

A subfamily of complex error functions

Shunsuke MOROSAWA

Department of Mathematics and Information Science,
Faculty of Science, Kochi University
morosawa@math.kochi-u.ac.jp

1 Introduction

A complex error function is a transcendental entire function given by the form

$$C_{a,b}(z) = a \int_0^z e^{-w^2} dw + b$$

with $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. It has two asymptotic values $\pm a\sqrt{\pi}/2 + b$ and has no other singular value. In [3], a subfamily of complex error functions given by the form

$$C_{a,\sqrt{B}}(z) = a \int_0^z e^{-w^2} dw + \sqrt{B}$$

with $a \in \mathbb{R} \setminus \{0\}$ and $B \in \mathbb{R}$ is considered. Hence the family is described by two real parameters. Fatou components of some functions of this family have common boundary curves. In this note, we consider a subfamily of complex error functions given by the form

$$f_a(z) = a \int_0^z e^{-w^2} dw$$

with $a \in \mathbb{C} \setminus \{0\}$. Hence the family is described by one holomorphic parameter. A well-known family of transcendental entire functions with one complex parameter is an exponential family. It is studied by Baker and Rippon [1], Devaney [2] and others.

2 Results

We say that f_a is hyperbolic if the orbit of each asymptotic value accumulates to attracting cyclic points. A connected component of the set of parameters

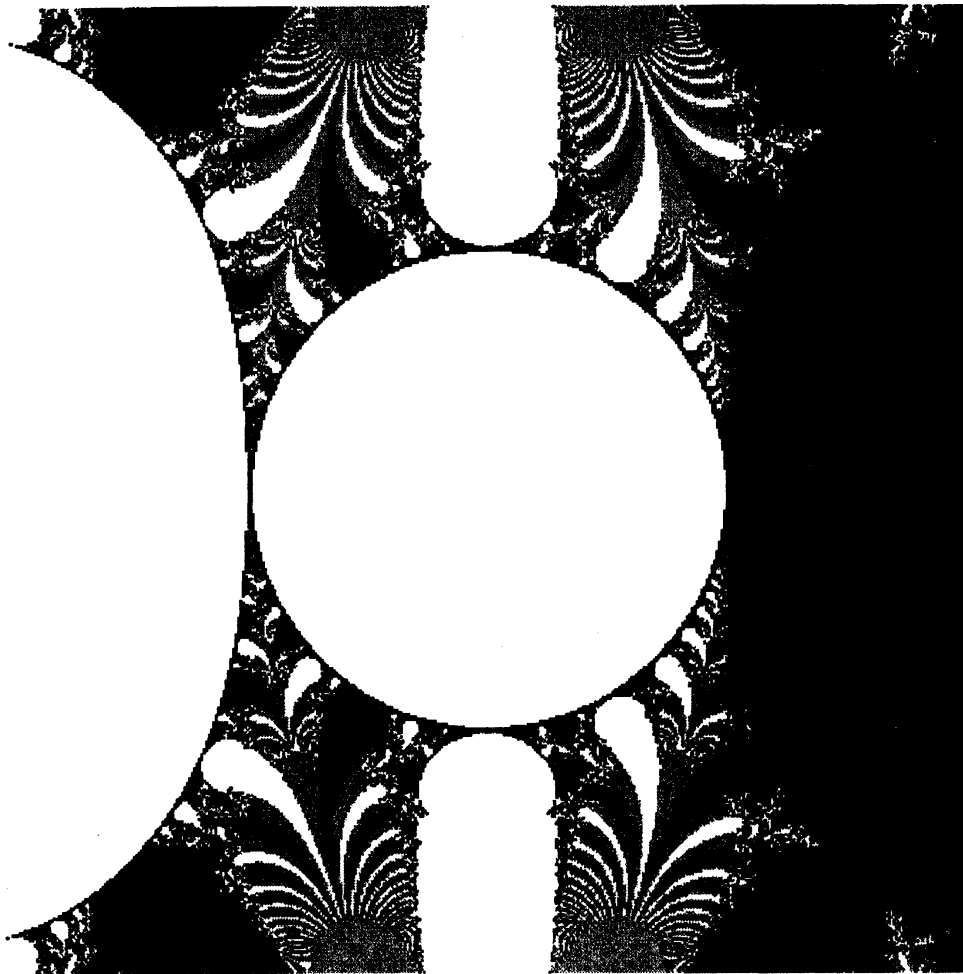


Figure 1: The parameter space of $f_a(z)$. The range shown is $|\Re a| \leq 2$, $|\Im a| \leq 2$. The disk in the center is A . Hyperbolic components of B_n are colored white and those of D_n are colored black.

for which f_a is hyperbolic is called a hyperbolic component. It is known that hyperbolic components are open.

We define subsets in the parameter space of f_a as follows:

$$\begin{aligned} A &= \{a \mid f_a \text{ has a completely invariant component.}\}, \\ B_n &= \{a \mid f_a \text{ has only one attracting cycle with the period } 2n.\}, \\ D_n &= \{a \mid f_a \text{ has two attracting cycles with the period } n.\}, \end{aligned}$$

for $n \in \mathbb{N}$.

If there exists a cycle $\{z_1, z_2, \dots, z_n\}$, then $\{-z_1, -z_2, \dots, -z_n\}$ is also a cycle from the equation

$$f_a(-z) = -f_a(z).$$

Furthermore, we see that if the cycle is attracting, repelling or indifferent, then so is the corresponding one, respectively. The Maclaurin expansion of

$Er(z) = f_1(z)$ is of the form

$$Er(z) = \int_0^z e^{-w^2} dw = z - \frac{z^3}{3} + \dots$$

Adding further investigation on properties of $Er(z)$, we have the following theorem.

Theorem 1. *Every hyperbolic component is contained in one of A , B_n and D_n . Furthermore, A is also described by $\{a \mid 0 < |a| < 1\}$. Each of B_1 and D_1 consists of only one component.*

By the arguments similar to those in [1], we have the following theorems.

Theorem 2. *Every hyperbolic component except A is simply-connected and unbounded.*

Theorem 3. *Each of B_n and D_n contains a component which is tangent to A .*

Cyclic Fatou components of the function belonging to a hyperbolic component tangent to A attach to each other at the origin. By the arguments similar to those in [3], we have the following theorem.

Theorem 4. *Fatou components of f_a belonging to a hyperbolic component tangent to A have common boundary curves.*

References

- [1] I. N. Baker and P. J. Rippon, Iteration of exponential functions, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, 9(1984), 47–77.
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- [3] S. Morosawa, Fatou components whose boundaries have a common curve, *Fund. Math.* 183(2004), 47–57.

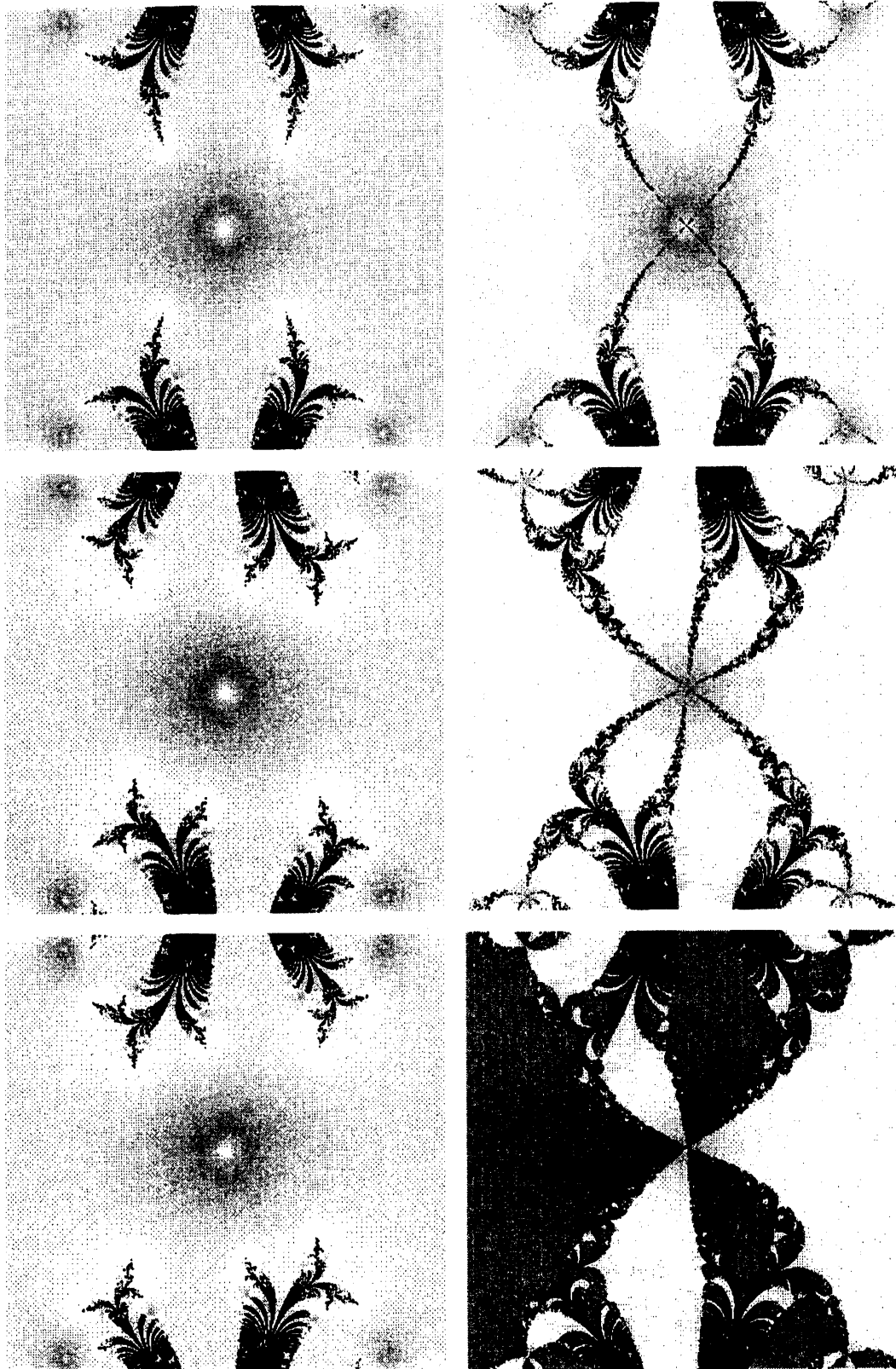


Figure 2: The Julia sets of $f_a(z)$. The range shown is $|\Re z| \leq 2$, $|\Im z| \leq 2$. Upper left: $a = 0.95i$. Upper right: $a = 1.05i$. Middle left: $a = 0.475 + 0.8227241i$. Middle right: $a = 0.55 + 0.952628i$. Lower left: $a = -0.475 + 0.8227241i$. Lower right: $a = -0.55 + 0.952628i$.