

Note on the homotopy stability of the space of non-resultant systems determined by a toric variety

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Abstract

For a fan Σ in $\mathbb{R}^m$ and r -tuple $D = (d_1, \dots, d_r)$ of positive integers, let $\text{Poly}_n^D(\mathbb{F})$ denote the certain affine variety over $\mathbb{F}$ called *the space of non-resultant systems of bounded multiplicity of type (Σ, n)* , where r is the number of one dimensional cones in Σ and X_Σ denotes the toric variety over $\mathbb{C}$ corresponds to the fan Σ . This space was first defined by B. Farb and J. Wolfson [10] when X_Σ is the complex projective space, and it was originally studied for investigating the homological densities of algebraic cycles in a manifold [11]. In this note, we shall report about the recent results concerning the homotopy stability of the space $\text{Poly}_n^{D, \Sigma}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{C}$. This result is based on the joint works with A. Kozłowski [19].

1 Basic definitions and notations

Let $\mathbb{N}$ be a set of all positive integers. For connected spaces X and Y , let $\text{Map}(X, Y)$ denote the space consisting of all continuous maps $f : X \rightarrow Y$ with the compact open topology. Let $\text{Map}^*(X, Y) \subset \text{Map}(X, Y)$ be the subspace of all base point preserving maps $f : (X, *) \rightarrow (Y, *)$. For a based homotopy class $D \in \pi_0(\text{Map}^*(X, Y)) = [X, Y]$, we denote by $\text{Map}_D^*(X, Y) \subset \text{Map}^*(X, Y)$ the path component containing the homotopy class D .

When X and Y are complex manifolds, let $\text{Hol}_D^*(X, Y) \subset \text{Map}_D^*(X, Y)$ denote the subspace consisting of all based holomorphic maps $f \in \text{Map}_D^*(X, Y)$. Then we have the natural inclusion

$$(1.1) \quad i_D : \text{Hol}_D^*(X, Y) \xrightarrow{\subset} \text{Map}_D^*(X, Y).$$

Now recall several definitions and notations.

Definition 1.1. (i) A *convex rational polyhedral cone* in $\mathbb{R}^m$ is a subset of $\mathbb{R}^m$ of the form

$$(1.2) \quad \sigma = \text{Cone}(S) = \text{Cone}(\mathbf{m}_1, \dots, \mathbf{m}_s) = \left\{ \sum_{k=1}^s \lambda_k \mathbf{m}_k : \lambda_k \geq 0 \right\}$$

for a finite set $S = \{\mathbf{m}_1, \dots, \mathbf{m}_s\} \subset \mathbb{Z}^m$. The dimension of σ is the dimension of the smallest subspace of $\mathbb{R}^m$ which contains σ .

(ii) A convex rational polyhedral cone σ is called *strongly convex* if $\sigma \cap (-\sigma) = \{\mathbf{0}_m\}$, where we set $\mathbf{0}_m = \mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^m$. A *face* τ of a convex rational polyhedral cone σ is a subset $\tau \subset \sigma$ of the form $\tau = \sigma \cap \{\mathbf{x} \in \mathbb{R}^m : L(\mathbf{x}) = 0\}$ for some linear form L on $\mathbb{R}^m$, such that $\sigma \subset \{\mathbf{x} \in \mathbb{R}^m : L(\mathbf{x}) \geq 0\}$.

If we set $\{k : 1 \leq k \leq s, L(\mathbf{m}_k) = 0\} = \{i_1, \dots, i_t\}$, we easily see that $\tau = \text{Cone}(\mathbf{m}_{i_1}, \dots, \mathbf{m}_{i_t})$. Hence, if σ is a strongly convex rational polyhedral cone, so is any of its faces.¹

(iii) Let Σ be a finite collection of strongly convex rational polyhedral cones in $\mathbb{R}^m$. Then it is called a *fan* (in $\mathbb{R}^m$) if the following two conditions (1.2.1) and (1.2.2) are satisfied:

(1.2.1) Every face τ of $\sigma \in \Sigma$ belongs to Σ .

(1.2.2) If $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a common face of each σ_k and $\sigma_1 \cap \sigma_2 \in \Sigma$.

(iv) An m dimensional irreducible normal variety X (over $\mathbb{C}$) is called a *toric variety* if it has a Zariski open subset $\mathbb{T}_{\mathbb{C}}^m = (\mathbb{C}^*)^m$ and the action of $\mathbb{T}_{\mathbb{C}}^m$ on itself extends to an action of $\mathbb{T}_{\mathbb{C}}^m$ on X . The most significant property of a toric variety is that it is characterized up to isomorphism entirely by its associated fan Σ . We denote by X_{Σ} the toric variety associated to a fan Σ . $\square$

Definition 1.2. Let K be a simplicial complex on the index set $[r] = \{1, 2, \dots, r\}$,² and let (X, A) be a pairs of based spaces.

(i) Let $I(K)$ denote the collection of subsets $\sigma \subset [r]$ defined by

$$(1.3) \quad I(K) = \{\sigma \subset [r] : \sigma \notin K\}.$$

(ii) Define the *polyhedral product* $\mathcal{Z}_K(X, A)$ with respect to K by

$$(1.4) \quad \mathcal{Z}_K(X, A) = \bigcup_{\sigma \in I(K)} (X, A)^{\sigma}, \quad \text{where} \\ (X, A)^{\sigma} = \{(x_1, \dots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma\}.$$

(iii) For each subset $\sigma = \{i_1, \dots, i_s\} \subset [r]$, let $L_{\sigma}(\mathbb{C}^n)$ denote the subspace of $\mathbb{C}^{nr}$ defined by

$$(1.5) \quad L_{\sigma}(\mathbb{C}^n) = \{(\mathbf{x}_1, \dots, \mathbf{x}_r) \in \mathbb{C}^{nr} : \mathbf{x}_i \in \mathbb{C}^n, \mathbf{x}_{i_1} = \dots = \mathbf{x}_{i_s} = \mathbf{0}_n\}$$

¹When S is the emptyset $\emptyset$, we set $\text{Cone}(\emptyset) = \{\mathbf{0}_m\}$ and we may also regard it as one of strongly convex rational polyhedral cones in $\mathbb{R}^m$.

²Let K be some set of subsets of $[r]$. Then the set K is called an *abstract simplicial complex* on the index set $[r]$ if the following condition holds: if $\tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$. In this paper by a simplicial complex K we always mean an *abstract simplicial complex*, and we always assume that a simplicial complex K contains the empty set $\emptyset$.

and let $L_n(\Sigma)$ denote the subspace of $\mathbb{C}^{nr}$ defined by

$$(1.6) \quad L_n(\Sigma) = \bigcup_{\sigma \in I(K)} L_\sigma(\mathbb{C}^n) = \bigcup_{\sigma \subset [r], \sigma \notin K} L_\sigma(\mathbb{C}^n).$$

Then it is easy to see that

$$(1.7) \quad \mathcal{Z}_K(\mathbb{C}^n, (\mathbb{C}^n)^*) = \mathbb{C}^{nr} \setminus L_n(\Sigma), \quad \text{where } (\mathbb{C}^n)^* = \mathbb{C}^n \setminus \{\mathbf{0}_n\}. \quad \square$$

Remark 1.3. It is well known that there are no holomorphic maps $\mathbb{CP}^1 = S^2 \rightarrow \mathbb{T}_{\mathbb{C}}^m$ except the constant maps, and that the fan Σ of $\mathbb{T}_{\mathbb{C}}^m$ is $\Sigma = \{\mathbf{0}_m\}$. Hence, without loss of generality we always assume that $X_\Sigma \neq \mathbb{T}_{\mathbb{C}}^m$ and that any fan Σ in $\mathbb{R}^m$ satisfies the condition $\{\mathbf{0}_m\} \subsetneq \Sigma$. $\square$

Definition 1.4. Let Σ be a fan in $\mathbb{R}^m$ such that $\{\mathbf{0}_m\} \subsetneq \Sigma$, and let

$$(1.8) \quad \Sigma(1) = \{\rho_1, \dots, \rho_r\}$$

denote the set of all one dimensional cones in Σ .

(i) For each $1 \leq k \leq r$, we denote by $\mathbf{n}_k \in \mathbb{Z}^m$ the primitive generator of ρ_k , such that $\rho_k \cap \mathbb{Z}^m = \mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k$. Note that $\rho_k = \text{Cone}(\mathbf{n}_k)$.

(ii) Let $\mathcal{K}_\Sigma$ denote the underlying simplicial complex of Σ defined by

$$(1.9) \quad \mathcal{K}_\Sigma = \left\{ \{i_1, \dots, i_s\} \subset [r] : \mathbf{n}_{i_1}, \mathbf{n}_{i_2}, \dots, \mathbf{n}_{i_s} \text{ span a cone in } \Sigma \right\}.$$

It is easy to see that $\mathcal{K}_\Sigma$ is a simplicial complex on the index set $[r]$.

(iii) Define the subgroup $G_\Sigma \subset \mathbb{T}_{\mathbb{C}}^r = (\mathbb{C}^*)^r$ by

$$(1.10) \quad G_\Sigma = \{(\mu_1, \dots, \mu_r) \in \mathbb{T}_{\mathbb{C}}^r : \prod_{k=1}^r (\mu_k)^{\langle \mathbf{n}_k, \mathbf{m} \rangle} = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^m\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^m u_k v_k$ for $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$.

(iv) Now consider the natural G_Σ -action on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ given by coordinate-wise multiplication, i.e.

$$(1.11) \quad (\mu_1, \dots, \mu_r) \cdot (\mathbf{x}_1, \dots, \mathbf{x}_r) = (\mu_1 \mathbf{x}_1, \dots, \mu_r \mathbf{x}_r)$$

for $((\mu_1, \dots, \mu_r), (\mathbf{x}_1, \dots, \mathbf{x}_r)) \in G_\Sigma \times \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$, where we set

$$(1.12) \quad \mu \mathbf{x} = (\mu x_1, \dots, \mu x_n) \quad \text{if } (\mu, \mathbf{x}) = (\mu, (x_1, \dots, x_n)) \in \mathbb{C} \times \mathbb{C}^n.$$

Then define the space $X_\Sigma(n)$ by the corresponding orbit space

$$(1.13) \quad X_\Sigma(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) / G_\Sigma. \quad \square$$

Remark 1.5. (i) Let Σ be a fan in $\mathbb{R}^m$ as in Definition 1.4. Then the fan Σ is completely determined by the pair $(\mathcal{K}_\Sigma, \{\mathbf{n}_k\}_{k=1}^r)$ (see [16, Remark 2.3] in detail).

(ii) Note that the group G_Σ acts on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ freely (cf. [25, Proposition 6.7]). Moreover, one can show that $X_\Sigma(n)$ is a toric variety. $\square$

The following theorem plays a crucial role in the proof of the main result of this paper.

Theorem 1.6 ([7], Theorem 2.1; [8], Theorem 3.1). *Let Σ be a fan in $\mathbb{R}^m$ as in Definition 1.4 and suppose that the set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$ (i.e. $\sum_{k=1}^r \mathbb{R} \cdot \mathbf{n}_k = \{\sum_{k=1}^r \lambda_k \mathbf{n}_k : \lambda_k \in \mathbb{R}\} = \mathbb{R}^m$).*

(i) *Then there is a natural isomorphism*

$$(1.14) \quad X_\Sigma \cong \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma(1).$$

(ii) *If $f : \mathbb{CP}^s \rightarrow X_\Sigma$ is a holomorphic map, then there exists an r -tuple $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ of non-negative integers satisfying the condition $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, and homogenous polynomials $f_i \in \mathbb{C}[z_0, \dots, z_s]$ of degree d_i ($i = 1, 2, \dots, r$) such that the polynomials $\{f_i\}_{i \in \sigma}$ have no common root except $\mathbf{0}_{s+1} \in \mathbb{C}^{s+1}$ for each $\sigma \in I(\mathcal{K}_\Sigma)$ and that the diagram*

$$(1.15) \quad \begin{array}{ccc} \mathbb{C}^{s+1} \setminus \{\mathbf{0}_{s+1}\} & \xrightarrow{(f_1, \dots, f_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \\ \gamma_s \downarrow & & \downarrow q_\Sigma \\ \mathbb{CP}^s & \xrightarrow{f} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma \end{array}$$

is commutative, where we identify $X_\Sigma = X_\Sigma(1)$ as in (1.14) and the two maps $\gamma_s : \mathbb{C}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \rightarrow \mathbb{CP}^s$ and $q_\Sigma : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^) \rightarrow X_\Sigma = X_\Sigma(1)$ denote the canonical Hopf fibering and the canonical projection induced from the identification (1.14), respectively. In this case, we call this holomorphic map f a holomorphic map of degree $D = (d_1, \dots, d_r)$ and we represent it as*

$$(1.16) \quad f = [f_1, \dots, f_r].$$

(iii) *If $g_i \in \mathbb{C}[z_0, \dots, z_s]$ is a homogenous polynomial of degree d_i ($1 \leq i \leq r$) such that $f = [f_1, \dots, f_r] = [g_1, \dots, g_r]$, there exists some element $(\mu_1, \dots, \mu_r) \in G_\Sigma$ such that $f_i = \mu_i \cdot g_i$ for each $1 \leq i \leq r$. Thus, the r -tuple $(f_1, \dots, f_r)$ of homogenous polynomials representing a holomorphic map f is determined uniquely up to G_Σ -action. $\square$*

From now on, let Σ be a fan in $\mathbb{R}^m$ as in Definition 1.4, and assume that X_Σ is simply connected and non-singular. Moreover, we shall assume the following condition holds.

$$(1.16.1) \quad \text{There is an } r\text{-tuple } D_* = (d_1^*, \dots, d_r^*) \in \mathbb{N}^r \text{ such that } \sum_{k=1}^r d_k^* \mathbf{n}_k = \mathbf{0}_m.$$

Remark 1.7. It follows from [9, Theorem 12.1.10] that X_Σ is simply connected if and only if the fan Σ satisfies the following condition (*):

$$(*) \quad \text{The set } \{\mathbf{n}_k\}_{k=1}^r \text{ of all primitive generators spans } \mathbb{Z}^m \text{ over } \mathbb{Z}, \text{ i.e.} \\ \sum_{k=1}^r \mathbb{Z} \cdot \mathbf{n}_k = \mathbb{Z}^m.$$

Thus, one can easily see that the set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$ if X_Σ is simply connected. In particular, we can see that X_Σ is simply connected if X_Σ is a compact smooth toric variety. $\square$

2 The space $\text{Poly}_n^{D,\Sigma}(\mathbb{F})$

Let X_Σ be a simply connected non-singular toric variety satisfying the condition (1.16.1), and from now on we identify $X_\Sigma = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma(1)$.

Now consider a base point preserving holomorphic map $f = [f_1, \dots, f_r] : \mathbb{CP}^s \rightarrow X_\Sigma$ for the case $s = 1$. In this case, we make the identification $\mathbb{CP}^1 = S^2 = \mathbb{C} \cup \infty$ and choose the points ∞ and $[1, 1, \dots, 1]$ as the base points of $\mathbb{CP}^1$ and X_Σ , respectively.

Then, by setting $z = \frac{z_0}{z_1}$, for each $1 \leq k \leq r$, we can view f_k as a monic polynomial $f_k(z) \in \mathbb{C}[z]$ of degree d_k in the complex variable z . Now by using Theorem 1.6 we can define the space of holomorphic maps as follows.

Definition 2.1. (i) For a field $\mathbb{F}$, let $P_{\mathbb{F}}^d$ denote the space of all $\mathbb{F}$ -coefficient monic polynomials $g(z) = z^d + a_1 z^{d-1} + \dots + a_{d-1} z + a_d \in \mathbb{F}[z]$ of degree d , and we set

$$(2.1) \quad P_{\mathbb{F}}^D = P_{\mathbb{F}}^{d_1} \times \dots \times P_{\mathbb{F}}^{d_r} \quad \text{if } D = (d_1, \dots, d_r) \in \mathbb{N}^r.$$

Note that there is a natural homeomorphism $\phi : P_{\mathbb{F}}^d \cong \mathbb{F}^d$ given by $\phi(z^d + \sum_{k=1}^d a_k z^{d-k}) = (a_1, \dots, a_d) \in \mathbb{F}^d$. When $\mathbb{F} = \mathbb{C}$, we write $P^D = P_{\mathbb{C}}^D$, and $P^d = P_{\mathbb{C}}^d$.

(ii) For any r -tuple $D = (d_1, \dots, d_r) \in \mathbb{N}^r$ satisfying the condition (1.16.1), we denote by $\text{Hol}_D^*(S^2, X_\Sigma)$ the space consisting of all r -tuples $f = (f_1(z), \dots, f_r(z)) \in P^D$ satisfying the following condition ($\dagger_\Sigma$):

($\dagger_\Sigma$) For any $\sigma = \{i_1, \dots, i_s\} \in I(\mathcal{K}_\Sigma)$, the polynomials $f_{i_1}(z), \dots, f_{i_s}(z)$ have no common root, i.e. $(f_{i_1}(\alpha), \dots, f_{i_s}(\alpha)) \neq \mathbf{0}_s = (0, \dots, 0)$ for any $\alpha \in \mathbb{C}$.

By identifying $X_\Sigma = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma$ and $\mathbb{CP}^1 = S^2 = \mathbb{C} \cup \infty$, one can define the natural inclusion map $i_D : \text{Hol}_D^*(S^2, X_\Sigma) \rightarrow \text{Map}^*(S^2, X_\Sigma) = \Omega^2 X_\Sigma$ by

$$(2.2) \quad i_D(f)(\alpha) = \begin{cases} [f_1(\alpha), f_2(\alpha), \dots, f_r(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1, 1, \dots, 1] & \text{if } \alpha = \infty \end{cases}$$

for $f = (f_1(z), \dots, f_r(z)) \in \text{Hol}_D^*(S^2, X_\Sigma)$, where we choose the points ∞ and $[1, 1, \dots, 1]$ as the base points of S^2 and X_Σ .

Since the representation of polynomials in $P^D = P_{\mathbb{C}}^D$ representing a base point preserving holomorphic map of degree D is uniquely determined, the space $\text{Hol}_D^*(S^2, X_\Sigma)$ can be identified with *the space of base point preserving holomorphic maps of degree D* . Moreover, since $\text{Hol}_D^*(S^2, X_\Sigma)$ is path-connected, the image of i_D is contained in a certain path-component of $\Omega^2 X_\Sigma$, which is denoted by $\Omega_D^2 X_\Sigma$. Thus we have a natural inclusion

$$(2.3) \quad i_D : \text{Hol}_D^*(S^2, X_\Sigma) \rightarrow \text{Map}_D^*(S^2, X_\Sigma) = \Omega_D^2 X_\Sigma. \quad \square$$

Now consider the space $\text{Poly}_n^{D,\Sigma}(\mathbb{F})$ for $\mathbb{F} = \mathbb{C}$. For this purpose, we need the following notation.

Definition 2.2. For a monic polynomial $f(z) \in \mathbb{P}^d$ of degree d , let $F_n(f)(z)$ denote the n -tuple of monic polynomials of the same degree d given by

$$(2.4) \quad F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \dots, f(z) + f^{(n-1)}(z)).$$

Note that a monic polynomial $f(z) \in \mathbb{P}^d$ has a root $\alpha \in \mathbb{C}$ of multiplicity $\geq n$ iff $F_n(f)(\alpha) = \mathbf{0}_n \in \mathbb{C}^n$. $\square$

Definition 2.3. Let $\mathbb{F}$ be a field with its algebraic closure $\overline{\mathbb{F}}$.

(i) For each $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, $n \in \mathbb{N}$ and a fan Σ in $\mathbb{R}^m$, let $\text{Poly}_n^{D, \Sigma}(\mathbb{F})$ denote the space of r -tuples $(f_1(z), \dots, f_r(z)) \in \mathbb{P}_{\mathbb{F}}^D$ of $\mathbb{F}$ -coefficients monic polynomials satisfying the following condition $(\dagger_{\Sigma, n})$:

$(\dagger_{\Sigma, n})$ For any $\sigma = \{i_1, \dots, i_s\} \in I(\mathcal{K}_{\Sigma})$, polynomials $f_{i_1}(z), \dots, f_{i_s}(z)$ have no common root $\alpha \in \overline{\mathbb{F}}$ of multiplicity $\geq n$ (but they may have common roots of multiplicity $< n$).

Note that the condition $(\dagger_{\Sigma})$ coincides with the condition $(\dagger_{\Sigma, n})$ if $(\mathbb{F}, n) = (\mathbb{C}, 1)$.

From now on, we only consider the case $\mathbb{F} = \mathbb{C}$.

(ii) When $\mathbb{F} = \mathbb{C}$ and $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_n$, define the map $i_D : \text{Poly}_n^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega^2 X_{\Sigma}(n)$ by

$$(2.5) \quad i_D(f)(\alpha) = \begin{cases} [F_n(f_1)(\alpha), F_n(f_2)(\alpha), \dots, F_n(f_r)(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [\mathbf{e}, \mathbf{e}, \dots, \mathbf{e}] & \text{if } \alpha = \infty \end{cases}$$

for $f = (f_1(z), \dots, f_r(z)) \in \text{Poly}_n^{D, \Sigma}(\mathbb{C})$ and $\alpha \in \mathbb{C} \cup \infty = S^2$, where the space $X_{\Sigma}(n)$ is the space defined as in (1.13) and we set $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{C}^n$.

Since $\text{Poly}_n^{D, \Sigma}(\mathbb{C})$ is connected, the image of i_D is contained some path-component of $\Omega^2 X_{\Sigma}(n)$, which is denoted by $\Omega_D^2 X_{\Sigma}(n)$. Thus we have the map

$$(2.6) \quad i_D : \text{Poly}_n^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega_D^2 X_{\Sigma}(n). \quad \square$$

Remark 2.4. (i) By using the classical theory of resultants, one can show that $\text{Poly}_n^{D, \Sigma}(\mathbb{F})$ is an affine variety over $\mathbb{F}$ which is the complement of the set of solutions of a system of polynomial equations (called a generalised resultant) with integer coefficients. This is why we call it *the space of non-resultant systems of bounded multiplicity of type (Σ, n)* .

(ii) When X_{Σ} is a simply connected non-singular toric variety (over $\mathbb{C}$) satisfying the condition (1.16.1), one can show that $\text{Poly}_n^{D, \Sigma}(\mathbb{C}) = \text{Hol}_D^*(S^2, X_{\Sigma})$ if $n = 1$ (see Definition 2.1 for the details). $\square$

3 The space $\text{Poly}_n^{D, \Sigma}(\mathbb{C})$

Before stating the results for the space $\text{Poly}_n^{D, \Sigma}(\mathbb{C})$, we need to define the positive integers $r_{\min}(\Sigma)$ and $d(D; \Sigma, n)$.

Definition 3.1. We say that a set $S = \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\}$ is a *primitive* if $\text{Cone}(S) \notin \Sigma$ and $\text{Cone}(T) \in \Sigma$ for any proper subset $T \subsetneq S$. Then we define $d(D, \Sigma, n)$ to be the positive integer given by

$$(3.1) \quad d(D; \Sigma, n) = (2nr_{\min}(\Sigma) - 3)\lfloor d_{\min}/n \rfloor - 2,$$

where $r_{\min}(\Sigma)$ and $d_{\min} = d_{\min}(D)$ are the positive integers given by

$$(3.2) \quad r_{\min}(\Sigma) = \min\{s \in \mathbb{N} : \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\} \text{ is primitive}\},$$

$$(3.3) \quad d_{\min} = d_{\min}(D) = \min\{d_1, d_2, \dots, d_r\}. \quad \square$$

Definition 3.2. Let $f : X \rightarrow Y$ be a based continuous map, and let $N_0 \in \mathbb{N}$ be a fixed positive integer.

(i) The map f is called a *homology (resp. homotopy) equivalence through dimension N_0* if the induced homomorphism

$$(3.4) \quad f_* : H_k(X; \mathbb{Z}) \rightarrow H_k(Y; \mathbb{Z}) \quad (\text{resp. } f_* : \pi_k(X) \rightarrow \pi_k(Y))$$

is an isomorphism for any $k \leq N_0$.

(ii) Similarly, the map f is called a *homology (resp. homotopy) equivalence up to dimension N_0* if the induced homomorphism f_* (given by (3.4)) is an isomorphism for any $k < N_0$ and an epimorphism for $k = N_0$. $\square$

For connected space X , let $\Omega_0^2 X$ denote the path-component of $\Omega^2 X$ which contains null-homotopic maps and recall the following result.

Theorem 3.3 ([16]). *Let X_Σ be an m dimensional simply connected non-singular toric variety such that the condition (1.16.1) holds. Then if $D = (d_1, \dots, d_r) \in \mathbb{N}^r$ and $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, the inclusion map*

$$i_D : \text{Hol}_D^*(S^2, X_\Sigma) \rightarrow \Omega_D^2 X_\Sigma \simeq \Omega_0^2 X_\Sigma \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^2, S^1)$$

is a homotopy equivalence through dimension $d(D; \Sigma, 1) = (2r_{\min}(\Sigma) - 3)d_{\min} - 2$ if $r_{\min}(\Sigma) \geq 3$ and a homology equivalence through dimension $d(D; \Sigma, 1) = d_{\min} - 2$ if $r_{\min}(\Sigma) = 2$. $\square$

The main result of this paper is a generalization of the above theorem (Theorem 3.3) to spaces of non-resultant systems of bounded multiplicity.

Theorem 3.4 ([19]). *Let $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, $n \geq 2$ and let X_Σ be an m dimensional simply connected non-singular toric variety such that the condition (1.16.1) holds.*

(i) *If $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, then the natural map*

$$i_D : \text{Poly}_n^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega_D^2 X_\Sigma(n) \simeq \Omega_0^2 X_\Sigma(n) \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through dimension $d(D; \Sigma, n)$.

(ii) *If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_D : \text{Poly}_n^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

which is a homotopy equivalence through dimension $d(D; \Sigma, n)$. $\square$

Corollary 3.5 ([19]). *Let $n \geq 2$, $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional compact non-singular toric variety over $\mathbb{C}$ such that the condition (1.16.1) holds. Let $\Sigma(1)$ denote the set of all one dimensional cones in Σ , and let Σ_1 be any fan in $\mathbb{R}^m$ such that $\Sigma(1) \subset \Sigma_1 \subsetneq \Sigma$.*

Then X_{Σ_1} is a non-compact non-singular toric subvariety of X_Σ and the following two statements hold:

(i) *If $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, the map*

$$i_D : \text{Poly}_n^{D, \Sigma_1}(\mathbb{C}) \rightarrow \Omega_D^2 X_{\Sigma_1}(n) \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through the dimension $d(D; \Sigma_1, n)$.

(ii) *If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_D : \text{Poly}_n^{D, \Sigma_1}(\mathbb{C}) \rightarrow \Omega^2 \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(D^{2n}, S^{2n-1})$$

which is a homotopy equivalence through dimension $d(D; \Sigma_1, n)$. □

Remark 3.6. We would like to study about the homotopy stability of the space $\text{Poly}_n^{D, \Sigma}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{R}$ in our future paper. However, note that the homotopy type of the space $\text{Poly}_n^{D, \Sigma}(\mathbb{R})$ was already studied when X_Σ is a complex projective space in [20] (cf. [21], [26], [27]). We would like to generalize these results for more general toric varieties X_Σ .

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References

- [1] M. Adamaszek, A. Kozłowski and K. Yamaguchi, Spaces of algebraic and continuous maps between real algebraic varieties, *Quart. J. Math.* **62** (2011), 771–790.
- [2] V. I. Arnold, Certain topological invariants of algebraic functions, (Russian), *Trudy Moskov. Obshch.* **21** (1970), 27–46
- [3] M. F. Atiyah and J. D. S. Jones, Topological aspects of Yang–Mills theory, *Commun. Math. Phys.* **59** (1978), 97–118.
- [4] V. M. Buchstaber and T. E. Panov, Torus actions and their applications in topology and combinatorics, *Univ. Lecture Note Series* **24**, Amer. Math. Soc. Providence, 2002.
- [5] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, The topology of rational functions and divisors of surfaces, *Acta Math.* **166** (1991), 163–221.

- [6] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, The homotopy type of rational functions, *Math. Z.* **207** (1993), 37–47.
- [7] D. A. Cox, The homogenous coordinate ring of a toric variety, *J. Algebraic Geometry* **4** (1995), 17–50.
- [8] D. A. Cox, The functor of a smooth toric variety, *Tohoku Math. J.* **47** (1995), 251–262.
- [9] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Math. **124**, Amer. Math. Soc., 2011.
- [10] B. Farb and J. Wolfson, Topology and arithmetic of resultants, I, *New York J. Math.* **22** (2016), 801–821.
- [11] B. Farb, J. Wolfson and M. M. Wood, Coincidences between homological densities, predicted by arithmetic, *Advances in Math.*, **352** (2019), 670–716.
- [12] M. A. Guest, The topology of the space of rational curves on a toric variety, *Acta Math.* **174** (1995), 119–145.
- [13] M. A. Guest, A. Kozłowski and K. Yamaguchi, Spaces of polynomials with roots of bounded multiplicity, *Fund. Math.* **116** (1999), 93–117.
- [14] A. Kozłowski and K. Yamaguchi, The homotopy type of spaces of resultants of bounded multiplicity, *Topology Appl.* **232** (2017), 112–139.
- [15] A. Kozłowski and K. Yamaguchi, The homotopy type of spaces of rational curves on a toric variety, *Topology Appl.* **249** (2018), 19–42.
- [16] A. Kozłowski and K. Yamaguchi, The homotopy type of the space of algebraic loops on a toric variety, *Topology Appl.* **300** (2021), Article ID: 107705.
- [17] A. Kozłowski and K. Yamaguchi, The homotopy type of real resultants with bounded multiplicity, *J. Math. Soc. Japan* **74** (2022), 1047–1077.
- [18] A. Kozłowski and K. Yamaguchi, Spaces of non-resultant systems of bounded multiplicity determined by a toric variety, *Topology Appl.*, **337** (2023), Article ID: 108626.
- [19] A. Kozłowski and K. Yamaguchi, The homotopy type of spaces of non-resultants of bounded multiplicity with real coefficients, preprint (arXiv:2212.05494).
- [20] A. Kozłowski and K. Yamaguchi, Homotopy stability of spaces of non-resultants of bounded multiplicity with real coefficients, preprint (arXiv:2305.00307).
- [21] J. Mostovoy, Spaces of rational maps and the Stone-Weierstrass Theorem, *Topology* **45** (2006), 281–293.

- [22] J. Mostovoy, Truncated simplicial resolutions and spaces of rational maps, *Quart. J. Math.* **63** (2012), 181–187.
- [23] J. Mostovoy and E. Munguia-Villanueva, Spaces of morphisms from a projective space to a toric variety, *Rev. Colombiana Mat.* **48** (2014), 41–53.
- [24] T. E. Panov, Geometric structures on moment-angle manifolds, *Russian Math. Surveys* **68** (2013), 503–568.
- [25] G. B. Segal, Configuration spaces and iterated loop spaces, *Invent. Math.* **21** (1973), 213–221.
- [26] G. B. Segal, The topology of spaces of rational functions, *Acta Math.* **143** (1979), 39–72.
- [27] V. A. Vassiliev, Complements of discriminants of smooth maps, *Topology and Applications*, Amer. Math. Soc., *Translations of Math. Monographs* **98**, 1992 (revised edition 1994).

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