

The most symmetric non-singular plane curves of degree $n < 8$

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0 Introduction

Throughout this paper k stands for the complex number field $\mathbb{C}$. A homogeneous polynomial $f(x, y, z) \in k[x, y, z]$ defines a plane algebraic curve $f = 0$, or $C(f)$ in the projective plane $\mathbb{P}^2$. A non-singular matrix $A \in GL(3, k)$ defines a projectivity (A) sending a point P with the homogeneous coordinates (x) to a point $(A)P$ with the homogeneous coordinates $(x({}^tA))$. Denote by $PGL(3, k)$ the group of projectivities in $\mathbb{P}^2$. Denote by $\text{Aut}(f)$ the projective automorphism group of f , namely $\text{Aut}(f) = \{(A) \in PGL(3, k); f_A \text{ is proportional to } f\}$, where $f_A(x, y, z) = f((x, y, z)({}^tA^{-1}))$. When $C(f)$ is non-singular and of degree n , i.e. $\deg f = n$, then $C(f)$ is a compact Riemann surface of genus $g = (n-1)(n-2)/2$. In this case we can consider the holomorphic automorphism group $\text{AUT}(f)$ of the Riemann surface $C(f)$. Clearly $\text{Aut}(f)$ is a subgroup of $\text{AUT}(f)$. If $\deg f \geq 4$ and $C(f)$ is non-singular, then $\text{Aut}(f) = \text{AUT}(f)$ [7, p.372], and $|\text{AUT}(f)| \leq 84(g-1)$ [5]. Therefore $|\text{Aut}(f)|$ is bounded above when $C(f)$ runs through non-singular plane curve of degree $n \geq 4$, n being fixed. As will be shown in the next section, the same is true for non-singular plane cubics.

Let an f in $k[x, y, z]$ be homogeneous. We call f singular or non-singular according as the curve $C(f)$ has a singular point or not. A non-singular curve $C(f)$ of degree n ($n \geq 3$) is the most symmetric, if it attains the maximum order of the projective automorphism groups for non-singular plane algebraic curves of degree n ($n \geq 3$). We often identify the polynomial f and the curve $C(f)$.

Our main results are the following Theorems 1, 3, and 5. Theorem 2 is well known [3, pp.348-349].

Theorem 1 *Let f be a non-singular plane cubic.*

- (1) $|\text{Aut}(f)| \leq 54$.
- (2) $|\text{Aut}(f)| = 54$ if and only if f is projectively equivalent to $x^3 + y^3 + z^3$.

Theorem 2 *Let f be a non-singular plane quartic.*

- (1) $|\text{Aut}(f)| \leq 168$.
- (2) $|\text{Aut}(f)| = 168$ if and only if f is projectively equivalent to the Klein quartic $x^3y + y^3z + z^3x$.

Theorem 3 *Let f be a non-singular plane quintic.*

- (1) $|\text{Aut}(f)| \leq 150$.
- (2) $|\text{Aut}(f)| = 150$ if and only if f is projectively equivalent to $x^5 + y^5 + z^5$.

Theorem 4 ([1]) *Let f be a non-singular plane sextic.*

- (1) $|\text{Aut}(f)| \leq 360$.
- (2) $|\text{Aut}(f)| = 360$ if and only if f is projectively equivalent to the Wiman sextic $10x^3y^3 + 9(x^5 + y^5)z - 45x^2y^2z^2 - 135xyz^4 + 27z^6$.

Theorem 5 *Let f be a non-singular plane septic.*

- (1) $|\text{Aut}(f)| \leq 294$.
- (2) $|\text{Aut}(f)| = 294$ if and only if f is projectively equivalent to $x^7 + y^7 + z^7$.

Our definitions and notations are as follows. Let $A, B \in GL(3, k)$, and $f \in k[x_1, x_2, x_3]$. We define $f_A \in k[x_1, x_2, x_3]$ as $f_A(x_1, x_2, x_3) = f([x_1, x_2, x_3]^t A^{-1})$ so that $(f_A)_B = f_{BA}$. Let G be a subset of the group $PGL(3, k)$ of projectivities of the projective plane $\mathbb{P}^2$. A homogeneous $f \in k[x, y, z]$ is called G -invariant, if $f_A \sim f$ for any $(A) \in G$. More generally, let H be an abstract group. By abuse of notation we call f is H -invariant, if there is a subgroup G of $PGL(3, k)$ such that 1) G and H are isomorphic, and 2) f is G -invariant. For a homogeneous $f \in k[x_1, x_2, x_3]$ $\text{Hess}(f)$ denotes the Hessian of f : $\text{Hess}(f) = \det[\frac{\partial^2}{\partial x_i \partial x_j} f]$. It is well known that, if f is non-singular, then the intersection $f \cap h$ coincides with the set of all flexes. It is also known that $\text{Aut}(f) \subset \text{Aut}(\text{Hess}(f))$. Finally $E_3 = [e_1, e_2, e_3]$ denotes the unit matrix of $GL(3, k)$, where e_j stands for the j -th column of E_3 . When two quantities a and b such as functions and matrices, $a \sim b$ means that a and b are proportional.

The cases of cubics, quintics, and septic are discussed in §1, §2, and §3 respectively. Proofs are not given in principle to make our report short.

1 Cubics

In this section we will prove Theorem 1. We begin with

Theorem 1.1 ([8], [6]) *Let $f = x^n + y^n + z^n$ ($n \geq 3$). Then $|\text{Aut}(f)| = 6n^2$.*

Theorem 1.2 *Let f be a non-singular plane cubic.*

- (1) $|\text{Aut}(f)| \leq 54$.
- (2) $|\text{Aut}(f)| = 54$ if and only if f is projectively equivalent to $x^3 + y^3 + z^3$.

Proof. As is known, f has a flex P . Without loss of generality we may assume that $P = (0, 1, 0)$ and that the tangent there is z . Namely $f(x, 1, z) = z + 2z(ax + bz) + Ax^3 + Bx^2z + Cxz^2 + Dz^3$, or equivalently $f = y^2z + 2yz(ax + bz) + Ax^3 + Bx^2z + Cxz^2 + Dz^3$. Substituting y for $y + ax + bz$, we get $f = y^2z + Ax^3 + Bx^2z + Cxz^2 + Dz^3$. So we may assume that $f = y^2z + x^3 + Bx^2z + Cxz^2$. As can be seen easily, f is non-singular if and only if $C(B^2 - 4C) \neq 0$. Let $G_P = \{(A) \in \text{Aut}(f); (A)P = P\}$, and assume $(A) \in G_P$. Since (A) fixes the tangent z at P as well, the rows of A take the form $[a_1, 0, c_1]$, $[a_2, 1, c_2]$, and $[0, 0, c_3]$ respectively up to constant multiplication. Since $f_{A^{-1}}$ contains none of monomials of degree 1 with respect to y , $a_2 = c_2 = 0$. Now $f_{A^{-1}} \sim f$, if and only if $a_1^3/c_3 = 1$, $3a_1^2c_1/c_3 + Ba_1^2 = B$, $3a_1c_1^2/c_3 + 2a_1c_1B + a_1C/c_3 = C$ and $c_1^3/c_3 + c_1^2B + c_1c_3C = 0$. From the first and the second equalities of these four equalities, we get $c_3 = a_1^3$ and $c_1 = a_1(1 - a_1^2)B/3$. So the third equality can be written as $(a_1^4 - 1)(-B^2/3 + C) = 0$. If $C \neq B^2/3$, then $|G_P| \leq 4$. If $C = B^2/3$, then the fourth equality can be written as $(1 - a_1^2)(1 + a_1^2 + a_1^4)B^3 = 0$. Note that $y^2z + x^3$ is singular. Hence, only when $C = B^2/3 \neq 0$, f is non-singular and $|G_P| = 6$. Since $|f \cap h| \leq 9$ by Bezout's theorem,

$$|\text{Aut}(f)|/|G_P| = |\text{Aut}(f)P| \leq 9.$$

So $|\text{Aut}(f)| \leq 54$, and the equality holds, if and only if $|G_P| = 6$ and $|\text{Aut}(f)P| = 9$. We have shown that $|G_P| = 6$ if and only if $C = B^2/3 \neq 0$, namely $f = y^2z + x^3 + Bx^2z + B^2xz^2/3$ with $B \neq 0$, which is projectively equivalent to $f' = y^2z + x^3 + x^2x + xz^2/3$. Consequently, if there exists a non-singular cubic f with $|\text{Aut}(f)| = 54$, then f is projectively equivalent to f' . This means the uniqueness of non-singular cubics satisfying $|\text{Aut}(f)| = 54$. On the other hand there exists such a cubic by Theorem 1.1

2 Quintics

In this section we will specify the most symmetric non-singular quintics (Theorems 2.2 and 2.22).

Theorem 2.1 (Hurwitz) Denote by $\text{AUT}(C)$ the holomorphic automorphism group of a compact Riemann surface C of genus $g \geq 2$. Let $g' = g - 1$. The possible values of the order $d = |\text{AUT}(C)|$ are

$$\begin{array}{cccccccc} 84g', & 48g', & 40g', & 36g', & 30g', & \frac{132}{5}g', & 24g', & \frac{156}{7}g', \\ 21g', & 20g', & \frac{96}{5}g', & \frac{56}{3}g', & \frac{204}{11}g', & 18g', & & \text{or less.} \end{array}$$

Proof. The author of [5] cites values down to $36g'$. For our purposes, however, other possible values are necessary. The idea of the proof given below is entirely due to [5]. According to [5] there exist integers $\hat{g} \geq 0$, $s \geq 3$, and $m_1 \geq m_2 \geq \dots \geq m_s \geq 2$ such that

$$2g' = d\{2(\hat{g} - 1) + \sum_{j=1}^s (1 - \frac{1}{m_j})\}.$$

If $\hat{g} \geq 2$, then $d \leq g'$. If $\hat{g} = 1$, then $d \leq 4g'$. Suppose $\hat{g} = 0$. Note that $2g' \geq d\{-2 + s/2\}$. If $s \geq 5$, then $d \leq 4g'$. If $s = 4$, then $m_1 \geq 3$ so that $2g' \geq d\{-2 + (1 - 1/3) + 3/2\} = d/6$, namely $d \leq 12g'$. Assume $s = 3$.

Suppose $m_3 \geq 4$. Then $2g' \geq d(1 - 3/4) = d/4$, namely $d \leq 8g'$. Suppose $m_3 = 3$. Then $m_1 \geq 4$. If $m_1 \geq 5$, then $2g' \geq d(1 - 1/5 - 1/3 - 1/3) = 2d/15$, namely $d \leq 15g'$. If $m_1 = 4$ and $m_2 = 4$, then $2g' = d(1 - 1/2 - 1/3) = d/6$, namely $d = 12g'$. If $m_1 = 4$ and $m_2 = 3$, then $2g' = d(1 - 1/4 - 2/3) = d/12$, namely $d = 24g'$. Suppose $m_3 = 2$. Then $m_2 \geq 3$. If $m_2 \geq 6$, then $2g' \geq d(1 - 2/6 - 1/2) = d/6$, namely $d \leq 12g'$.

Let $m_2 = 5$. If $m_1 \geq 6$, then $2g' \geq d(1 - 1/6 - 1/5 - 1/2) = 2d/15$, namely $d \leq 15g'$.

If $m_1 = 5$, then $2g' = d(1 - 2/5 - 1/2) = d/10$, namely $d = 20g'$.

Let $m_2 = 4$. Then $m_1 \geq 5$. If $m_1 \geq 8$, then $2g' \geq d(1 - 1/8 - 1/4 - 1/2) = d/8$, namely $d \leq 16g'$.

If $m_1 = 7$, then $2g' = d(1 - 1/7 - 3/4) = 3d/28$, namely $d = 56g'/3$.

If $m_1 = 6$, then $2g' = d(1 - 1/6 - 3/4) = d/12$, namely $d = 24g'$.

If $m_1 = 5$, then $2g' = d(1 - 1/5 - 3/4) = d/20$, namely $d = 40g'$.

Let $m_2 = 3$. Then $m_1 \geq 7$. If $m_1 \geq 19$, then $2g' \geq d(1/6 - 1/19) = 13d/114$, namely $d \leq 228g'/13$.

If $m_1 = 18$, then $2g' = d(1/6 - 1/18) = 2d/18$, namely $d = 18g'$.

If $m_1 = 17$, then $2g' = d(1/6 - 1/17) = 11d/102$, namely $d = 204g'/11g'$.

If $m_1 = 16$, then $2g' = d(1/6 - 1/16) = 5d/48$, namely $d = 96g'/5$.

If $m_1 = 15$, then $2g' = d(1/6 - 1/15) = d/10$, namely $d = 20g'$.

If $m_1 = 14$, then $2g' = d(1/6 - 1/14) = 2d/21$, namely $d = 21g'$.

If $m_1 = 13$, then $2g' = d(1/6 - 1/13) = 7d/78$, namely $d = 156g'/7$.

If $m_1 = 12$, then $2g' = d(1/6 - 1/12) = d/12$, namely $d = 24g'$.

If $m_1 = 11$, then $2g' = d(1/6 - 1/11) = 5d/66$, namely $d = 132g'/5$.

If $m_1 = 10$, then $2g' = d(1/6 - 1/10) = d/15$, namely $d = 30g'$.

If $m_1 = 9$, then $2g' = d(1/6 - 1/9) = d/18$, namely $d = 36g'$.

If $m_1 = 8$, then $2g' = d(1/6 - 1/8) = d/24$, namely $d = 48g'$.

If $m_1 = 7$, then $2g' = d(1/6 - 1/7) = d/42$, namely $d = 84g'$.

Let f be a non-singular plane quintic, hence $C(f)$ is a compact Riemann surface of genus $g = 6$. From now on let $g' = g - 1 = 5$ throughout this section. Then possible values of $|\text{Aut}(f)|$ are

$84g' = 4 \cdot 3 \cdot 5 \cdot 7$, $48g' = 16 \cdot 3 \cdot 5$, $40g' = 8 \cdot 5^2$, $36g' = 4 \cdot 3^2 \cdot 5$, $30g' = 2 \cdot 3 \cdot 5^2$ or less.

We will prove the following theorem by showing that $|\text{Aut}(f)|$ cannot be equal to none of $84g'$, $48g'$, $40g'$, and $36g'$.

Theorem 2.2 *If f is a non-singular plane quintic, then $|\text{Aut}(f)| \leq 150$.*

A proof of this theorem will be given after a series of lemmas and propositions.

Let ε be a primitive n -th root of 1 ($n \geq 3$). A cyclic subgroup G_n of order n in $PGL(3, k)$ is clearly conjugate to either $G_{01} = \langle (\text{diag}[1, 1, \varepsilon]) \rangle$ or $G_{ij} = \langle (\text{diag}[1, \varepsilon^i, \varepsilon^j]) \rangle$

for some $1 \leq i < j \leq n-1$ satisfying the greatest common divisor $(i, j, n) = 1$.

Lemma 2.3 *Let notations be as above. Suppose that $1 \leq i < j \leq n-1$, $1 \leq i' < j' \leq n-1$, and $(i, j, n) = (i', j', n) = 1$. Then G_{ij} is conjugate to $G_{i'j'}$ if and only if there exists an $1 \leq m \leq n-1$ with $(m, n) = 1$ and a permutation $\sigma \in S_3$ such that*

$$\text{diag}[\varepsilon_{\sigma(1)}, \varepsilon_{\sigma(2)}, \varepsilon_{\sigma(3)}] \sim \text{diag}[1, \varepsilon^{i'}, \varepsilon^{j'}],$$

where $[\varepsilon_1, \varepsilon_2, \varepsilon_3] = [1, \varepsilon^{im}, \varepsilon^{jm}]$.

Lemma 2.4 *Let ε be a primitive 7-th root of 1. A subgroup G_7 of $\text{PGL}(3, k)$ is isomorphic to $\mathbb{Z}_7$ if and only if G_7 is conjugate to one of the following subgroups of $\text{PGL}(3, k)$: $G_{01} = \langle (\text{diag}[1, 1, \varepsilon]) \rangle$, $G_{12} = \langle (\text{diag}[1, \varepsilon, \varepsilon^2]) \rangle$, $G_{13} = \langle (\text{diag}[1, \varepsilon, \varepsilon^3]) \rangle$.*

Lemma 2.5 *Let $f_1, \dots, f_n$ be non-zero homogeneous polynomials of the same degree such that $f_{jA} = \lambda_j f_j$ ($j = 1, 2, \dots, n$) for an $A \in \text{GL}(3, k)$ with mutually distinct λ_j . Then a linear combination $f = c_1 f_1 + \dots + c_n f_n \neq 0$ satisfies $f_A = \lambda f$ for some $\lambda \in k$ if and only if $c_j \neq 0$ except for just one value of j .*

The following proposition implies that $|\text{Aut}(f)| = 84g' = 4 \cdot 3 \cdot 5 \cdot 7$ is impossible for any non-singular quintic f .

Proposition 2.6 *A $\mathbb{Z}_7$ -invariant quintic has a singular point.*

Proof. Let ε be a primitive 7-th root of 1, and denote by A_j ($j = 1, 2, 3$) the matrices $\text{diag}[1, 1, \varepsilon]$, $\text{diag}[1, \varepsilon, \varepsilon^2]$ and $\text{diag}[1, \varepsilon, \varepsilon^3]$ respectively. Then a quintic satisfying $f_{A_j^{-1}} = \varepsilon^n f$ for some $0 \leq n \leq 6$ turns out to be singular. Indeed, let $f'(x, y, z)$ be a homogeneous polynomial of degree $d \geq 2$. Then $(1, 0, 0)$ is a singular point of $C(f)$, if and only if none of monomials x^d , $x^{d-1}y$ and $x^{d-1}z$ appears in f' . We summarize the values i such that $m_{A_j^{-1}} = \varepsilon^i m$ for each j and the special nine monomials m in the following table.

	x^5	x^4y	x^4z	y^5	y^4x	y^4z	z^5	z^4x	z^4y
(1)	0	0	1	0	0	1	5	4	4
(2)	0	1	2	5	4	6	3	1	2
(3)	0	1	3	5	4	0	1	5	6

From this table we can easily see that a quintic $C(f)$ satisfying $f_{A_j^{-1}} = \varepsilon^n f$ for some $0 \leq n \leq 6$ has a singular point $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$.

A finite group of order $48g'$ or $40g'$ contains a subgroup of order 8. Such a group is isomorphic to one of the following five groups [4, p.51–52]:

- 1) $\mathbb{Z}_8$
- 2) $\mathbb{Z}_2 \times \mathbb{Z}_4$
- 3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- 4) Q_8 , which is generated by a and b such that $a^4 = 1$, $b^2 = a^2$, and $ba = a^{-1}b$.
- 5) D_8 , which is generated by a and b such that $a^4 = 1$, $b^2 = 1$, and $ba = a^{-1}b$.

We may safely omit the proof of

Lemma 2.7 Let ε be a primitive 8-th root of 1. A subgroup G_8 of $PGL(3, k)$ is isomorphic to $\mathbb{Z}_8$, if and only if G_8 is conjugate to one of the following 4 subgroups of $PGL(3, k)$:

$$\begin{aligned} G_{01} &= \langle (\text{diag}[1, 1, \varepsilon]) \rangle, & G_{12} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^2]) \rangle, \\ G_{13} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^3]) \rangle, & G_{14} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^4]) \rangle. \end{aligned}$$

Proposition 2.8 Let f be a $\mathbb{Z}_8$ -invariant quintic.

- (1) f is non-singular if and only if it is projectively equivalent to $f' = x^5 + Bx^3z^2 + xz^4 + y^4z$ with $B^2 - 4 \neq 0$.
 (2) $|\text{Aut}(f')| \leq 148$.

Lemma 2.9 Let $p \neq 3$ be a prime and ε be a primitive p -th root of 1. Then a subgroup G of $PGL(3, k)$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ if and only if G is conjugate to $G_{p^2} = \langle (\text{diag}[1, \varepsilon, 1]), (\text{diag}[1, 1, \varepsilon]) \rangle$.

The following lemma is due to Hiroaki Taniguchi.

Lemma 2.10 (Taniguchi) Let p be a prime, let ε be a primitive p -th root of 1 and let G_{p^2} be as in Lemma 2.9. If $f(x, y, z)$ is a G_{p^2} -invariant homogeneous polynomial of degree d with $p \nmid d$, then f is reducible.

Proof. Let $A = \text{diag}[1, \varepsilon, 1]$, and $B = \text{diag}[1, 1, \varepsilon]$. Assume $f_A = \varepsilon^i f$ and $f_B = \varepsilon^j f$ for some $i, j \in \{0, 1, \dots, p-1\}$. If $i > 0$, then y divides f . Similarly if $j > 0$, then z divides f . If $i = j = 0$, then x divides f , because f is a linear combination of monomials $x^{d_1}y^{d_2}z^{d_3}$ with $d_2 \equiv d_3 \equiv 0 \pmod{p}$ so that $d_1 = n - d_2 - d_3 \not\equiv 0 \pmod{p}$.

Proposition 2.11 A $\mathbb{Z}_2 \times \mathbb{Z}_4$ -invariant quintic is singular.

Proof. A $\mathbb{Z}_2 \times \mathbb{Z}_4$ -invariant quintic is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -invariant quintic. Such a quintic is reducible by Lemma 2.9 and Lemma 2.10.

Proposition 2.12 No subgroup of $PGL(3, k)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 2.13 Let G_8 be a subgroup of $PGL(3, k)$.

- (1) G_8 is isomorphic to Q_8 if and only if it is conjugate to

$$\langle (\text{diag}[1, \sqrt{-1}, \sqrt{-1}^3]), ([e_1, e_3, e_2]\text{diag}[1, \sqrt{-1}, \sqrt{-1}]) \rangle.$$

- (2) G_8 is isomorphic to D_8 if and only if it is conjugate to

$$\langle (\text{diag}[1, \sqrt{-1}, \sqrt{-1}^3]), ([e_1, e_3, e_2]) \rangle.$$

Proposition 2.14 (1) A Q_8 -invariant quintic, if any, is singular.

(2) A D_8 -invariant quintic, if any, is singular.

A group of order $36g'$ contains a subgroup of order 9 by Sylow's theorem. Such a group is isomorphic to either $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ [4]. By Lemma 2.3 we get

Lemma 2.15 *Let ε be a primitive 9-th root of 1. A subgroup G_9 of $PGL(3, k)$ is isomorphic to Z_9 , if and only if it is conjugate to one of the following three subgroups:*

$$G_{01} = \langle (\text{diag}[1, 1, \varepsilon]) \rangle, \quad G_{12} = \langle (\text{diag}[1, \varepsilon, \varepsilon^2]) \rangle, \quad G_{13} = \langle (\text{diag}[1, \varepsilon, \varepsilon^3]) \rangle.$$

Proposition 2.16 *A Z_9 -invariant quintic is singular.*

Lemma 2.17 *Let ω be a primitive third root of 1. A subgroup G_9 of $PGL(3, k)$ is isomorphic to $Z_3 \times Z_3$ if and only if it is conjugate to one of the following two groups:*

$$G_{01} = \langle (\text{diag}[1, 1, \omega]), (\text{diag}[1, \omega, 1]) \rangle, \quad G_{12} = \langle (\text{diag}[1, \omega, \omega^2]), ([e_2, e_3, e_1]) \rangle.$$

Proposition 2.18 *A $Z_3 \times Z_3$ -invariant quintic is singular.*

Proof of Theorem 2.2 Let f be a non-singular quintic, and let $d = |\text{Aut}(f)|$. Recall that

$$84g' = 4 \cdot 3 \cdot 5 \cdot 7, \quad 48g' = 16 \cdot 3 \cdot 5, \quad 40g' = 8 \cdot 25, \quad 36g' = 4 \cdot 5 \cdot 9.$$

By Proposition 2.6 we get $d \neq 84g'$. The inequalities $d \neq 48g'$, $40g'$ follow from Propositions 2.8, 2.11, 2.12 and 2.14. Finally Propositions 2.16 and 2.18 imply $d \neq 36g'$.

We note that $30g' = 2 \cdot 3 \cdot 25$. A group of order 25 is isomorphic to Z_{25} or $Z_5 \times Z_5$ [4].

Lemma 2.19 *Let ε be a primitive 25-th root of 1. A subgroup G_{25} of $PGL(3, k)$ is isomorphic to Z_{25} if and only if it is conjugate to one of the following subgroups:*

$$\begin{aligned} G_{01} &= \langle (\text{diag}[1, 1, \varepsilon]) \rangle, & G_{12} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^2]) \rangle, & G_{13} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^3]) \rangle, \\ G_{14} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^4]) \rangle, & G_{15} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^5]) \rangle, & G_{1,10} &= \langle (\text{diag}[1, \varepsilon, \varepsilon^{10}]) \rangle. \end{aligned}$$

Proof. By Lemma 2.3 we can classify subgroups $G_{ij} = \langle (\text{diag}[1, \varepsilon^i, \varepsilon^j]) \rangle$ ($1 \leq i < j \leq 24$ with the greatest common divisor $(i, j, 5) = 1$) up to conjugacy, using computer.

Proposition 2.20 *A Z_{25} -invariant quintic is singular.*

Proposition 2.21 *A $Z_5 \times Z_5$ -invariant non-singular quintic is projectively equivalent to $x^5 + y^5 + z^5$.*

Theorem 2.22 *A non-singular quintic f satisfying $|\text{Aut}(f)| = 150$ is projectively equivalent to $x^5 + y^5 + z^5$.*

Proof. Propositions 2.20 and 2.21 imply the theorem.

3 Septics

Let $g = 15$, the genus of non-singular plane septic (i.e. a curve of degree 7), and let $g' = g - 1 = 14$. By Theorem 1.1 $|\text{Aut}(x^7 + y^7 + z^7)| = 21g'$. If f is a non-singular plane septic, then $|\text{Aut}(f)|$ may take values

$$\begin{aligned} 84g' &= 8 \cdot 3 \cdot 49, & 48g' &= 32 \cdot 3 \cdot 7, & 40g' &= 16 \cdot 5 \cdot 7, & 36g' &= 8 \cdot 9 \cdot 7, \\ 30g' &= 4 \cdot 3 \cdot 5 \cdot 7, & 24g' &= 16 \cdot 3 \cdot 7, & \frac{156}{7}g' &= 8 \cdot 3 \cdot 13, & 21g' &= 2 \cdot 3 \cdot 49 \end{aligned}$$

or less by Theorem 2.1. The eight values above are multiples of 8 except for $30g'$ and $21g'$. As we remarked in §2, a group of order 8 is isomorphic to one of the following five groups: $\mathbf{Z}_8$, $\mathbf{Z}_2 \times \mathbf{Z}_4$, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, Q_8 and D_8 . No subgroup of $PGL(3, k)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ by Proposition 2.12. As for a quintic we have following Propositions 3.1 and 3.2

Proposition 3.1 *A $\mathbf{Z}_8$ -invariant septic is singular.*

Proposition 3.2 *A $\mathbf{Z}_2 \times \mathbf{Z}_4$ -invariant septic is singular.*

Proposition 3.3 (1) *A Q_8 -invariant septic, if any, is singular.*

(2) *A D_8 -invariant septic, if any, is singular.*

Theorem 3.4 *The maximum value of $|\text{Aut}(f)|$ for a non-singular septic f is equal to either $30g'$ or $21g'$.*

Proof. By Propositions 3.1, 3.2 and 3.3 the order $|\text{Aut}(f)|$ does not belong to $\{84g', 48g', 40g', 36g', 30g', 24g', \frac{156}{7}g'\} \setminus \{30g'\}$. Meanwhile $|\text{Aut}(x^7 + y^7 + z^7)| = 21g'$ by Theorem 1.1.

We will show that $|\text{Aut}(f)| \neq 30g'$ for any non-singular septic. Note that $30g' = 4 \cdot 3 \cdot 5 \cdot 7$. As we notice in the proof of Proposition 3.2,

Proposition 3.5 *A $\mathbf{Z}_2 \times \mathbf{Z}_2$ -invariant septic is singular.*

Suppose that there exists a non-singular septic f' such that $|\text{Aut}(f')| = 30g'$. Denote by G' the finite group $\text{Aut}(f')$. By Proposition 3.5 Sylow 2-group of G' is isomorphic to $\mathbf{Z}_4$. So we can apply the following theorem to G' .

Theorem 3.6 ([4, p.146]) *If the Sylow subgroups of a finite group G of order n are all cyclic, then it is generated by two elements a and b with defining relations:*

$$\begin{aligned} a^i &= 1, & b^j &= 1, & b^{-1}ab &= a^r, \\ ij &= n, \\ \gcd(i, (r-1)j) &= 1, \\ r^j &\equiv 1 \pmod{i}. \end{aligned}$$

For our group G' of order $420 = 4 \cdot 3 \cdot 5 \cdot 7$, possible pairs of $\{i, j\}$ in Theorem 3.6 are the followings (note that $\gcd(i, j) = 1$ if $r > 1$):

$$\{1, 420\}, \{4, 105\}, \{3, 140\}, \{5, 84\}, \{7, 60\}, \{12, 35\}, \{20, 21\}, \{28, 15\}.$$

In particular G' has an element of order 10, 12 or 15.

Lemma 3.7 *Let ε be a primitive 10-th root of 1. A subgroup G_{10} of $PGL(3, k)$ is isomorphic to Z_{10} if and only if G_{10} is conjugate to one of the following subgroups:*

$$\begin{aligned} &< (\text{diag}[1, 1, \varepsilon]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^2]) >, \\ &< (\text{diag}[1, \varepsilon, \varepsilon^3]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^5]) >. \end{aligned}$$

Proposition 3.8 *A Z_{10} -invariant septic f is singular.*

Lemma 3.9 *Let ε be a primitive 12-th root of 1. A subgroup G_{12} of $PGL(3, k)$ is isomorphic to Z_{12} if and only if G_{12} is conjugate to one of the following subgroups:*

$$\begin{aligned} &< (\text{diag}[1, 1, \varepsilon]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^2]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^3]) >, \\ &< (\text{diag}[1, \varepsilon, \varepsilon^4]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^5]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^6]) >. \end{aligned}$$

Proposition 3.10 *If f is a Z_{12} -invariant non-singular septic, then $|\text{Aut}(f)| \neq 30g' = 420$.*

Lemma 3.11 *Let ε be a primitive 15-th root of 1. A subgroup G_{15} of $PGL(3, k)$ is isomorphic to Z_{15} if and only if it is conjugate to one of the following subgroups:*

$$\begin{aligned} &< (\text{diag}[1, 1, \varepsilon]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^2]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^3]) >, \\ &< (\text{diag}[1, \varepsilon, \varepsilon^4]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^5]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^6]) >. \end{aligned}$$

Proposition 3.12 *A Z_{15} -invariant septic f is singular.*

Theorem 3.13 $|\text{Aut}(f)| \leq 21g' = 294$.

Proof. Propositions 3.8, 3.10, and 3.12 imply that $|\text{Aut}(f)|$ cannot be equal to $30g'$. By Theorem 3.4 we get the desired inequality.

Finally we will show that non-singular septics f with $|\text{Aut}(f)| = 21g' = 2 \cdot 3 \cdot 49$ are unique.

Lemma 3.14 *Let ε be a primitive 49-th root of 1. A subgroup G_{49} of $PGL(3, k)$ is isomorphic to Z_{49} , if and only if it is conjugate to one of the following subgroups:*

$$\begin{aligned} &< (\text{diag}[1, 1, \varepsilon]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^2]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^3]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^4]) >, \\ &< (\text{diag}[1, \varepsilon, \varepsilon^5]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^6]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^7]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^{14}]) >, \\ &< (\text{diag}[1, \varepsilon, \varepsilon^{18}]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^{19}]) >, &< (\text{diag}[1, \varepsilon, \varepsilon^{21}]) >. \end{aligned}$$

Proof. In view of Lemma 2.3 we can classify subgroups $\langle (\text{diag}[1, \varepsilon^i, \varepsilon^j]) \rangle$ ($1 \leq i < j \leq 48$) up to conjugacy, using computer.

Proposition 3.15 *A $\mathbb{Z}_{49}$ -invariant septic f is singular.*

Proposition 3.16 *A $\mathbb{Z}_7 \times \mathbb{Z}_7$ -invariant septic f is non-singular if and only if f is projectively equivalent to $x^7 + y^7 + z^7$.*

Proof. Let $A = \text{diag}[1, 1, \varepsilon]$ and $B = \text{diag}[1, \varepsilon, 1]$. By Lemma 2.9 a subgroup G of $\text{PGL}(3, k)$ is isomorphic to $\mathbb{Z}_7 \times \mathbb{Z}_7$, if and only if G is conjugate to $\langle (A), (B) \rangle$. A septic f satisfying $f_{A^{-1}} = \varepsilon^i f$ and $f_{B^{-1}} = \varepsilon^j f$, if any, is a singular except for the case $i = j = 0$. In the exceptional case f is a linear combination of x^7, y^7 and z^7 .

Theorem 3.17 *A non-singular plane septic f with $|\text{Aut}(f)| = 21g' = 2 \cdot 9 \cdot 2$ is projectively equivalent to $x^7 + y^7 + z^7$.*

Proof. The theorem is a trivial consequence of Propositions 3.15 and 3.16.

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