^{(k)})^{-1}(K_\Omega^{\delta, L}))_{\delta, L > 0}$, a family of seminorms $\|\cdot\|_\Omega^{\delta, L}$ ($\delta, L > 0$) defined by

$$\|\hat{\varphi}\|_\Omega^{\delta, L} := \sup_{\underline{\xi} \in (\mathfrak{r}_\Omega^{(k)})^{-1}(K_\Omega^{\delta, L})} |(\mathfrak{p}_\Omega^{(k)})^* \hat{\varphi}(\underline{\xi})| \quad \text{for } \hat{\varphi} \in \hat{\mathcal{R}}_\Omega^{(k)}$$

induces a structure of Fréchet space on $\hat{\mathcal{R}}_\Omega^{(k)}$.

§ 3. Iterated convolution product

In this section, we introduce “iterated convolution product of index s ” and give an estimate for it useful for application.

§ 3.1. Rooted trees

Definition 3.1. A (non-planar) rooted tree $T = (V, E)$ is a connected and simply connected set of edges and vertices in which one vertex is distinguished, where V

(resp. E) is the set of vertices (resp. edges) of T . We call such a vertex ρ the root of T . We regard a rooted tree as a directed graph by an arborescent orientation, i.e., each edge is directed away from the root. Then, the root is a unique minimal vertex in the arborescent order $\leq$, i.e., $\rho \leq v$ holds for any $v \in V$. A leaf of T is a maximal vertex in the order. A rooted forest is a disjoint union of rooted trees. The empty graph $\emptyset$ is regarded as a rooted forest. The set of isomorphism classes of rooted trees (resp. rooted forests) as directed graphs is denoted by $\mathcal{T}$ (resp. $\mathcal{F}$).

Remark. By abuse of notation, we use representatives to represent isomorphism classes of trees and forests since the names of the vertices are not important in the following discussion.

The degree $\deg(F)$ of a forest $F = (V, E) \in \mathcal{F}$ is defined by the cardinality of V . By the definition, $F \in \mathcal{F}$ is given by the disjoint union of $T_1, \dots, T_\ell \in \mathcal{T}$ and it is simply denoted by $T_1 \cdots T_\ell$. The grafting operator $B^+ : \mathcal{F} \rightarrow \mathcal{T}$ is defined as follows: for $F = T_1 \cdots T_\ell$ ($T_1, \dots, T_\ell \in \mathcal{T}$), $B^+(F) = T'$ is defined by adding to F a vertex ρ' (the root of T') and edges $e_j = \rho' \rightarrow \rho_j$ ($j = 1, \dots, \ell$) from ρ' to ρ_j , where ρ_j is the root of T_j . The trimming operator $B^- : \mathcal{T} \rightarrow \mathcal{F}$ is defined as follows: for $T \in \mathcal{T}$, $B^-(T)$ is defined by removing the root ρ of T and the edges whose tail is ρ .

§ 3.2. Iterated convolution product

We take $T = (V, E) \in \mathcal{T}$. For each $v \in V$, we associate a subtree $T_v = (V_v, E_v)$ of T , where $V_v = \{u \in V \mid v \leq u\}$ and E_v is the set of edges whose tail is in V_v . We see T_v is a subtree of T whose root is v and $B^-(T_v)$ is written as $B^-(T_v) = T_{v_1} \cdots T_{v_\ell}$ for $v_1, \dots, v_\ell \in V$. Let $\hat{f}_v \in \xi^{1-1/s} \mathbb{C}\{\xi\}$ and $\hat{\varphi}_v \in \mathbb{C}\{\xi\}$ be assigned to each vertex $v \in V$. Then, we inductively define $\{\hat{\psi}_v\}_{v \in V}$ from $\{\hat{f}_v\}_{v \in V}$ and $\{\hat{\varphi}_v\}_{v \in V}$ by the following rules:

$$(3.1) \quad \hat{\psi}_v := \hat{\varphi}_v \cdot \hat{f}_v \text{ when } v \text{ is a leaf of } T,$$

$$(3.2) \quad \hat{\psi}_v := \hat{\varphi}_v \cdot (\hat{f}_v *_s \hat{\psi}_{v_1} *_s \cdots *_s \hat{\psi}_{v_\ell}) \text{ when } B^-(T_v) = T_{v_1} \cdots T_{v_\ell},$$

where $\cdot$ denotes the Cauchy product.

Definition 3.2. Given $T \in \mathcal{T}$, $\{\hat{f}_v\}_{v \in V} \subset \xi^{1-k} \mathbb{C}\{\xi\}$ and $\{\hat{\varphi}_v\}_{v \in V} \subset \mathbb{C}\{\xi\}$, we call $\hat{\psi}_T := \hat{\psi}_\rho$ defined by the rules (3.1) and (3.2) *iterated convolution product* (of index s) of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$.

We now state our main result:

Theorem 3.3 ([1]). *Let Ω be a discrete filtered set satisfying $\Omega = \Omega^\circ$ and let $\delta, L > 0$ be reals such that $2\delta < R(\Omega)$. Then, there exist $c, \delta' > 0$ such that, for every*

$T = (V, E) \in \mathcal{T}$, $\{\hat{f}_v\}_{v \in V} \subset \xi^{1-k} \hat{\mathcal{R}}_{\emptyset}^{(k)}$ and $\{\hat{\varphi}_v\}_{v \in V} \subset \hat{\mathcal{R}}_{\Omega}^{(k)}$, the iterated convolution product $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ satisfies the following: $\hat{\psi}_T \in \xi^{d-k} \hat{\mathcal{R}}_{\Omega^{*\ell}}^{(k)}$ and

$$(3.3) \quad \|\xi^{k-d} \hat{\psi}_T\|_{\Omega^{*\ell}}^{\delta, L} \leq \frac{c^d}{\Gamma(d/k)} \prod_{v \in V} \|\hat{\varphi}_v\|_{\Omega}^{\delta', L} \|\xi^{k-1} \hat{f}_v\|_{\emptyset}^{\delta', L},$$

where $d = \deg(T)$ and ℓ is the number of the leaves of T .

§ 4. Application

We consider the following nonlinear ordinary differential equation:

$$(4.1) \quad x^{k+1} \frac{d\Phi}{dx} = F(x, \Phi)$$

at $x = 0$, where $F(x, \Phi) \in \mathbb{C}^n \{x, \Phi\}$ is an analytic function at the origin satisfying $F(0, \Phi) = \partial_{\Phi} F(0, 0) \Phi$ with $\det(\partial_{\Phi} F(0, 0)) \neq 0$. Then, (4.1) has a unique formal series solution $\Phi = \sum_{j=1}^{\infty} \Phi_j x^j \in \mathbb{C}^n[[x]]_{1/k}$. We rewrite (4.1) as follows:

$$(4.2) \quad \left[x^{k+1} \frac{d}{dx} - \partial_{\Phi} F(0, 0) \right] \Phi = \tilde{F}(x, \Phi),$$

where $\tilde{F}(x, \Phi) = F(x, \Phi) - F(0, \Phi)$. Since

$$\mathfrak{B}_{1/k} \left[x^{k+1} \frac{d\Phi}{dx} \right] = k \xi^k \mathfrak{B}_{1/k}[\Phi],$$

we obtain the following equation for $\mathfrak{B}_{1/k}[\Phi](\xi) = \hat{\Phi}(\xi)$:

$$(4.3) \quad \hat{\Phi}(\xi) = P^{-1}(\xi) \cdot \mathfrak{B}_{1/k}[\tilde{F}(x, \Phi)],$$

where $P^{-1}(\xi)$ is the inverse matrix of $P(\xi) := k\xi^k - \partial_{\Phi} F(0, 0)$. Then, poles are visible at $\Sigma := \{\xi \in \mathbb{C} \mid \det P(\xi) = 0\}$, i.e., $k\xi^k - \lambda_j = 0$ ($j = 1, \dots, n$), where $\{\lambda_j\}_{j=1}^n$ are eigenvalues of $\partial_{\Phi} F(0, 0)$, in the coefficients of the right side of (4.3). Therefore, we expect $\hat{\Phi}(\xi)$ has singular points at Σ . However, the singular locus is not so simple. The reason is that singular points are newly generated from Σ by the convolution product and the nonlinearity of (4.3) makes the sheet structure of the Riemann surface of $\hat{\Phi}(\xi)$ much more complicated. (See [4], [5], [6], [7], [8], [10], [11], [12], [13].) When $k = 1$, $\Omega^{*\infty}$ -continuity of $\hat{\Phi}(\xi)$ was verified in [6], where $\Omega = \Omega(\Sigma)$ is a discrete filtered set defined by

$$(4.4) \quad \Omega(\Sigma)_L := \{\xi \in \Sigma \mid |\xi| \leq L\} \quad (L \geq 0).$$

The key estimate in the proof is Theorem 4.6 in [6], which is similar to Theorem 4.1 for $k = 1$. Then, using a decomposition of $\widehat{\Phi}(\xi)$ by iterated convolution products of index $1/k$ as in Lemma 5.4 in [6] and Theorem 4.1 to validate the convergence of the sum, we obtain the following

Theorem 4.1 ([1]). *Let Ω be a discrete filtered set defined by $\Omega = \Omega(\Sigma)^\circ$. Then, each element of $\widehat{\Phi}(\xi)$ is in $\xi^{1-k} \widehat{\mathcal{R}}_{\Omega^{*\infty}}^{(k)}$.*

The details of the proof will be given in [1].

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