

MULTIPLIER IDEAL SHEAVES AND INTEGRAL INVARIANTS

AKITO FUTAKI
TOKYO INSTITUTE OF TECHNOLOGY
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1. INTRODUCTION

This talk is based on joint a work with Yuji Sano.

Let M be a compact complex manifold with $c_1(M) > 0$, i.e. a Fano manifold, with $\dim M = m$.

The first Chern class $c_1(M)$ is represented as a de Rham class by a closed positive $(1, 1)$ -form

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^m g_{ij} dz^i \wedge d\bar{z}^j,$$

with (g_{ij}) a positive definite Hermitian matrix.

It is well known, or by definition, that

$$d\omega = 0 \iff \omega \text{ is a Kähler form.}$$

We regard $c_1(M)$ as a Kähler class (the space of Kähler forms).

On the other hand, by the theory of characteristic classes (Chern-Weil Theory), $c_1(M)$ is represented by a **Ricci form**

$$\text{Ric}_\omega := -\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det(g_{ij})$$

and its coefficient

$$R_{ij} := -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij})$$

is called the **Ricci curvature**.

DEF : ω is called a **Kähler-Einstein metric** if

$$\text{Ric}_\omega = \omega$$

or equivalently

$$R_{ij} = g_{ij}.$$

But in general $\text{Ric}_\omega \neq \omega$, and we have for some smooth function h

$$\text{Ric}_\omega = \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial} h.$$

Problem : Find another $\tilde{\omega}$ such that

$$\text{Ric}_{\tilde{\omega}} = \tilde{\omega}.$$

If we put

$$\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \varphi,$$

the Einstein equation

$$\text{Ric}_{\tilde{\omega}} = \tilde{\omega}$$

is equivalent to the complex Monge-Ampère equation

$$\frac{\det(g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j})}{\det(g_{ij})} = e^{-\varphi + h}.$$

Thus, starting from arbitrary $\omega \in c_1(M)$, finding a Kähler-Einstein metric with $\tilde{\omega} \in c_1(M)$ is reduced to solving the non-linear PDE

$$\frac{\det(g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j})}{\det(g_{ij})} = e^{-\varphi + h}.$$

Conjecture (Yau-Tian-Donaldson) :

The existence of a Kähler-Einstein metric will be equivalent to GIT stability (K-polystability).

2. OBSTRUCTIONS

On the one hand there are **obstructions** to $\exists$ of K-E metrics by Matsushima, the speaker, Bando-Mabuchi, Chen-Tian-Donaldson-Stoppa-Mabuchi, ... as below.

Matsushima (1956) : If M admits a K-E metric then the Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector fields is reductive.

Futaki (1983) : $\exists$ Lie algebra character $f : \mathfrak{h}(M) \rightarrow \mathbb{C}$ such that if $\exists$ K-E metric then $f = 0$. This f is called the so-called “Futaki invariant”, and the precise definition will be given below.

Bando-Mabuchi (1987) : K -energy is bounded from below.

Chen-Tian, Donaldson, Stoppa, Mabuchi, ... : Existence of K-E $\implies$ K-stability.

The definition of K-stability is roughly stated as follows.

Definition

M is K-stable. $\iff$

For all $\mathbb{C}^*$ -equivariant degenerations (test configurations) of M , the central fiber has positive Donaldson's Futaki invariant. (The minus of the Futaki invariant is the invariant used as the analogy to the numerical criterion of GIT.)

M is K-polystable. $\iff$

For all $\mathbb{C}^*$ -equivariant degenerations (test configurations) of M , the central fiber has non-negative Futaki invariant, and the equality occurs only when the test configuration is a product $M \times \mathbb{C}$ with non-trivial $\mathbb{C}^*$ -action on M . (In this case Futaki invariant necessarily vanishes because we may also consider the opposite $\mathbb{C}^*$ -action.)

Definition of f : Recall that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

$$\rho_\omega = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det(g_{i\bar{j}}),$$

and

$$\rho_\omega - \omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} h, \quad h \in C^\infty(M).$$

Then f is defined by

$$f(X) = \int_M X h \omega^m$$

for $X \in \mathfrak{h}(M)$.

Theorem (1) f is independent of $\omega \in c_1(M)$.
 (2) $f \neq 0$ implies nonexistence of KE metric.

The definition of f was reformulated by Donaldson only using algebraic geometry in a way that can be applied to schemes. But I will not go into the detail here.

3. KNOWN EXISTENCE RESULTS

So far, I talked about obstructions. Next, I turn to **Existence Results** of K-E metrics, due to Siu, Tian, Nadel and their variants.

Siu (1988) : Enough symmetries $\implies \exists$ K-E metric .

Tian (1987) : $\alpha(M) > \frac{m}{m+1} \implies \exists$ K-E metric.

Nadel (1988) :

$\nexists$ of K-E metric $\implies \exists$ of proper multiplier ideal sheaf.

i.e. $\nexists$ of proper multiplier ideal sheaf $\implies \exists$ of K-E metric.

Demailly-Kollàr(2001) :

Simplification of Nadel's arguments, applications to orbifolds.

Boyer-Galicki, Kollàr :

Applications to Sasaki-Einstein metrics.

Demailly-Kollàr version of multiplier ideal sheaves

Let ψ be an ω_g -plurisubharmonic function, i.e., a real-valued upper semi-continuous function satisfying $\omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi \geq 0$ in the current sense. The **multiplier ideal sheaf with respect to ψ** is the ideal sheaf defined by the following presheaf

$$(U, \mathcal{I}(\psi)) = \{f \in \mathcal{O}(U) \mid \int_U |f|^2 e^{-\psi} dV < \infty\}$$

where U is an open subset of M .

To prove the existence of KE metric, we consider the family of Monge-Ampère equations

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-t\varphi + h}$$

for $t \in [0, 1]$.

If there is no KE metric, there exists $t_0 < 1$ such that $\{\varphi_t\}$ is the solution $\{\varphi_t\}_{0 \leq t < t_0}$ and that

$$\inf_M (\varphi_t - \sup_M \varphi_t) \rightarrow -\infty$$

as $t \rightarrow t_0$. Note that solutions exist on open set of t 's by Banach space implicit function theorem.

This is because of

Theorem(Yau)

If $\{\varphi_t\}$ is bounded in C^0 then $\{\varphi_t\}$ is bounded in C^3 .

Thus if $\{\varphi_t\}$ is bounded in C^0 then $\{\varphi_t\}$ is uniformly bounded and equi-continuous up to second order derivatives.

Thus by Ascoli-Arzelà, a suitable subsequence $\{\varphi_t\}$ converges to the solution φ_{t_0} of the Monge-Ampère equation for $t = t_0$. Then the set of t 's such that a solution φ_t exists is a non-empty open and closed subset of t 's. Thus we have a solution for $t = 1$. This is a contradiction because we assume there is no KE metric.

Therefore we must have

$$\inf_M (\varphi_t - \sup_M \varphi_t) \rightarrow -\infty$$

as $t \rightarrow t_0$.

Let M be a Fano manifold of dimension m .

Let G be a compact subgroup of $\text{Aut}(M)$.

Assume that M does not have a G -invariant Kähler-Einstein metric.

Let $\epsilon \in (m/(m+1), 1)$. This number $(m/(m+1))$ arises from an analytic inequality for Monge-Ampère equations, called the Harnack inequality. This is too much to talk about here, and the audience should take it granted as something necessary from PDE theory.

Then there exists a sequence $\{\varphi_{t_k}\}_{k=1}^\infty$ such that

$t_k \rightarrow t_0$ as $k \rightarrow \infty$,

there exists $\varphi_\infty = \lim_{k \rightarrow \infty} (\varphi_{t_k} - \sup_M \varphi_{t_k})$ in L^1 -topology, which is an ω_g -psh function, and

$\mathcal{I}(\varphi_\infty)$ is a proper multiplier ideal sheaf, i.e, $\mathcal{I}(\varphi_\infty)$ is neither 0 nor $\mathcal{O}_M$.

4. THE RELATION BETWEEN THE MIS AND THE INVARIANT f

Now I turn to the question I want to raise in **This talk** :

What is the relation between the MIS and the invariant f ?

There has been an answer to this question by Nadel stated as

Theorem (Nadel, 1995)

Suppose M does not admit a K-E metric, and let V be the support of the MIS.

For $v \in \mathfrak{h}(M)$ with $f(v) = 0$ we have

$$V \not\subset \text{Zero}^+(v) := \{p \in \text{Zero}(v) \mid \Re((\text{div}(v))(p)) > 0\}.$$

Here $\text{div}(v)\text{vol}_g = \mathcal{L}_v \text{vol}_g$. Notice that $\text{div}(v)$ is independent of the choice of g along $\text{Zero}(v)$.

We extend this in several ways.

to get some more informations on Fano manifolds,

to show the existence of MIS for Kähler-Ricci solitons,

to study the MIS arising from the non-convergence of Kähler-Ricci flow and study the relation between MIS and f .

So, we study three types of MIS.

KE-MIS : due to Nadel, arising from the failure of solving Monge-Ampère equations for **Kähler-Einstein** metrics by continuity method.

KRS-MIS : Arising from the failure of solving Monge-Ampère equations for **Kähler-Ricci solitons** by continuity method.

KRF-MIS : Arising from the failure of convergence of **Kähler-Ricci flow**.

Let M be a Fano manifold,

G be a compact subgroup of $\text{Aut}(M)$,

T^r maximal torus of G .

For any G -invariant Kähler metric g with

$$\omega_g := \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \in c_1(M)$$

consider the Hamiltonian T^r -action with the moment map $\mu_g : M \rightarrow \mathfrak{t}^r$.

For $\xi \in \mathfrak{t}^r$ we put

$$D^{\leq 0}(\xi) := \{y \in \mu(M) \mid \langle y, \xi \rangle \leq 0\}.$$

Theorem (Futaki-Sano)

Suppose M does not admit a K-E metric, and let V be the support of the KE-MIS. Let $\xi \in \mathfrak{t}^r \subset \mathfrak{h}(M)$ satisfy $f(v_\xi) > 0$ where v_ξ is the holomorphic vector field corresponding to ξ . Then

$$\mu_g(V) \not\subset D^{\leq 0}(\xi)$$

for any G -invariant Kähler metric g in $c_1(M)$.

Corollary

Let M be the one-point blow-up of $\mathbb{CP}^2$. Then V is the exceptional divisor. This V destabilizes slope stability in the sense of Ross-Thomas.

Definition (Slope stability w.r.t. $V \subset M$):

Put $\mathcal{M} = \text{blow up of } M \times \mathbb{C} \text{ along } V \times 0$.

The central fiber is regarded as a degeneration of M .

Compute Donaldson's algebraic reformulation of f .

If it has the right sign for any V , then M is said to be **slope stable**.

Outline of Proof of Theorem 2

Let $h \in C^\infty(M)$ satisfy $\text{Ric}_g - \omega_g = i\partial\bar{\partial}h$.

Suppose

$$\frac{\det(g_{ij} + \varphi_{ij})}{\det(g_{ij})} = e^{-t\varphi + h}$$

has solutions only for $t \in [0, t_0)$, $t_0 < 1$.

Then we have a MIS with support V .

Fact 1 : (Nadel, based on earlier estimates by Siu and Tian)

Let $K \subset M - V$ be a compact subset. Then

$$\int_K \omega_{g_t}^m \rightarrow 0$$

as $t \rightarrow t_0$.

Fact 2 :

$$\mu_g(p) \in D^{\leq 0}(\xi) \iff (\text{div}(v_\xi))(p) \geq 0$$

where

$$\text{div}(v_\xi)(e^h \omega^m) = \mathcal{L}_{v_\xi}(e^h \omega^m).$$

Fact 3 :

$$\frac{t}{t-1} f(v_\xi) = \int_M \text{div}(v_\xi) \omega_t^m.$$

By Fact 3 and our assumption $f(v_\xi) > 0$, we have for $t \in (\delta, t_0)$ with $t_0 < 1$

$$\int_M \text{div}(v_\xi) \omega_t^m = \frac{t}{t-1} f(v_\xi) < -C$$

with $C > 0$ independent of t . Suppose $\mu_g(V) \subset D^{\leq 0}(\xi) = \{\text{div}(v_\xi) \geq 0\}$.

Choose $\epsilon > 0$ small and put

$$W := \{p \in M | \text{div}(v_\xi)(p) \leq -\epsilon\}.$$

Then $W \subset M - V$ compact. Apply Fact 1 to W to get

$$\int_{W_\epsilon} \omega_{g_t}^m \rightarrow 0$$

as $t \rightarrow t_0$.

But then

$$\begin{aligned} -C \geq \int_M \operatorname{div}(v_\xi) \omega_t^m &= \int_M \operatorname{div}(v_\xi) \omega_t^m + \int_{W_\epsilon} \operatorname{div}(v_\xi) \omega_t^m \\ &\geq -2 \operatorname{vol}(M, g) \end{aligned}$$

as $t \rightarrow t_0$, a contradiction ! This completes the proof.

KRS-MIS

Let M be a Fano manifold. Choose $\omega_g \in c_1(M)$. Let $v \in \mathfrak{h}_r(M)$ be in the reductive part $\mathfrak{h}_r(M)$ of $\mathfrak{h}(M)$.

Definition : We say (g, v) is a Kähler-Ricci soliton
 $\iff \operatorname{Ric}(\omega_g) - \omega_g = \mathcal{L}_v(\omega_g)$.

Then $\Im(v)$ is necessarily Killing.

Start with an initial metric g^0 with $\omega_0 := \omega_{g^0} \in c_1(M)$. Let h_0 and $\varphi_{v,0}$ be the smooth functions such that

$$\begin{aligned} \operatorname{Ric}(\omega_0) - \omega_0 &= i\partial\bar{\partial}h_0, \quad \int_M e^{h_0} \omega_0^m = \int_M \omega_0^m, \\ i_v \omega_0 &= i\bar{\partial} \varphi_{v,0}, \quad \int_M e^{\theta_{v,0}} \omega_0^m = \int_M \omega_0^m. \end{aligned}$$

Consider for $t \in [0, 1]$

$$\det(g_{ij}^0 + \varphi_{tij}) = \det(g_{ij}^0) e^{h_0 + \theta_{v,0} + \varphi_{v,t} - t\varphi_t}.$$

The solution for $t = 1$ gives the Kähler-Ricci soliton.

Zhu has shown that $t = 0$ always has a solution.

Implicit function theorem shows for some $\epsilon > 0$, all $t \in [0, \epsilon)$ have a solution.

Suppose we only have solutions on $[0, t_\infty)$, $t_\infty < 1$.

Let $\varphi_{v,g}$ satisfy

$$i_v \omega_g = i\bar{\partial} \varphi_{v,g}, \quad \int_M e^{\theta_{v,g}} \omega_g^m = \int_M \omega_g^m.$$

Definition :

$$f_v(w) = \int_M w(h_g - \varphi_{v,g}) e^{\theta_{v,g}} \omega_g^m$$

This f_v is independent of g with $\omega_g \in c_1(M)$.

Theorem (Tian-Zhu) There exists a unique $v \in \mathfrak{h}_r(M)$ such that

$$f_v(w) = 0 \text{ for all } w \in \mathfrak{h}_r(M).$$

Theorem (F-Sano)

Let K be the compact subgroup such that $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{h}_r(M)$. Suppose there is no KRS. Then we get MIS and its support V_v satisfies

$$V_v \not\subset Z^+(w) \text{ for } \forall w \in \mathfrak{h}_r(M).$$

We can apply this to prove the existence of KRS on the one point blow-up of $\mathbb{CP}^2$.

KRF-MIS

Theorem (Phong-Sesum-Sturm)

One gets MIS from the failure of convergence of normalized Kähler-Ricci flow:

$$\frac{\partial g}{\partial t} = -\text{Ric}(g) + g.$$

If we put $g_{tij} = g_{ij} + \varphi_{tij}$ the Ricci flow is equivalent to

$$\begin{aligned} \frac{\partial \varphi_t}{\partial t} &= \log \frac{\det(g_{ij} + \varphi_{tij})}{\det(g_{ij})} + \varphi_t - h_0 \\ \varphi_0 &= c_0 \end{aligned}$$

Yanir Rubinstein modified Phong-Sesum-Sturm's MIS using the idea of Demailly-Kollár:

$$\varphi_t - \int_M \varphi_t \omega^m \longrightarrow \varphi_\infty \quad \text{almost psh}$$

as $t \rightarrow \infty$.

Let V_γ be the MIS for $\psi = \varphi_\infty$, $\gamma \in (\frac{m}{m+1}, 1)$, defined by

$$(U, \mathcal{I}(\psi)) = \{ f \in \mathcal{O}_M(U) \mid \int_U |f|^2 e^{-\psi} \omega_g^m < \infty \}.$$

This MIS satisfies

$$H^q(M, \mathcal{I}(\psi)) = 0 \quad \text{for } \forall q > 0.$$

Yuji Sano's work:

Let M be a toric Fano manifold, and put

$$\begin{aligned} T_{\mathbb{R}} &= T^m, & T_{\mathbb{C}} &= (\mathbb{C}^*)^m, \\ N_{\mathbb{R}} &= J\mathfrak{t}_{\mathbb{R}}. \end{aligned}$$

Let $W(M) = N(T_{\mathbb{C}})/T_{\mathbb{C}}$ be the Weyl group.

Theorem (Wang-Zhu) There exists a KRS (g_{KRS}, v_{KRS}) .

Theorem (Sano) Suppose $\dim N_{\mathbb{R}}^{W(M)} = 1$. Let $t = \exp(tv_{KRS})$, $0 < t < 1$ and ω a $T_{\mathbb{R}}$ -invariant Kähler form. Then the support of Rubinstein's KRF-MIS of exponent t is equal to the support of the MIS of exponent t obtained from the Kähler potentials of $\{(1-t)\omega\}$.

Using this Sano computed the support of KRF-MIS for various t on some examples.