

On angular estimate of analytic functions

By

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Abstract

In this paper, we obtained the following result

$$|\arg(zp'(z) - p(z))| > \frac{\pi}{2}(\beta + \frac{2}{\pi}\tan^{-1}(-\beta))$$

$$\Rightarrow |\arg p(z)| < \frac{\pi}{2}\beta$$

where β and $p(z)$ satisfy the conditions of Theorem 1.

Introduction and Results.

If $f(z)$ and $g(z)$ are analytic in the unit disk $E = \{z : |z| < 1\}$, then $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if $g(z)$ is univalent in E , $f(0) = g(0)$ and $f(E) \subset g(E)$.

In this paper we need the following lemma.

Lemma 1. Let $p(z)$ be analytic in E , $p(0) = 1$, $p(z) \neq 0$ in E and suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\beta$$

where $0 < \beta$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta$$

where

$$k \geq 1 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\beta$$

and

$$k \leq -1 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\beta$$

We owe this lemma to [2].

In [1], Miller and Mocanu proved the following theorem.

Theorem A. Let $\beta_0 = 1.21872 \dots$ be the solution of

$$\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta$$

and let

$$\alpha = \alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\beta$$

for $0 < \beta < \beta_0$.

If $p(z)$ is analytic in E , with $p(0) = 1$, then

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha$$

$$\Rightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^\beta$$

or

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2}\alpha \quad E$$

$$\Rightarrow |\arg p(z)| < \frac{\pi}{2}\beta \quad E.$$

Applying Theorem A, we can obtain many interesting results.

Corresponding Theorem A, we will obtain a result which is probably useful to obtain some results for meromorphic functions.

Theorem 1. Let $p(z)$ be analytic in E , $p(0) = 1$, $p(z) \neq 0$ in E and suppose that

$$|\arg(zp'(z) - p(z))| > \frac{\pi}{2}(\beta + \frac{2}{\pi}\tan^{-1}(-\beta)) \quad \text{in } E$$

where $0 < \beta$ and the branche of $\tan^{-1}(-\beta)$ is restricted in $\frac{\pi}{2} < \tan^{-1}(-\beta) < \pi$.

Then we have

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{for } E.$$

Proof. If there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\beta,$$

then from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\beta k$$

where

$$(1) \quad k \geq 1 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\beta$$

and

$$(2) \quad k \leq -1 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\beta.$$

When $\arg p(z_0) = \frac{\pi}{2}\beta$, then from (1), we have

$$\begin{aligned} \arg(z_0 p'(z_0) - p(z_0)) &= \arg p(z_0) \left(\frac{z_0 p'(z_0)}{p(z_0)} - 1 \right) \\ &= \frac{\pi}{2}\beta + \arg(i\beta k - 1) \\ &\leq \frac{\pi}{2}\beta + \tan^{-1}(-\beta) \\ &= \frac{\pi}{2}(\beta + \frac{2}{\pi} \tan^{-1}(\beta)) \end{aligned}$$

and if $\arg p(z_0) = -\frac{\pi}{2}\beta$, then from (2), we also have

$$\begin{aligned} \arg(z_0 p'(z_0) - p(z_0)) &= \arg p(z_0) \left(\frac{z_0 p'(z_0)}{p(z_0)} - 1 \right) \\ &= -\frac{\pi}{2}\beta + \arg(i\beta k - 1) \\ &\geq -\frac{\pi}{2}\beta - \tan^{-1}(-\beta) \\ &= -\frac{\pi}{2}(\beta + \frac{2}{\pi} \tan^{-1}(\beta)). \end{aligned}$$

These contradict the assumption of the theorem and therefore this completes the proof.

Remark. It is trivial that $\frac{\pi}{2}(\beta + \frac{2}{\pi} \tan^{-1}(-\beta)) > \pi$ for $0 < \beta$.

Applying the same method as the proof of Theorem 1 and from Lemma 1, we can generalize Theorem A as the following.

Theorem A'. Let $p(z)$ be analytic in E , $p(0) = 1$, $p(z) \neq 0$ in E and suppose that

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2}(\beta + \frac{2}{\pi} \tan^{-1}\beta) \quad \text{in } E$$

where $0 < \beta$.

Then we have

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{in } E.$$

References

- [1] S.S Miller and P.T.Mocanu , Marx-Strohhäcker differential subordination systems, Proc. Amer. Math. Soc., 99(3), 527-534(1987).
- [2] M.Nunokawa, On the order of strongly convex functions, Proc. Japan Acad., 69. A, No.7, 234-237(1993).

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