

Families of K3 Surfaces in the Smooth Fano 3-folds

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1 Introduction

Varieties assume irreducible, reduced, algebraic and complex.

X : smooth Fano 3-fold, i.e., $-K_X$ is ample.

$B_2(X)$: the second Betti number of X ,

$P(\Delta)$: (projective) toric variety associated to a fan $\Delta \subset \mathbb{R}^3$.

Classification of smooth Fano 3-folds

$B_2(X) \geq 2$: Mori-Mukai [3][4],

$X = P(\Delta)$: Batyrev [1], Watanabe-Watanabe [5].

Family of K3 surfaces

$S \in |-K_X|$: general member. $\rightsquigarrow S$: K3 surface,
 $|-K_X|$: family of K3 surfaces.

Definition 1 The Picard lattice $\text{Pic}(\mathcal{F})$ of a family $\mathcal{F}$ of K3 surfaces is the Picard group of generic member with a cup product.

Definition 2 A surface S is Gorenstein K3 if $K_S \sim 0$, $h^1(S, \mathcal{O}_S) = 0$ and S has at worst RDP (S is birational to a K3 surface).

$\mathcal{F}, \mathcal{F}'$: families of K3 surfaces in the smooth Fano 3-folds.

PROBLEM. Are the families $\mathcal{F}$ and $\mathcal{F}'$ “generically birationally corresponding” if $\text{Pic}(\mathcal{F}) \simeq \text{Pic}(\mathcal{F}')$?

2 Preliminaries Fix $\mathbb{P}^3 \supset H \supset C$ $\begin{smallmatrix} \text{line} \\ \text{plane} \end{smallmatrix}$ irreducible smooth cubic

$X' := \text{Bl}_l(\mathbb{P}^3)$, $X := \text{Bl}_C(\mathbb{P}^3)$.

$\rightsquigarrow \begin{cases} * X' \text{ is toric and } X \text{ is not toric.} \\ * \text{Pic}(|-K_{X'}|) \simeq \text{Pic}(|-K_X|). \\ * X \text{ depends on the blow-up centre.} \end{cases}$

PROBLEM. (special case) Are any Gorenstein K3 surfaces in $|-K_{X'}|$ birational to Gorenstein K3 in $|-K_X|$? (see Theorem)

5 References

- [1] Batyrev, V. V., Toroidal Fano 3-folds, Math. USSR Izvestija **19** (1982), 13–25.
- [2] Mori, S., Mukai, S., Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math., **36** (1981), 147–162.
- [3] Mori, S., Mukai, S., Classification of Fano 3-folds with $B_2 \geq 2$ (Erratum), Manuscripta Math., **110** (2003), 407.
- [4] Watanabe, K., Watanabe, M., The classification of Fano 3-folds with torus embeddings, Tokyo J. Math. **5** (1982), 37–48.

3 Main Result

Families of K3 surfaces in the smooth toric Fano 3-folds

We obtain a partial answer to the Problem:

Proposition The Picard lattices of families of K3 surfaces in the smooth toric Fano 3-folds are mutually distinct.

$\rightsquigarrow$ Families of K3 surfaces in the smooth toric Fano 3-folds are mutually distinct.

Families of K3 surfaces $|-K_{X'}|$ and $|-K_X|$

$\begin{cases} * \mathcal{F}_1 := \{S_1 \in |-K_{\text{Bl}_l(\mathbb{P}^3)}|; S_1 \text{ is Gorenstein K3}\}, \\ * \mathcal{F}_2 := \{S_2 \in \bigcup_{C \subset \mathbb{P}^3} |-K_{\text{Bl}_C(\mathbb{P}^3)}|; S_2 \text{ is Gorenstein K3}\}, \end{cases}$
 where $C \subset \mathbb{P}^3$ in the union run all the elliptic curves.

$\begin{cases} * V := \left\{ (S, H) \right. \\ \left. \in |-K_{\mathbb{P}^3}| \times |\mathcal{O}_{\mathbb{P}^3}(1)| \right\} \left| \begin{array}{l} H \subset \mathbb{P}^3 : \text{plane,} \\ S : \text{Gorenstein K3,} \\ S \cap H = (\text{line}) \cup (\text{smooth cubic}) \end{array} \right\} \end{cases}$

Theorem There exists a correspondence $(V, \mathcal{F}_1, \mathcal{F}_2)$ between the families $\mathcal{F}_1$ and $\mathcal{F}_2$ of K3 surfaces.

The projections $V \rightarrow \mathcal{F}_i$ are surjective.

The period domains of the families are isomorphic.

Sketch of proof. $C' \neq C$: elliptic curve.

Step 1. Any Gorenstein K3 surface $S_1 \in |-K_{\text{Bl}_C(\mathbb{P}^3)}|$ is birational to a Gorenstein K3 $S_2 \in |-K_{\text{Bl}_{C'}(\mathbb{P}^3)}|$.

Step 2. Any Gorenstein K3 surface $S' \in |-K_{X'}|$ is birational to a Gorenstein K3 surface $S \in |-K_X|$.

Step 3. Define R by $S_1 \sim_R S_2$ if $S_1 \in \mathcal{F}_1$ is birational to $S_2 \in \mathcal{F}_2$. The relation R defines a correspondence $(V, \mathcal{F}_1, \mathcal{F}_2)$.

4 Remarks

- (1) By the Theorem, we are expecting to compute the Gromov-Witten invariants of the family $|-K_X|$ of K3 surfaces via the family $|-K_{X'}|$.
- (2) We expect another proof of the Theorem via a “small toric degeneration” of non-toric Fano 3-fold X .
- (3) We may try to construct correspondences among the 88 families of K3 surfaces in the smooth Fano 3-folds.