

ON EQUIVARIANT PERTURBATIVE INVARIANTS IN 3-DIMENSION BY MORSE THEORY

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1. INTRODUCTION

Around 1992, Axelrod–Singer and Kontsevich independently developed the method to obtain (mathematical) topological invariants of 3-manifolds by perturbative expansion of Witten’s path integral (Chern–Simons perturbation theory, [1, 5]). The invariant is a series of terms corresponding to Feynman diagrams such that each term is given by integration over the configuration space of a 3-manifold. This is known to be very strong, for example, the expansion around the trivial connection dominates all $\mathbb{Q}$ -valued Ohtsuki finite type invariants for integral homology 3-spheres ([7]). In this note, we explain about our attempt to construct ‘equivariant invariant’ of 3-manifolds with the first Betti number 1.

Around 2008, Ohtsuki constructed an equivariant refinement of the LMO invariant¹ for 3-manifolds with the first Betti number 1 ([12, 13]), which pioneered a new direction of perturbative invariants of 3-manifolds. Inspired by Ohtsuki’s work, Lescop constructed an equivariant refinement of Chern–Simons perturbation theory for 3-manifolds with the first Betti number 1 for the 2-loop graphs by using a method similar to Marché ([9, 11]). Lescop’s construction is as follows.

Let M be a closed 3-manifold with $H_1(M) = \mathbb{Z}$. The *equivariant configuration space* $\text{Conf}_{K_2}(M)$ is defined as the set of tuples (x_1, x_2, γ) , $x_1, x_2 \in M$, satisfying the following conditions.

- (1) $x_1 \neq x_2$.
- (2) γ is the relative bordism class of paths $c : [0, 1] \rightarrow M$ that go from x_1 to x_2 .

The natural map $\text{Conf}_{K_2}(M) \rightarrow \text{Conf}_2(M) = M \times M \setminus \Delta_M$ that forgets γ is an infinite cyclic covering. Instead of removing the diagonal Δ_M in the definition of $\text{Conf}_2(M)$, consider the blowing-up along Δ_M , namely replacing Δ_M with its normal sphere bundle, to obtain a compactification $\overline{\text{Conf}}_2(M)$ of $\text{Conf}_2(M)$. Similarly, by blowing-up along the preimage of Δ_M in the space of tuples (x_1, x_2, γ) satisfying only (2) above, we obtain the ‘closure’ $\overline{\text{Conf}}_{K_2}(M)$ of $\text{Conf}_{K_2}(M)$.

Lescop defined an invariant of 3-manifolds M with $b_1(M) = 1$ by an equivariant intersection theory in $\overline{\text{Conf}}_{K_2}(M)$. The principal term of it is given by the equivariant triple intersection $\langle Q, Q, Q \rangle_{\mathbb{Z}}$ for a fundamental 4-chain

$$Q \in C_4(\overline{\text{Conf}}_{K_2}(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t, t^{-1}]} \mathbb{Q}(t)$$

which satisfies a certain boundary condition (*equivariant propagator*)².

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¹LMO invariant is defined combinatorially by using Kontsevich’s link invariant and is known to be universal among finite type invariants of homology 3-spheres

²The Poincaré dual of $[Q] \in H_4(\overline{\text{Conf}}_2(M), \partial \overline{\text{Conf}}_2(M); \mathbb{Q}(t))$ generates $H^2(\overline{\text{Conf}}_2(M); \mathbb{Q}(t)) \cong \mathbb{Q}(t)$. The meaning of the word ‘propagator’ here differs from the usual one.

Lescop proved the existence of an equivariant propagator by means of homology theoretic arguments. We developed a notion of ‘Z-paths’ (we previously called ‘AL-paths’) in a surface bundle M over S^1 and gave an explicit equivariant propagator by the natural map from the moduli space of Z-paths to configuration space ([16]). By using the equivariant propagator, we construct an invariant of fiberwise Morse functions on M ([17]). The construction of the invariant can be applied to a construction of a perturbative isotopy invariant of knots in M , which is useful for the study of finite type invariants of knots in M ([18]).

2. MODULI SPACE OF Z-PATHS

We define the moduli space of Z-paths and its ‘closure’.³

2.1. Z-path. Let M be an oriented closed 3-manifold. Assume that M admits a structure of an oriented fiber bundle $\kappa : M \rightarrow S^1$. We say that a C^∞ map $f : M \rightarrow \mathbb{R}$ is a *fiberwise Morse function* if the restriction $f_s = f|_{\kappa^{-1}(s)} : \kappa^{-1}(s) \rightarrow \mathbb{R}$ is Morse for each $s \in S^1$ (known to exist for every κ). The totality of the critical points of f_s , $s \in S^1$, forms a 1-submanifold of M (closed braid) and we call each component of the 1-submanifold a *critical locus*. Let ξ be the gradient of f along the fibers, namely, the one whose restriction to each fiber over $s \in S^1$ is $\text{grad } f_s$. Let $\Sigma(\xi)$ denote the union of all critical loci of ξ . For a critical locus p of a fiberwise Morse function f , the *descending/ascending manifold* are defined respectively by

$$\begin{aligned}\mathcal{D}_p(\xi) &= \{x \in M \mid \lim_{t \rightarrow -\infty} \Phi_{-\xi}^t(x) \in p\} \\ \mathcal{A}_p(\xi) &= \{x \in M \mid \lim_{t \rightarrow \infty} \Phi_{-\xi}^t(x) \in p\}\end{aligned}$$

where $\Phi_{-\xi}^t : M \rightarrow M$ is the flow of $-\xi$.

Let $\tilde{\kappa} : \tilde{M} \rightarrow \mathbb{R}$ be the pullback of κ by the projection $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = S^1$.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\kappa}} & \mathbb{R} \\ \pi \downarrow & & \downarrow \\ M & \xrightarrow{\kappa} & S^1 \end{array}$$

The induced map $\pi : \tilde{M} \rightarrow M$ on the total space is an infinite cyclic covering. The function $\tilde{f} = f \circ \pi : \tilde{M} \rightarrow \mathbb{R}$ is a fiberwise Morse function (for a fiber bundle over $\mathbb{R}$). Let $\tilde{\xi}$ denote the gradient for $\tilde{f}$ along the fibers. By replacing ξ with $\tilde{\xi}$, the critical locus, its descending/ascending manifolds are defined similarly.

We say that an embedding $\sigma : [\mu, \nu] \rightarrow \tilde{M}$ is *horizontal* if $\text{Im } \sigma$ is included in a single fiber of $\tilde{\kappa}$ and say that it is *vertical* if $\text{Im } \sigma$ is included in a single critical locus of $\tilde{f}$. A horizontal (resp. vertical) embedding $\sigma : [\mu, \nu] \rightarrow \tilde{M}$ is *descending* if $\tilde{f}(\sigma(\mu)) \geq \tilde{f}(\sigma(\nu))$ (resp. $\tilde{\kappa}(\sigma(\mu)) \geq \tilde{\kappa}(\sigma(\nu))$). A horizontal embedding $\sigma : [\mu, \nu] \rightarrow \tilde{M}$ is a *flow-line* of $\tilde{\xi}$ if for each $t \in (\mu, \nu)$, $d\sigma_t(\frac{\partial}{\partial t})$ is a positive multiple of $(-\tilde{\xi})_{\sigma(t)}$.

Definition 2.1. Let $x, y \in \tilde{M}$ be such that $\tilde{\kappa}(x) \geq \tilde{\kappa}(y)$. A *Z-path* from x to y is a sequence $\gamma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ satisfying the following six conditions.

³The definition in this note differs slightly from that of [16].

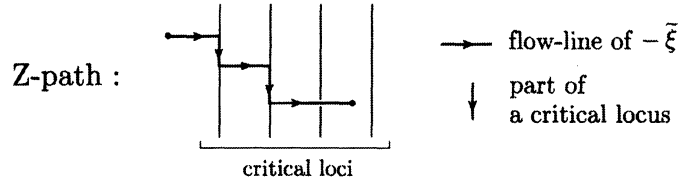


FIGURE 1. Z-path

- (1) For each i , σ_i is an embedding $[\mu_i, \nu_i] \rightarrow \widetilde{M}$ (μ_i, ν_i are real numbers such that $\mu_i \leq \nu_i$) and it is either horizontal or vertical.
- (2) For each i , σ_i is descending.
- (3) If σ_i is horizontal, then σ_i is a flow-line of $\widetilde{\xi}$. If it is vertical, then $\mu_i < \nu_i$.
- (4) $\sigma_1(\mu_1) = x$, $\sigma_n(\nu_n) = y$.
- (5) $\sigma_i(\nu_i) = \sigma_{i+1}(\mu_{i+1})$ for $1 \leq i < n$.
- (6) If σ_i is horizontal (resp. vertical) and if $i < n$, then σ_{i+1} is vertical (resp. horizontal).

We say that two Z-paths are *equivalent* if they are related by piecewise reparametrizations. We call a sequence of paths of the form $\pi \circ \gamma = (\pi \circ \sigma_1, \dots, \pi \circ \sigma_n)$ for a Z-path γ in $\widetilde{M}$ a Z-path in M .

Let $\mathcal{M}_2^Z(\widetilde{\xi})$ be the set of all equivalence classes of Z-paths in $\widetilde{M}$. This has a natural structure of a noncompact manifold with corners. Let t denote the covering translation of the covering $\pi : \widetilde{M} \rightarrow M$ that induces the translation $x \mapsto x - 1$ in $\mathbb{R}$. This induces diagonal $\mathbb{Z}$ -actions $\gamma \mapsto t^n \gamma$, $(x, y) \mapsto (t^n x, t^n y)$ on $\mathcal{M}_2^Z(\widetilde{\xi})$ and $\widetilde{M} \times \widetilde{M}$. We denote the quotient spaces $\mathcal{M}_2^Z(\widetilde{\xi})/\mathbb{Z}$ and $(\widetilde{M} \times \widetilde{M})/\mathbb{Z}$ respectively by $\mathcal{M}_2^Z(\widetilde{\xi})_{\mathbb{Z}}$ and $\widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$. We consider another $\mathbb{Z}$ -action on the quotient spaces, denoted by t^n by abuse of notation, as

$$t([x \times y]) = [x \times ty].$$

For a gradient ξ along the fiber for a fiberwise Morse function f , let $\hat{\xi}$ denote the nonsingular vector field $\xi + \text{grad } \kappa$ on M . Let $s_{\hat{\xi}} : M \rightarrow ST(M)$ (ST denotes the unit tangent bundle) be the section given by $-\hat{\xi}/\|\hat{\xi}\|$.

Theorem 2.2 ([16]). *Let Σ be an oriented connected closed surface and let M be the mapping torus of an orientation preserving diffeomorphism $\varphi : \Sigma \rightarrow \Sigma$. Let $\widetilde{\Delta}_M \subset \widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$ be the preimage of the diagonal Δ_M of $M \times M$.*

- (1) *There is a natural ‘closure’ $\overline{\mathcal{M}}_2^Z(\widetilde{\xi})_{\mathbb{Z}}$ of $\mathcal{M}_2^Z(\widetilde{\xi})_{\mathbb{Z}}$ that is a countable union of compact manifolds with corners.*
- (2) *Suppose that κ induces an isomorphism $H_1(M)/\text{Torsion} \cong H_1(S^1)$. Let $\bar{b} : \overline{\mathcal{M}}_2^Z(\widetilde{\xi})_{\mathbb{Z}} \rightarrow \widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$ be the map that assigns the endpoints. Let $Bl_{\bar{b}^{-1}(\widetilde{\Delta}_M)}(\overline{\mathcal{M}}_2^Z(\widetilde{\xi})_{\mathbb{Z}})$ be the blow-up of $\overline{\mathcal{M}}_2^Z(\widetilde{\xi})_{\mathbb{Z}}$ along $\bar{b}^{-1}(\widetilde{\Delta}_M)$. Then $\bar{b}$ induces a map*

$$Bl_{\bar{b}^{-1}(\widetilde{\Delta}_M)}(\overline{\mathcal{M}}_2^Z(\widetilde{\xi})_{\mathbb{Z}}) \rightarrow \overline{\text{Conf}}_{K_2}(M)$$

that represents a 4-dimensional $\mathbb{Q}(t)$ -chain $Q(\tilde{\xi})$ of $\overline{\text{Conf}}_{K_2}(M)$. Moreover, the following identity in $H_3(\partial\overline{\text{Conf}}_{K_2}(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t, t^{-1}]} \mathbb{Q}(t)$ holds.

$$[\partial Q(\tilde{\xi})] = [s_{\tilde{\xi}}(M)] + \frac{t\zeta'_{\varphi}}{\zeta_{\varphi}}[ST(M)|_K],$$

where ζ_{φ} is the Lefschetz zeta function for φ and K is a knot such that $\kappa_*([K])$ is the positive generator of $H_1(S^1)$. Furthermore, there is a product $P(t)$ of cyclotomic polynomials such that $P(t)\Delta(M)Q(\tilde{\xi})$ is a $\mathbb{Q}[t, t^{-1}]$ -chain ($\Delta(M)$ is the Alexander polynomial of M).

2.2. Closure of the moduli space of Z-paths. We define the space $\mathcal{M}_2(\tilde{\xi})$ of horizontal paths in $\tilde{M}$ by

$$\mathcal{M}_2(\tilde{\xi}) = \{(x, y) \in \tilde{M} \times \tilde{M}; \tilde{\kappa}(x) = \tilde{\kappa}(y), y = \Phi_{-\tilde{\xi}}^t(x) \text{ for some } t > 0\}.$$

Let $b : \mathcal{M}_2(\tilde{\xi}) \rightarrow \tilde{M} \times \tilde{M}$ denote the inclusion map. For a continuous parameter $s \in S$ such as real numbers, we denote the sum $\bigcup_{s \in S} V_s$ by $\int_{s \in S} V_s$ and if the parameter is at most countable, then we denote it by $\sum_{s \in S} V_s$ or $V_{s_1} + V_{s_2} + \dots$ etc.

For a generic $\tilde{\xi}$, the intersection $\mathcal{D}_p(\tilde{\xi}) \cap \mathcal{A}_q(\tilde{\xi})$ is transversal and hence is a smooth manifold. There is a free $\mathbb{R}$ -action on $\mathcal{D}_p(\tilde{\xi}) \cap \mathcal{A}_q(\tilde{\xi})$ by $x \mapsto \Phi_{-\tilde{\xi}}^T(x)$ ($T \in \mathbb{R}$). We put

$$\mathcal{M}_{pq}(\tilde{\xi}) = (\mathcal{D}_p(\tilde{\xi}) \cap \mathcal{A}_q(\tilde{\xi})) / \mathbb{R}.$$

Proposition 2.3. *There is a natural closure $\overline{\mathcal{M}}_2(\tilde{\xi})$ of $\mathcal{M}_2(\tilde{\xi})$ and the extension $\bar{b} : \overline{\mathcal{M}}_2(\tilde{\xi}) \rightarrow \tilde{M} \times \tilde{M}$ of b such that for a generic $\tilde{\xi}$ the following hold ($\Delta_S \subset S \times S$ denotes the diagonal for any set S).*

- (1) $\overline{\mathcal{M}}_2(\tilde{\xi}) - \bar{b}^{-1}(\Delta_{\tilde{M}})$ is a manifold with corners.
- (2) $\bar{b}$ induces a diffeomorphism $\text{Int } \overline{\mathcal{M}}_2(\tilde{\xi}) \rightarrow \mathcal{M}_2(\tilde{\xi})$.
- (3) The codimension r stratum of $\overline{\mathcal{M}}_2(\tilde{\xi}) - \bar{b}^{-1}(\Delta_{\tilde{M}})$ corresponds to broken flow-lines that are broken r times at critical points. The codimension r stratum of $\overline{\mathcal{M}}_2(\tilde{\xi}) - \bar{b}^{-1}(\Delta_{\tilde{M}})$ for $r \geq 1$ is canonically diffeomorphic to

$$\begin{cases} \int_{s \in \mathbb{R}} \sum_{q_1 \in \Sigma(\tilde{\xi})} \mathcal{A}_{q_1}(\tilde{\xi}_s) \times \mathcal{D}_{q_1}(\tilde{\xi}_s) - \sum_{q_1 \in \Sigma(\tilde{\xi})} \Delta_{q_1} & (r = 1) \\ \int_{s \in \mathbb{R}} \sum_{\substack{q_1, \dots, q_r \in \Sigma(\tilde{\xi}) \\ q_1, \dots, q_r \text{ distinct}}} \mathcal{A}_{q_1}(\tilde{\xi}_s) \times \mathcal{M}_{q_1 q_2}(\tilde{\xi}_s) \times \dots \times \mathcal{M}_{q_{r-1} q_r}(\tilde{\xi}_s) \times \mathcal{D}_{q_r}(\tilde{\xi}_s) & (r \geq 2) \end{cases}$$

The formula for the codimension r stratum ($r \geq 2$) in Proposition 2.3 can be rewritten as follows.

$$\int_{s \in \mathbb{R}} X(s) \times \underbrace{\Omega(s) \times \dots \times \Omega(s)}_{r-1} \times {}^t Y(s).$$

Here, if $\Sigma(\tilde{\xi}) = \{p_1, p_2, \dots, p_N\}$, then

$$\begin{aligned} X(s) &= (\mathcal{A}_{p_1}(\tilde{\xi}_s) \ \mathcal{A}_{p_2}(\tilde{\xi}_s) \ \cdots \ \mathcal{A}_{p_N}(\tilde{\xi}_s)), \\ Y(s) &= (\mathcal{D}_{p_1}(\tilde{\xi}_s) \ \mathcal{D}_{p_2}(\tilde{\xi}_s) \ \cdots \ \mathcal{D}_{p_N}(\tilde{\xi}_s)), \\ \Omega(s) &= \begin{pmatrix} \emptyset & \mathcal{M}_{p_1 p_2}(\tilde{\xi}_s) & \mathcal{M}_{p_1 p_3}(\tilde{\xi}_s) & \cdots & \mathcal{M}_{p_1 p_N}(\tilde{\xi}_s) \\ \mathcal{M}_{p_2 p_1}(\tilde{\xi}_s) & \emptyset & \mathcal{M}_{p_2 p_3}(\tilde{\xi}_s) & \cdots & \mathcal{M}_{p_2 p_N}(\tilde{\xi}_s) \\ \mathcal{M}_{p_3 p_1}(\tilde{\xi}_s) & \mathcal{M}_{p_3 p_2}(\tilde{\xi}_s) & \emptyset & & \mathcal{M}_{p_3 p_N}(\tilde{\xi}_s) \\ \vdots & \vdots & & \ddots & \vdots \\ \mathcal{M}_{p_N p_1}(\tilde{\xi}_s) & \mathcal{M}_{p_N p_2}(\tilde{\xi}_s) & \mathcal{M}_{p_N p_3}(\tilde{\xi}_s) & \cdots & \emptyset \end{pmatrix} \end{aligned}$$

and the direct product of matrices is defined by replacing multiplications and sums with direct products and disjoint unions, respectively.

Proposition 2.4. *Let p be a critical locus of $\tilde{\xi}$ and let $\overline{\mathcal{D}}_p(\tilde{\xi}) = \bar{b}^{-1}(p \times \widetilde{M})$, $\overline{\mathcal{A}}_p(\tilde{\xi}) = \bar{b}^{-1}(\widetilde{M} \times p)$. For a generic $\tilde{\xi}$, the following are satisfied.*

- (1) $\overline{\mathcal{D}}_p(\tilde{\xi})$ (resp. $\overline{\mathcal{A}}_p(\tilde{\xi})$) is a manifold with corners.
- (2) $\bar{b}$ induces a diffeomorphism $\text{Int } \overline{\mathcal{D}}_p(\tilde{\xi}) \rightarrow \mathcal{D}_p(\tilde{\xi})$ (resp. $\text{Int } \overline{\mathcal{A}}_p(\tilde{\xi}) \rightarrow \mathcal{A}_p(\tilde{\xi})$).
- (3) The codimension r stratum of ${}^t\overline{Y} = (\overline{\mathcal{D}}_{p_1}(\tilde{\xi}) \ \overline{\mathcal{D}}_{p_2}(\tilde{\xi}) \ \cdots \ \overline{\mathcal{D}}_{p_N}(\tilde{\xi}))$ (resp. $\overline{X} = (\overline{\mathcal{A}}_{p_1}(\tilde{\xi}) \ \overline{\mathcal{A}}_{p_2}(\tilde{\xi}) \ \cdots \ \overline{\mathcal{A}}_{p_N}(\tilde{\xi}))$) for $r \geq 1$ is canonically diffeomorphic to

$$\int_{s \in \mathbb{R}} \underbrace{\Omega(s) \times \cdots \times \Omega(s)}_r \times {}^tY(s) \quad (\text{resp.} \quad \int_{s \in \mathbb{R}} X(s) \times \underbrace{\Omega(s) \times \cdots \times \Omega(s)}_r)$$

Proposition 2.5. *Let p, q be critical loci of $\tilde{\xi}$ and let $\overline{\mathcal{M}}_{pq}(\tilde{\xi}) = \bar{b}^{-1}(p \times q)$. For a generic $\tilde{\xi}$, the following hold.*

- (1) $\overline{\mathcal{M}}_{pq}(\tilde{\xi})$ is a manifold with corners.
- (2) There is a natural diffeomorphism $\text{Int } \overline{\mathcal{M}}_{pq}(\tilde{\xi}) \rightarrow \mathcal{M}_{pq}(\tilde{\xi})$.
- (3) The codimension r stratum of $\overline{\Omega} = ((1 - \delta_{ij}) \cdot \overline{\mathcal{M}}_{p_i p_j}(\tilde{\xi}))$ for $r \geq 1$ is canonically diffeomorphic to

$$\int_{s \in \mathbb{R}} \underbrace{\Omega(s) \times \cdots \times \Omega(s)}_{r+1}.$$

A fiberwise space over a space B is a pair of a space E and a continuous map $\phi : E \rightarrow B$. A fiber over a point $s \in B$ is $E(s) = \phi^{-1}(s)$ ([2]). For two fiberwise spaces $E_1 = (E_1, \phi_1)$ and $E_2 = (E_2, \phi_2)$ over B , a fiberwise product $E_1 \times_B E_2$ is defined as the following subspace of $E_1 \times E_2$:

$$E_1 \times_B E_2 = \int_{s \in B} E_1(s) \times E_2(s).$$

Namely, $E_1 \times_B E_2$ is the pullback of $E_1 \xrightarrow{\phi_1} B \xleftarrow{\phi_2} E_2$.

For a sequence $A_i = (A_i, \phi_i)$, $\phi_i : A_i \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, n$) of fiberwise spaces over $\mathbb{R}$, we define its iterated integrals as

$$\begin{aligned} \int_{\mathbb{R}} A_1 A_2 \cdots A_n &= \int_{s_1 > s_2 > \cdots > s_n} A_1(s_1) \times A_2(s_2) \times \cdots \times A_n(s_n) \\ &= (\phi_1 \times \cdots \times \phi_n)^{-1}(\{(s_1, \dots, s_n) \in \mathbb{R}^n \mid s_1 > \cdots > s_n\}), \\ \overline{\int}_{\mathbb{R}} A_1 A_2 \cdots A_n &= \int_{s_1 \geq s_2 \geq \cdots \geq s_n} A_1(s_1) \times A_2(s_2) \times \cdots \times A_n(s_n) \\ &= (\phi_1 \times \cdots \times \phi_n)^{-1}(\{(s_1, \dots, s_n) \in \mathbb{R}^n \mid s_1 \geq \cdots \geq s_n\}) \end{aligned}$$

For a matrix $P = (A_{ij})$ of fiberwise spaces over $\mathbb{R}$, we define a fiber of $s \in \mathbb{R}$ by $P(s) = (A_{ij}(s))$. Then iterated integrals for matrices of fiberwise spaces over $\mathbb{R}$ can be defined by similar formulas as above.

We define matrices X, Y, Ω of fiberwise spaces over $\mathbb{R}$ by

$$\begin{aligned} X &= (\mathcal{A}_{p_1}(\tilde{\xi}) \quad \mathcal{A}_{p_2}(\tilde{\xi}) \quad \cdots \quad \mathcal{A}_{p_N}(\tilde{\xi})), \quad Y = (\mathcal{D}_{p_1}(\tilde{\xi}) \quad \mathcal{D}_{p_2}(\tilde{\xi}) \quad \cdots \quad \mathcal{D}_{p_N}(\tilde{\xi})), \\ \Omega &= ((1 - \delta_{ij})\mathcal{M}_{p_i p_j}(\tilde{\xi}))_{1 \leq i, j \leq N}. \end{aligned}$$

Then the space of Z-paths in $\widetilde{M}$ is rewritten by means of the iterated integrals as follows.

$$\mathcal{M}_2^Z(\tilde{\xi}) = \mathcal{M}_2(\tilde{\xi}) + \int_{\mathbb{R}} X {}^t Y + \int_{\mathbb{R}} X \Omega {}^t Y + \int_{\mathbb{R}} X \Omega \Omega {}^t Y + \cdots$$

We would like to define the ‘closure’ of this space.

Lemma 2.6. *For a generic $\tilde{\xi}$, the space $\overline{\int_{\mathbb{R}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_n {}^t \overline{Y}}$ is the disjoint union of finitely many manifolds with corners, and the closure of its codimension 1 stratum is given by the following formula.*

$$\begin{aligned} &\overline{\int_{\mathbb{R}} (\partial \overline{X}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_n {}^t \overline{Y}} + \sum_{i=1}^n \overline{\int_{\mathbb{R}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{i-1} (\partial \overline{\Omega}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-i} {}^t \overline{Y}} + \overline{\int_{\mathbb{R}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_n (\partial {}^t \overline{Y})} \\ &+ \overline{\int_{\mathbb{R}} (\overline{X} \times_{\mathbb{R}} \overline{\Omega}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-1} {}^t \overline{Y}} + \sum_{i=1}^{n-1} \overline{\int_{\mathbb{R}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{i-1} (\overline{\Omega} \times_{\mathbb{R}} \overline{\Omega}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-i-1} {}^t \overline{Y}} + \overline{\int_{\mathbb{R}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-1} (\overline{\Omega} \times_{\mathbb{R}} {}^t \overline{Y})}. \end{aligned}$$

For $n \geq 0$, let S_n (resp. T_n) denote the first line (resp. the second line) of the formula in Lemma 2.6.

Lemma 2.7. *There is a natural stratification preserving diffeomorphisms*

$$\begin{aligned} \partial \overline{X} &\cong \overline{X} \times_{\mathbb{R}} \overline{\Omega}, \quad \partial {}^t \overline{Y} \cong \overline{\Omega} \times_{\mathbb{R}} {}^t \overline{Y}, \\ \partial \overline{\Omega} &\cong \overline{\Omega} \times_{\mathbb{R}} \overline{\Omega}, \quad \partial \overline{\mathcal{M}}_2(\tilde{\xi}) \cong \Delta_{\widetilde{M}} + \overline{X} \times_{\mathbb{R}} {}^t \overline{Y}. \end{aligned}$$

These induce, for $n \geq 0$, a stratification preserving diffeomorphism

$$S_n \cong T_{n+1}.$$

Let $S_{-1} \subset \partial \overline{\mathcal{M}}_2(\tilde{\xi})$ be the face that corresponds to $\overline{X} \times_{\mathbb{R}} {}^t \overline{Y}$ by the diffeomorphism of Lemma 2.7.

Definition 2.8.

$$\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi}) = \left[\overline{\mathcal{M}}_2(\tilde{\xi}) + \int_{\mathbb{R}} \overline{X}^t \overline{Y} + \int_{\mathbb{R}} \overline{X} \overline{\Omega}^t \overline{Y} + \int_{\mathbb{R}} \overline{X} \overline{\Omega} \overline{\Omega}^t \overline{Y} + \dots \right] / \sim$$

Here, for each $n \geq 0$, we identify S_{n-1} with T_n by the diffeomorphism of Lemma 2.7. $\mathbb{Z}$ acts on $\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})$ by $(x_1, x_2, \dots, x_n) \mapsto (tx_1, tx_2, \dots, tx_n)$. We put

$$\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})_{\mathbb{Z}} = \overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi}) / \mathbb{Z}.$$

Outline of the proof of Theorem 2.2. By fixing orientations on the manifold pieces in the stratified space $\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})_{\mathbb{Z}}$, the map $\bar{b} : \overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})_{\mathbb{Z}} \rightarrow \widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$ represents a $\mathbb{Q}(t)$ -chain of $\widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$. (The proof that the coefficients are rational functions is an analogue of the proof of the rationality of Novikov complexes by Pajitnov ([14, 15]).) By Lemmas 2.6, 2.7 and by checking the orientations on the gluing parts, it turns out that the boundary of $\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})_{\mathbb{Z}}$ concentrates on the lift $\tilde{\Delta}_M$ of the diagonal Δ_M . Hence the boundary of $B\ell_{\bar{b}^{-1}(\tilde{\Delta}_M)}(\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})_{\mathbb{Z}})$ consists of $\mathbb{Z}$ -paths with endpoints agree (in $\widetilde{M}$) and of closed $\mathbb{Z}$ -paths (in M). One sees that in the homology class of the boundary of $B\ell_{\bar{b}^{-1}(\tilde{\Delta}_M)}(\overline{\mathcal{M}}_2^{\mathbb{Z}}(\tilde{\xi})_{\mathbb{Z}})$, the part for closed $\mathbb{Z}$ -paths corresponds to the logarithmic derivative of the Lefschetz zeta function.

3. PERTURBATION THEORY FOR $\hat{\Lambda}$ -COEFFICIENTS

Put $\Lambda = \mathbb{Q}[t, t^{-1}]$, $\hat{\Lambda} = \mathbb{Q}(t)$. Since a recipe for the perturbation theory for Lie algebra local coefficient systems is given by Axelrod-Singer, Kontsevich ([1, 5]), it is expected that one can obtain a perturbative invariant with $\hat{\Lambda}$ -coefficients if there is an appropriate propagator with $\hat{\Lambda}$ -coefficients. The $\mathbb{Q}(t)$ -chain given by the moduli space of $\mathbb{Z}$ -paths can be considered as an appropriate equivariant propagator and we use it.

3.1. $\hat{\Lambda}$ -colored graph. We call a finite connected graph with edges oriented a graph. A *vertex-orientation* of a graph is an assignment of cyclic order of edges incident to each vertex. For a vertex-oriented graph Γ , a $\hat{\Lambda}$ -coloring of Γ is a mapping $\phi : \text{Edges}(\Gamma) \rightarrow \hat{\Lambda}$.

Definition 3.1 (Garoufalidis-Rozansky [4]).

$$\mathcal{A}_n(\hat{\Lambda}) = \frac{\text{span}_{\mathbb{Q}}\{\Gamma : 3\text{-valent, } 2n \text{ vertices, } \hat{\Lambda}\text{-colored vertex-oriented graphs}\}}{\text{AS, IHX, Orientation reversal, Linearity, Holonomy}}$$

The diagrammatic identities are as follows:

- AS (Axelrod-Singer):** A diagram showing a vertex with three edges, one of which is a loop, equal to the negative of a similar diagram with the loop on a different edge.
- IHX (Ihara-Xavier):** A diagram showing a vertex with three edges, one of which is a loop, equal to the negative of a diagram where the loop is crossed by another edge.
- Orientation Reversal:** A diagram showing a vertical edge with a downward arrow labeled $p(t)$ equal to a vertical edge with an upward arrow labeled $p(t^{-1})$.
- Linearity:** A diagram showing a vertical edge with an upward arrow labeled $p + \alpha q$ equal to the sum of a vertical edge with an upward arrow labeled p and a vertical edge with an upward arrow labeled $\alpha(q)$.
- Holonomy:** A diagram showing a vertex v with three edges labeled p, q, r and indices $(1), (2), (3)$ equal to a similar diagram where the edges are labeled $t^{\pm 1}p, t^{\pm 1}q, t^{\pm 1}r$.

3.2. Equivariant configuration space and equivariant intersection. Let $\kappa : M \rightarrow$

S^1 be a fiber bundle and let $\Theta = 1 \xrightarrow{(1)} \xrightarrow{(2)} 2 \xrightarrow{(3)}$. We define M^Θ as

$$M^\Theta := \{(x_1, x_2; \gamma_1, \gamma_2, \gamma_3) \mid x_1, x_2 \in M, \\ \gamma_i : \text{homotopy class of } c_i : [0, 1] \rightarrow S^1 \text{ such that} \\ c_i(0) = \kappa(x_1), c_i(1) = \kappa(x_2)\}.$$

When $H_1(M) = \mathbb{Z}$, the homotopy class γ_i of c_i is the same thing as the relative bordism class of the lift $\tilde{c}_i : [0, 1] \rightarrow M$ of c_i . The equivariant configuration space $\overline{\text{Conf}}_\Theta(M)$ for Θ is defined by

$$\overline{\text{Conf}}_\Theta(M) := \text{Bl}(M^\Theta, \text{preimage of } \Delta_M),$$

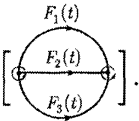
where $\text{Bl}(X, A)$ is the blow-up of a (real) manifold X along a submanifold A . The projection $\overline{\text{Conf}}_\Theta(M) \rightarrow \overline{\text{Conf}}_2(M)$ is a $\mathbb{Z}^3$ -covering and we have $\pi_0(\overline{\text{Conf}}_\Theta(M)) \approx H^1(\Theta; \mathbb{Z}) = [\Theta, S^1]$.

By extending the intersections of chains by $\hat{\Lambda}$ -linearity, we define the multilinear form

$$Q_1 \otimes Q_2 \otimes Q_3 \mapsto \langle Q_1, Q_2, Q_3 \rangle_\Theta \in C_0(\overline{\text{Conf}}_\Theta(M); \mathbb{Q}) \otimes_{\Lambda^{\otimes 3}} \hat{\Lambda}^{\otimes 3}$$

($\Lambda^{\otimes 3} = \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$, $\hat{\Lambda}^{\otimes 3} = \mathbb{Q}(t_1) \otimes_{\mathbb{Q}} \mathbb{Q}(t_2) \otimes_{\mathbb{Q}} \mathbb{Q}(t_3)$) for ‘generic’ 4-dimensional $\hat{\Lambda}$ -chains Q_1, Q_2, Q_3 in $\overline{\text{Conf}}_{K_2}(M)$.

We define the trace $\text{Tr}_\Theta : \hat{\Lambda}^{\otimes 3} \rightarrow \mathcal{A}_1(\hat{\Lambda})$ for $\hat{\Lambda}$ -colored graphs by

$$\text{Tr}_\Theta(F_1(t_1) \otimes F_2(t_2) \otimes F_3(t_3)) = \left[\bigoplus_{\text{graph}} \right]$$


This induces the following map.

$$\text{Tr}_\Theta : H_0(\overline{\text{Conf}}_\Theta(M); \mathbb{Q}) \otimes_{\Lambda^{\otimes 3}} \hat{\Lambda}^{\otimes 3} \rightarrow H_0(\overline{\text{Conf}}_2(M); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{A}_1(\hat{\Lambda}) = \mathcal{A}_1(\hat{\Lambda}).$$

Similarly, for a 3-valent graph Γ with $2n$ vertices ($3n$ edges), we obtain

$$\begin{aligned} & Q_1 \otimes Q_2 \otimes \cdots \otimes Q_{3n} \quad (Q_i: \text{codimension 2 } \hat{\Lambda}\text{-chain in } \overline{\text{Conf}}_{K_2}(M) \text{ or in } M) \\ \mapsto & \langle Q_1, Q_2, \dots, Q_{3n} \rangle_\Gamma \in C_0(\overline{\text{Conf}}_\Gamma(M); \mathbb{Q}) \otimes_{\Lambda^{\otimes 3n}} \hat{\Lambda}^{\otimes 3n} \\ \mapsto & \text{Tr}_\Gamma \langle Q_1, Q_2, \dots, Q_{3n} \rangle_\Gamma \in H_0(\overline{\text{Conf}}_{2n}(M); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{A}_n(\hat{\Lambda}) = \mathcal{A}_n(\hat{\Lambda}). \end{aligned}$$

Definition 3.2. Let $\kappa_i : M \rightarrow S^1$ be an oriented surface bundle such that $\kappa_i \simeq \kappa$. Let $f_i : M \rightarrow \mathbb{R}$ be an oriented ⁴fiberwise Morse function (w.r.t. κ_i), let ξ_i be the gradient for κ_i along the fibers ($i = 1, 2, \dots, 3n$). We define Z_n as follows.

$$Z_n := \sum_{\Gamma} \text{Tr}_\Gamma \langle Q(\tilde{\xi}_1), Q(\tilde{\xi}_2), \dots, Q(\tilde{\xi}_{3n}) \rangle_\Gamma \in \mathcal{A}_n(\hat{\Lambda}).$$

The sum is over all (labeled) 3-valent graphs with $2n$ vertices.

⁴Namely, $\mathcal{D}_p(\xi_i)$ is oriented for each critical locus p .

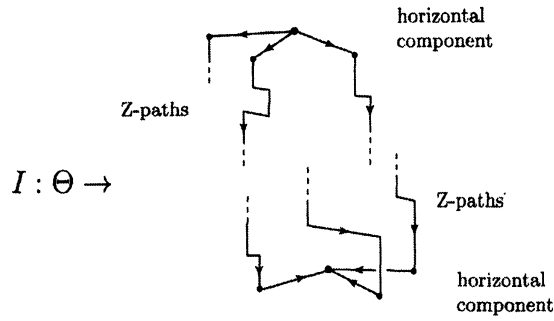


FIGURE 2. Z-graph

Theorem 3.3 ([17]).

$$\widehat{Z}_n = Z_n - Z_n^{\text{anomaly}}(\bar{\rho}_W) \in \mathcal{A}_n(\widehat{\Lambda})$$

is an invariant of $(M, \mathfrak{s}, [\kappa], [f])$. ($Z_n^{\text{anomaly}}(\bar{\rho}_W)$ is a term obtained by counting affine graphs in a rank 3 vector bundle over some compact 4-manifold W such that $\partial W = M$. Here,

- (1) $\mathfrak{s}$ is a spin structure on M .
- (2) $[\kappa] \in H^1(M)$ is the homotopy class of κ .
- (3) $[f]$ is the ‘concordance class’ of an oriented fiberwise Morse function $f : M \rightarrow \mathbb{R}$. (Oriented fiberwise Morse functions f_0 and f_1 are concordant if there is a generic homotopy $F : M \times [0, 1] \rightarrow \mathbb{R}$ between f_0 and f_1 such that for each birth-death locus, its projection to $S^1 \times [0, 1]$ is a simple closed curve and is not nullhomotopic.)

To get an invariant of $(M, [\kappa])$, one must show that $\widehat{Z}_n$ does not depend on the choice of concordance class of oriented fiberwise Morse functions, namely, that $\widehat{Z}_n$ is invariant under a generic homotopy of oriented fiberwise Morse functions. However, as suggested by the definition of concordance, the topology of the moduli space of Z-paths may change if there is a birth-death locus whose projection on $S^1 \times [0, 1]$ is nullhomotopic. We guess that the restriction of the homotopy to concordances might be too strong.

Though, this is sufficient to study finite type isotopy invariants of knots in a 3-manifold ([18]). Thanks to the definition of $\widehat{Z}_n$ by Z-paths, Theorem 3.3 can be proved by a standard argument (constructing a cobordism between moduli spaces on the endpoints) without difficulty.

3.3. Z-graph. Now we explain that Z_n can be defined by counting certain graphs. In the following, we only consider the graph $\Gamma = \Theta$ for simplicity.

Definition 3.4. Put $\Sigma = \kappa^{-1}(0)$. For $(a_1, a_2, a_3) \in \mathbb{Z}^3$, we define

$$\mathcal{M}_{\Theta(a_1, a_2, a_3)}^Z(\Sigma; \xi_1, \xi_2, \xi_3)$$

as the set of maps $I : \Theta \rightarrow M$ such that

- (1) i -th edge is a Z-path for ξ_i .
- (2) $\#(i\text{-th edge of } I) \cap \Sigma = a_i$ (count with signs)

We call such a map $I : \Theta \rightarrow M$ a *Z-graph*.

This definition is an analogue of the flow-graphs considered in Fukaya’s Morse homotopy theory [3]. The following lemma can be proved by a transversality argument as in [3].

Lemma 3.5. For a generic κ_i, ξ_i ($i = 1, 2, 3$), the moduli space $\mathcal{M}_{\Theta(a_1, a_2, a_3)}^{\mathbb{Z}}(\Sigma; \xi_1, \xi_2, \xi_3)$ is a compact oriented 0-dimensional manifold ($\forall (a_1, a_2, a_3) \in \mathbb{Z}^3$)

Proposition 3.6. Choose κ_i, ξ_i ($i = 1, 2, 3$) generically as in the Lemma. Put

$$F_{\Theta} := \sum_{(a_1, a_2, a_3) \in \mathbb{Z}^3} \# \mathcal{M}_{\Theta(a_1, a_2, a_3)}^{\mathbb{Z}}(\Sigma; \xi_1, \xi_2, \xi_3) t_1^{a_1} t_2^{a_2} t_3^{a_3}.$$

Then there exist a polynomial $P(t_1, t_2, t_3) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$ and a product $C(t) \in \Lambda$ of cyclotomic polynomials such that

$$\begin{aligned} F_{\Theta} &= \frac{P(t_1, t_2, t_3)}{C(t_1)C(t_2)C(t_3)\Delta(t_1)\Delta(t_2)\Delta(t_3)} \\ &= \langle Q(\tilde{\xi}_1), Q(\tilde{\xi}_2), Q(\tilde{\xi}_3) \rangle_{\Theta} \end{aligned}$$

holds. ($\Delta(t)$ is the Alexander polynomial of M)

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