

MORE ON OPERATOR MONOTONE FUNCTIONS

HAMED NAJAFI

ABSTRACT. We investigate some properties of operator monotone functions. In particular, we show that if f is a non-constant operator monotone function on an interval J and A, B are self-adjoint operators with spectra in J such that $A > B$, then $f(A) > f(B)$. As an application we extend the celebrated Löwner–Heinz inequality.

1. INTRODUCTION

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ equipped with the operator norm $\|\cdot\|$. An operator $A \in \mathbb{B}(\mathcal{H})$ is called *positive* if $\langle Ax, x \rangle \geq 0$ holds for every $x \in \mathcal{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. Also for self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \succ B$ if $\langle Ax, x \rangle > \langle Bx, x \rangle$ holds for all non-zero elements $x \in \mathcal{H}$. Also $A > B$ if $A \geq B$ and $A - B$ is invertible.

A continuous real valued function f defined on an interval J is called operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all self adjoint operators A, B acting on a Hilbert space with spectra in J .

The Löwner theorem says that a function f is operator monotone on an interval J if and only if f has an analytic continuation to the upper half plane Π_+ such that f maps Π_+ into itself. If $f(t)$ is an operator monotone function on (a, b) , then clearly $f\left(\frac{2t-a-b}{b-a}\right)$ is operator monotone on $(-1, 1)$, so in this paper we study the family of operator monotone functions on $(-1, 1)$.

Let $\mathcal{K}$ denote the family of all operator monotone functions on $(-1, 1)$ such that $f(0) = 0$ and $f'(0) = 1$. Hansen and Pedersen [8] showed that $\mathcal{K}$ is a compact convex subset of the space of all bounded functions on $(-1, 1)$ with pointwise convergence topology and that the extreme points of $\mathcal{K}$ are of the form $f_\lambda(t) = \frac{t}{1-\lambda t}$ with $|\lambda| < 1$. They [8] also proved that every $f \in \mathcal{K}$ can be represented as

$$f(t) = \int_{-1}^1 \frac{t}{1-\lambda t} d\mu(\lambda),$$

where μ is a positive measure on $(-1, 1)$, see also [3].

The Löwner–Heinz inequality says that, $f(x) = x^r$ ($0 < r \leq 1$) is operator monotone on $[0, \infty)$. Löwner proved the inequality for matrices. Heinz proved it for positive

operators acting on a Hilbert space of arbitrary dimension. Based on the C^* -algebra theory, Pedersen [14] gave a shorter proof of the inequality.

There exist several operator norm inequalities each of which is equivalent to the Löwner-Heinz inequality. One of them is $\|A^r B^r\| \leq \|AB\|^r$, called the Cördes inequality in the literature, in which A and B are positive operators and $0 < r \leq 1$. A generalization of the Cördes inequality for operator monotone functions is given in [5]. It is shown in [2] that this norm inequality is related to the Finsler structure of the space of positive invertible elements.

Kwong [10] showed that if $A > B$ ($A \succ B$, resp.), then $A^r > B^r$ ($A^r \succ B^r$, resp.) for $0 < r \leq 1$. Uchiyama [15] showed that for every non-constant operator monotone function f on an interval J , $A \succ B$ implies $f(A) \succ f(B)$ for all self-adjoint operators A, B with spectra in J .

There are several extensions of the Löwner-Heinz inequality. The Furuta inequality [6], which states that if $A \geq B \geq 0$, then for $r \geq 0$, $(A^{r/2} A^p A^{r/2})^{1/q} \geq (A^{r/2} B^p A^{r/2})^{1/q}$ holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$, is known as an exquisite extension of the Löwner-Heinz inequality; Also Ando [1] extended the Löwner-Heinz inequality for a pair of J -selfadjoint matrices.

Let Ω be a open subset of $\mathbb{C}$. A set $\mathcal{F} \subseteq C(\Omega)$ is bounded if for each compact subset $K \subseteq \Omega$, $\sup\{\|f\|_K : f \in \mathcal{F}\} < \infty$. The Montel theorem states that if $\mathcal{F}$ is a bounded subset of the set $A(\Omega)$ of all analytic functions on Ω , then $\mathcal{F}$ is a normal family, i.e, each sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ converging uniformly on each compact subset of Ω .

2. THE RESULTS

Throughout this note, let $\Omega = \Pi_+ \cup \Pi_- \cup (-1, 1)$, where Π_- is the lower half plan.

Theorem 2.1. *The family $\mathcal{K}$ is bounded in $A(\Omega)$, so it is a normal family.*

Proof. Let S be the convex hull of $\{f_\lambda : |\lambda| < 1\}$ where $f_\lambda(t) = \frac{t}{1-\lambda t}$. By Krein-Millman's theorem, $\mathcal{K}$ is the closed convex hull of it's extreme points, so $\overline{S} = \mathcal{K}$. Fix $K \subseteq \Omega$ as a compact set. Then $h(\lambda, z) = |1 - \lambda z|$ is continuous on $[-1, 1] \times K$ and so takes its minimum value. It should be noticed that the minimum value m of h on $[-1, 1] \times K$ is nonzero. Put $M_K := \sup\{|z| : z \in K\}$. Then

$$|f_\lambda(z)| = \frac{|z|}{|1 - \lambda z|} \leq \frac{M_K}{m}$$

If $g = \sum_{i=1}^n c_i f_{\lambda_i} \in \mathcal{S}$, then

$$|g(z)| = \left| \sum_{i=1}^n c_i f_{\lambda_i}(z) \right| \leq \sum_{i=1}^n c_i |f_{\lambda_i}(z)| \leq \sum_{i=1}^n c_i \frac{M_k}{m} = \frac{M_k}{m},$$

whence $\|g\|_K \leq M_K$. Now assume that $g \in \mathcal{K}$ is arbitrary. There exists $\{f_n\}$ in $\mathcal{S}$ such that $f_n(t) \rightarrow g(t)$ for each $t \in (-1, 1)$. Since S is bounded, the sequence $\{f_n\}$ is bounded. By Montel's theorem there exists a subsequence $\{f_{n_j}\}$ converging to g'

in uniform compact convergence topology on Ω . Since $g = g'$ on $(-1, 1)$, we have $g(z) = g'(z)$ for each $z \in \Omega$. Hence

$$|g(z)| = |g'(z)| = \lim_{n_j \rightarrow \infty} |f_{n_j}(z)| \leq \frac{M_K}{m}.$$

Therefore $\mathcal{K}$ is a normal family. $\square$

Proposition 2.2. *Let $f \in \mathcal{K}$ and $f(-1, 1) \subseteq (-1, 1)$. Then $f(t) = t$ for each $t \in (-1, 1)$.*

Proof. Since $f(-1, 1) \subseteq (-1, 1)$, so $f^n = f \circ f \cdots \circ f \in \mathcal{K}$. Hence by Theorem (2.11), f^n has a convergent subsequence that converges to a function $h \in \mathcal{K}$. Assume that $f(t_0) < t_0$ for some $t_0 \in (-1, 1)$. Hence $\{f^{(n)}(t_0)\}$ is an increasing sequence converging to $h(t_0)$. Thus

$$h(f(t_0)) = \lim_{n \rightarrow \infty} f^n(f(t_0)) = \lim_{n \rightarrow \infty} f^{n+1}(t_0) = h(t_0)$$

Since h is one-one, we infer that $f(t_0) = t_0$, which is a contradiction and this completes the proof. $\square$

Remark 2.3. We can prove Proposition 2.2 directly as follows.

It follows from

$$f(t) = \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda),$$

that

$$-1 \leq \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda) \leq 1 \quad (-1 < t < 1).$$

Since for each λ the integrand $\frac{t}{1 - \lambda t}$ is positive and increasing on $0 < t < 1$, by the Lebesgue's monotone convergence theorem

$$\int_{-1}^1 \frac{1}{1 - \lambda} d\mu(\lambda) = \lim_{t \rightarrow 1^-} \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda) \leq 1.$$

Similarly we have

$$\int_{-1}^1 \frac{-1}{1 + \lambda} d\mu(\lambda) = \lim_{t \rightarrow -1^+} \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda) \geq -1.$$

Thus we have

$$\begin{aligned} \int_{-1}^1 \frac{1}{1 - \lambda^2} d\mu(\lambda) &= \frac{1}{2} \int_{-1}^1 \left(\frac{1}{1 - \lambda} + \frac{1}{1 + \lambda} \right) d\mu(\lambda) \\ &\leq 1 = \int_{-1}^1 1 d\mu(\lambda). \end{aligned}$$

From this it follows that $\frac{1}{1 - \lambda^2} = 1$ almost everywhere with respect to μ , Thus $\mu\{0\} = 1$, which implies $f(t) = t$. $\square$

Corollary 2.4. *Let f be an odd operator monotone function on $(-1, 1)$ and A is a bounded linear operator on a Hilbert space with spectrum in $(-1, 1)$. Then $f(|A|) \geq f'(0)|A|$.*

Proof. If $f(t_0) < f'(0)t_0$ for some $t_0 \in (0, 1)$, then $f_1(t) = \frac{1}{f'(0)t_0}f(t_0t) \in \mathcal{K}$ and $f_1(-1, 1) \subseteq (-1, 1)$, so, by Proposition (2.2), we have $f_1(1) = 1$, which is a contradiction. Hence

$$f(|t|) \geq f'(0)|t|, \quad t \in (-1, 1) \quad (2.1)$$

Therefore $f(|A|) \geq f'(0)|A|$. $\square$

Remark 2.5. A direct proof of (2.1) reads as follows. Notice that $f(0) = 0$. Hence

$$f(t) = f'(0) \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda). \quad (2.2)$$

Since $f(t) = -f(-t)$, we obtain

$$\int_{-1}^1 \frac{1}{1 - \lambda t} d\mu(\lambda) = \int_{-1}^1 \frac{1}{1 + \lambda t} d\mu(\lambda).$$

Thus

$$\begin{aligned} \int_{-1}^1 \frac{1}{1 - \lambda t} d\mu(\lambda) &= \frac{1}{2} \int_{-1}^1 \left(\frac{1}{1 - \lambda t} + \frac{1}{1 + \lambda t} \right) d\mu(\lambda) \\ &= \int_{-1}^1 \frac{1}{1 - (\lambda t)^2} d\mu(\lambda) \geq \int_{-1}^1 \frac{1}{1 - (\lambda t)^2} d\mu(\lambda) = 1. \end{aligned}$$

(2.2) yields $|f(t)| \geq f'(0)|t|$. $\square$

In the sequel we need the following lemma.

Lemma 2.6. [3, Lemma 2.4] *If f is an operator monotone function on an interval (a, b) , then $f^{2p+1}(t) \geq 0$ for all $p = 0, 1, 2, \dots$ and all $a < t < b$.*

Corollary 2.7. *Let f be an odd operator monotone function on $(-1, 1)$. Then f is concave on $(-1, 0)$ and convex on $(0, 1)$.*

Proof. Without loss of generality we may assume that $f \in \mathcal{K}$. We shall show that f is convex on $(0, 1)$. The proof of Lemma 4.1 of [8] shows that $f'(t) \geq \frac{f(t)^2}{t^2}$. It follows from Corollary (2.4) that $f'(t) \geq 1$ for each $t \in (0, 1)$. Therefore

$$f''(0) = \lim_{t \rightarrow 0^+} \frac{f'(t) - f'(0)}{t} = \lim_{t \rightarrow 0^+} \frac{f'(t) - 1}{t} \geq 0.$$

By Lemma (2.6), $f^{(3)}(t) \geq 0$ for all $t \in (-1, 1)$, so $f''(t) \geq 0$ for all $t \in (0, 1)$ since f'' is monotone. Hence f is a convex function on $(0, 1)$. Since f is an odd function, f is concave on $(-1, 0)$. $\square$

Theorem 2.8. *An odd operator monotone function on $(-1, 1)$ is of the form*

$$f(t) = f'(0) \int_{-1}^1 \frac{t}{1 - (\lambda t)^2} d\mu(\lambda), \quad (2.3)$$

where μ is a probability measure on $(-1, 1)$.

Proof. As before, we may assume that $f \in \mathcal{K}$. The function f can be represented as a power series $f(t) = \sum_{n=1}^{\infty} a_n t^n$, which is convergent for $|t| < 1$, cf. [3]. Since f is odd, $a_{2n} = 0$ for all n . Due to f is operator monotone, there is a probability measure μ on $(-1, 1)$ such that

$$f(t) = \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda) = \int_{-1}^1 \sum_{n=1}^{\infty} t(\lambda t)^{n-1} d\mu(\lambda) = \sum_{n=1}^{\infty} t^{n+1} \int_{-1}^1 \lambda^n d\mu(\lambda)$$

Therefore $a_{2n} = \int_{-1}^1 \lambda^{2n-1} d\mu(\lambda) = 0$ and so

$$f(t) = \int_{-1}^1 \sum_{n=1}^{\infty} t(\lambda t)^{2n-1} d\mu(\lambda) = \int_{-1}^1 \frac{t}{1 - (\lambda t)^2} d\mu(\lambda).$$

If f is of the form (2.3), then it is trivially odd. In addition,

$$f(t) = \int_{-1}^1 \frac{t}{1 - (\lambda t)^2} d\mu(\lambda) = \frac{1}{2} \int_{-1}^1 \frac{t}{1 - \lambda t} + \frac{t}{1 + \lambda t} d\mu(\lambda) = \frac{1}{2}(g(t) - g(-t)),$$

where $g(t) = \int_{-1}^1 \frac{t}{1 - \lambda t} d\mu(\lambda)$. Hence f is an odd operator monotone function on $(-1, 1)$. $\square$

We start main results with the following useful lemma.

Lemma 2.9. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be invertible positive operators such that $A - B \geq m > 0$. Then*

$$B^{-1} - A^{-1} \geq \frac{m}{(\|A\| - m) \|A\|}. \quad (2.4)$$

Proof. Since $f(t) = \frac{1}{t}$ is a decreasing operator monotone function on $[0, \infty)$ we have $B^{-1} \geq (A - m)^{-1}$. On the other hand

$$\begin{aligned} (A - m)^{-1} &\geq A^{-1} + \frac{m}{(\|A\| - m) \|A\|} \\ \Leftrightarrow (A^{-1} + \frac{m}{(\|A\| - m) \|A\|})(A - m) &\leq 1 \\ \Leftrightarrow \frac{A^2}{(\|A\| - m) \|A\|} - \frac{mA}{(\|A\| - m) \|A\|} &\leq 1 \\ \Leftrightarrow A^2 - mA &\leq (\|A\| - m) \|A\| \\ \Leftrightarrow \|A^2 - mA\| &\leq (\|A\| - m) \|A\|. \end{aligned}$$

There exists $\lambda_0 \in \text{sp}(A)$ such that $\|A\| = \lambda_0$. Since $A \geq m > 0$, we have

$$\begin{aligned} \|A^2 - mA\| &= \max\{\lambda : \lambda \in \text{sp}(A^2 - mA)\} \\ &= \max\{\lambda^2 - m\lambda : \lambda \in \text{sp}(A)\} \\ &= \lambda_0^2 - m\lambda_0 \\ &= (\|A\| - m)\|A\|. \end{aligned}$$

So $B^{-1} \geq (A - m)^{-1} \geq A^{-1} + \frac{m}{(\|A\| - m)\|A\|}$. □

Proposition 2.10. *Let f be a non-constant operator monotone function on an interval J and A, B be self-adjoint operators with spectra in J such that $A > B$. Then $f(A) > f(B)$.*

Proof. Without loss of generality we assume that $J = (-1, 1)$. Let $A, B \in \mathbb{B}(\mathcal{H})$ be self-adjoint operators with spectra in $(-1, 1)$ and $A - B$ is positive and invertible. So there exists $m > 0$ such that $A - B \geq m > 0$. Put $f_\lambda(t) = \frac{t}{1-\lambda t}$ for each λ with $|\lambda| < 1$. We shall show that $f_\lambda(A) - f_\lambda(B)$ is bounded below and so invertible. It is clear that the claim is true for $\lambda = 0$. If $0 < \lambda < 1$, then $(1 - \lambda B) - (1 - \lambda A) = \lambda(A - B) > \lambda m > 0$ as well as $1 - \lambda B$ and $1 - \lambda A$ are positive invertible operators. Since

$$\frac{t}{1 - \lambda t} = \frac{-1}{\lambda} + \frac{1}{\lambda} \left(\frac{1}{1 - \lambda t} \right),$$

by Lemma 2.9, we have

$$\begin{aligned} f_\lambda(A) - f_\lambda(B) &= \frac{1}{\lambda} \left(\frac{1}{1 - \lambda A} - \frac{1}{1 - \lambda B} \right) \\ &\geq \frac{1}{\lambda} \left(\frac{\lambda m}{(\|1 - \lambda B\| - \lambda m) \|1 - \lambda B\|} \right) \quad (\text{by (2.9)}) \\ &= \frac{m}{(\|1 - \lambda B\| - \lambda m) \|1 - \lambda B\|} > 0 \end{aligned}$$

A similar argument shows that

$$f_\lambda(A) - f_\lambda(B) \geq \frac{m}{(\|1 - \lambda A\| + \lambda m) \|1 - \lambda A\|} > 0$$

for each $-1 < \lambda < 0$. Since f is operator monotone on $(-1, 1)$, it can be represented as

$$f(t) = f(0) + f'(0) \int_{-1}^1 f_\lambda(t) d\mu(\lambda),$$

where μ is a nonzero positive measure on $(-1, 1)$. Since f is nonconstant, $f'(0) > 0$, [3, Lemma 2.3]. Hence

$$\begin{aligned} f(A) - f(B) &= f'(0) \int_{-1}^1 \left(\frac{A}{1 - \lambda A} - \frac{B}{1 - \lambda B} \right) d\mu(\lambda) \\ &= f'(0) \int_{-1}^1 (f_\lambda(A) - f_\lambda(B)) d\mu(\lambda) \\ &\geq f'(0) \int_{-1}^1 m_\lambda d\mu(\lambda), \end{aligned}$$

where

$$m_\lambda = \frac{m}{(\|1 - \lambda B\| - \lambda m) \|1 - \lambda B\|}$$

if $0 \leq \lambda < 1$, and

$$m_\lambda = \frac{m}{(\|1 - \lambda A\| + \lambda m) \|1 - \lambda A\|}$$

if $-1 < \lambda < 0$. Since μ is a nonzero positive measure and $m_\lambda > 0$, we have

$$f(A) - f(B) \geq f'(0) \int_{-1}^1 m_\lambda d\mu(\lambda) > 0.$$

Therefore $f(A) > f(B)$. □

Theorem 2.11. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators such that $A - B \geq m > 0$ and $0 < r \leq 1$. Then*

$$A^r - B^r \geq \|A\|^r - (\|A\| - m)^r.$$

Proof. Let $0 < r < 1$. It is known that

$$t^r = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{t}{\lambda + t} \lambda^{r-1} d\lambda, \quad (2.5)$$

in which $0 < r < 1$, see e.g. [4, Chapter V]. First note that,

$$\begin{aligned} \frac{A}{\lambda + A} - \frac{B}{\lambda + B} &= \lambda \left(\frac{1}{\lambda + B} - \frac{1}{\lambda + A} \right) \\ &\geq \frac{\lambda m}{(\|A + \lambda\| - m) \|A + \lambda\|} \quad \text{by (2.4)} \\ &= \frac{\lambda m}{(\|A\| + \lambda - m)(\|A\| + \lambda)} \end{aligned}$$

for each $\lambda > 0$. By using (2.5) we have

$$\begin{aligned} A^r - B^r &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \left(\frac{A}{\lambda + A} - \frac{B}{\lambda + B} \right) d\lambda \\ &\geq \frac{\sin(r\pi)}{\pi} \int_0^\infty \left(\frac{m\lambda^r}{(\|A\| + \lambda - m)(\|A\| + \lambda)} \right) d\lambda, \end{aligned}$$

We need to compute

$$I = \int_0^\infty \frac{\lambda^r}{(\lambda + \|A\|)(\lambda + (\|A\| - m))} d\lambda$$

where $0 < r < 1$. We will need the branch cut for $z^r = \rho^r e^{ir\theta}$, in which $z = \rho e^{i\theta}$ and $0 \leq \theta \leq 2\pi$. Consider

$$\int_C \frac{z^r}{(z + \|A\|)(z + (\|A\| - m))} dz,$$

where the keyhole contour C consists of a large circle C_R of radius R , a small circle C_ϵ of radius ϵ and two lines just above and below the branch cuts $\theta = 0$; see Figure 1. The contribution from C_R is $O(R^{r-2})2\pi R = O(R^{r-1}) = 0$ as $R \rightarrow \infty$. Similarly the contribution from C_ϵ is zero as $\epsilon \rightarrow 0$. The contribution from just above the branch cut and from just below the branch cut is I and $-e^{2r\pi i}I$, respectively, as $\epsilon \rightarrow 0$ and $R \rightarrow \infty$. Hence, taking the limits as $\epsilon \rightarrow 0$ and $R \rightarrow \infty$,

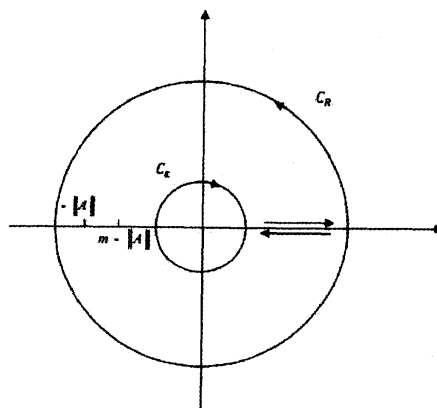


FIGURE 1. Keyhole contour

$$\begin{aligned} (1 - e^{2r\pi i})I &= \int_C \frac{z^r}{(z + \|A\|)(z + (\|A\| - m))} dz \\ &= -2\pi i e^{r\pi i} \left(\frac{\|A\|^r - (\|A\| - m)^r}{\|A\| - (\|A\| - m)} \right) \end{aligned}$$

by the Cauchy residue theorem. So

$$I = \frac{\pi}{m \sin(r\pi)} (\|A\|^r - (\|A\| - m)^r).$$

Therefore

$$\begin{aligned} A^r - B^r &\geq \frac{\sin(r\pi)}{\pi} \int_0^\infty \left(\frac{m\lambda^r}{(\|A\| + \lambda - m)(\|A\| + \lambda)} \right) d\lambda \\ &= \|A\|^r - (\|A\| - m)^r. \end{aligned}$$

□

Corollary 2.12. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators such that $A - B \geq m > 0$. Then*

$$\log A - \log B \geq \log \|A\| - \log(\|A\| - m).$$

Proof. Put $f_n(t) = n(t^{\frac{1}{n}} - 1)$ on $[0, \infty)$. Then the sequence $\{f_n\}$ uniformly converges to $\log t$ on any compact subset of $(0, \infty)$. Hence

$$\begin{aligned} \log A - \log B &= \lim_{n \rightarrow \infty} f_n(A) - f_n(B) \\ &\geq \lim_{n \rightarrow \infty} n(\|A\|^{\frac{1}{n}} - (\|A\| - m)^{\frac{1}{n}}) \\ &= \log \|A\| - \log(\|A\| - m). \end{aligned}$$

□

REFERENCES

1. T. Ando, *Löwner inequality of indefinite type*, Linear Algebra Appl., **385** (2004), 73–80.
2. E. Andruchow, G. Corach and D. Stojanoff, *Geometrical significance of Löwner–Heinz inequality*, Proc. Amer. Math. Soc. **128** (2000), no. 4, 1031–1037.
3. J. Bendat and S. Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79** (1955), 58–71.
4. R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
5. J. Fujii and M. Fujii, *A norm inequality for operator monotone functions*, Math. Japon. **35** (1990), no. 2, 249–252.
6. T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+r)q \geq p+2r$* , Proc. Amer. Math. Soc., **101** (1987), 85–88.
7. T. Furuta, *Norm inequalities equivalent to Löwner–Heinz theorem*, Rev. Math. Phys. **1** (1989), 135–137.
8. F. Hansen and G. Pedersen, *Jensen’s inequality for operators and Löwner’s theorem*, Math. Ann. **258** (1981/82), no. 3, 229–241.
9. E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.
10. M.K. Kwong, *Inequalities for the powers of nonnegative Hermitian operators*, Proc. Amer. Math. Soc. **51** (1975), 401–406.
11. C. Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177–216.
12. M.S. Moslehian and H. Najafi, *An extension of the Löwner–Heinz inequality*, Linear Algebra Appl. **437** (2012), no. 9, 2359–2365.
13. M.S. Moslehian, H. Najafi and M. Uchiyama, *A normal family of operator monotone functions*, Hokkaido Math. J., to appear.
14. G.K. Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc. **36** (1972), 309–310.
15. M. Uchiyama, *Strong monotonicity of operator functions*, Integral Equations Operator Theory **37** (2000), no. 1, 95–105.

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. Box 1159, MASHHAD 91775, IRAN.

E-mail address: hamednajafi20@gmail.com