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Vojta's Conjecture and Dynamics

By

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Abstract

Vojta's conjecture is a deep conjecture in Diophantine geometry, giving a quantitative description of how the geometry of a variety controls the Diophantine approximation of its rational points. This paper discusses some consequences of the conjecture to arithmetic dynamics on $\mathbb{P}^n$, in particular on finiteness of integer points in an orbit of a point under a morphism. Some unconditional results without assuming the conjecture are also mentioned.

§ 1. Introduction

In [12], Vojta formulated a precise dictionary between value distribution theory and Diophantine approximations and used it to translate Griffiths' conjecture in complex analysis to a powerful height inequality [12, Main Conjecture 3.4.3] for rational points of smooth varieties. Vojta's conjecture is a quantitative attempt at how the geometry controls the arithmetic, and it is very deep: its special cases include Schmidt's subspace theorem (Schmidt [8] and Schlickewei [7]) and two of Faltings' well-known theorems (Mordell's conjecture [2] for curves and Lang's conjecture [3] for abelian varieties). The conjecture also implies other profound conjectures, including the *abc* conjecture and the Bombieri–Lang conjecture. This article discusses the consequences of the conjecture to arithmetic dynamics.

The arithmetic dynamics studies arithmetic properties of iteration of self-maps. That is, we let $\phi : X \longrightarrow X$ where X is an algebraic variety defined over a number field, and study n-fold composition $\phi^{(n)} = \phi \circ \cdot \circ \phi$. For example, if we define the *orbit* of P under ϕ to be $\mathcal{O}_\phi(P) = \{P, \phi(P), \phi^{(1)}(P), \ldots\}$, then one could ask how many points

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defined over $\mathbb{Q}$ have finite orbits. This is a highly active field, but just as in complex dynamics, most of the research has focused on maps on $\mathbb{P}^1$. The central result of this article is on finiteness of integers in an orbit of a rational point under an endomorphism of $\mathbb{P}^N$, assuming Vojta's conjecture.

Theorem 1.1. *Let ϕ be an endomorphism on $\mathbb{P}^N$ defined over $\mathbb{Q}$ of degree d, namely $\phi^*\mathscr{O}(1) \cong \mathscr{O}(d)$. Fix a coordinate system on $\mathbb{P}^N$ and let H be the divisor where $X_0 = 0$. Given a point $P \in \mathbb{P}^N(\mathbb{Q})$, write $P = [a_0 : \cdots : a_N]$ with $a_i \in \mathbb{Z}$ having no common divisors. Similarly, we write $\phi^{(m)}(P) = [a_0^{(m)} : \cdots : a_N^{(m)}]$. Assuming Vojta's conjecture, we have the following:*

(a) *If there exists an n with $d^n > N+1$ such that $\left(\phi^{(n)}\right)^*(H)$ is a normal crossings divisor, and if $P \in \mathbb{P}^N(\mathbb{Q})$ is such that any infinite subset of the orbit $\mathcal{O}_\phi(P)$ is Zariski dense, then the orbit of P only contains finitely many integer points $\mathbb{Z}(\mathbb{P}^N \backslash H)$, that is, only finitely many m satisfies $a_0^{(m)} = \pm 1$.*

(b) *If the same condition holds for all n, then for P satisfying the property that any infinite subset of the orbit is dense, we have*

$$\lim_{m \longrightarrow \infty} \frac{\log \left| a_0^{(m)} \right|}{\log \max_i \left| a_i^{(m)} \right|} = 1$$

We will explain the necessary background for this theorem in the following section. As will be discussed there, an analogous theorem has already been proved by Silverman [10] (Theorem 2.4 below) on $\mathbb{P}^1$, without assuming Vojta's conjecture. Section 3 contains the proof of Theorem 1.1, together with some examples and additional remarks. The last section is devoted to special cases for which results like Theorem 1.1 can be proved without assuming Vojta's conjecture.

§ 2. Preliminaries

Vojta's conjecture is an inequality of local and global height functions, so we first introduce heights. For more details, see [1, 5, 9].

Let k be a number field, M_k be the set of places of k, and for each $v \in M_k$, we let $|\cdot|_v$ to be the representative whose restriction to $\mathbb{Q}$ is the $\frac{[k_v:\mathbb{Q}_v]}{[k:\mathbb{Q}]}$-th power of the normalized absolute value. We will often use the additive notation, $v(x) = -\log |x|_v$ for $x \neq 0$. For a Cartier divisor D on a projective variety X defined over k, we can define a global (Weil) height function $h_D : X(\overline{k}) \longrightarrow \mathbb{R}$. The global height function is geometric in the sense that for two linearly equivalent divisors D_1 and D_2, the heights h_{D_1} and h_{D_2} differ by a bounded function on $X(\overline{k})$.

The local height function $\lambda_D(-,v)$ on the other hand, is more arithmetic in nature and it is the v-adic part of the global height function. More specifically, it is a real-valued function on $X(\overline{k})\backslash|D|$ such that $\sum_{v\in M_k}\lambda_D(P,v) = h_D(P)$ up to a bounded function when $P\notin|D|$. When D is locally defined by a function f, the local height $\lambda_D(P,v)$ is roughly $-\log|f(P)|_v$. Note that this blows up when P is on $|D|$, and it is "big" when the point P is v-adically "close" to the divisor D. One can make everything precise by endowing the line bundle $\mathscr{L}(D)$ corresponding to the divisor D a locally M_k-bounded metric, and then defining the local height function as the absolute value of a meromorphic section.

In this article, we will only need heights on projective space $\mathbb{P}^N$. The global height function h_D when D has degree d is

$$h_D([x_0:\cdots:x_N]) = -d\sum_{v\in M_k}\min[v(x_0),\ldots,v(x_N)]$$

up to a bounded function. Since the global height is invariant under linear equivalence, this only depends on the degree of the divisor. A local height function for the divisor $D=(X_0)$ is given by

$$\lambda_{(X_0)}([x_0:\cdots:x_N],\,v) \;=\; v(x_0)-\min[v(x_0),\ldots,v(x_N)],$$

and more generally for the divisor defined by a homogeneous polynomial f of degree d,

$$\lambda_{(f)}([x_0:\cdots:x_N],\,v) \;=\; v(f(x_0,\ldots,x_N)) \;-\; d\min[v(x_0),\ldots,v(x_N)]. \tag{2.1}$$

This greatly depends on v and the divisor. For example, on $\mathbb{P}^1$,

$$\lambda_{[0:1]}([2^n:1],\,v_2) = n\log 2 \qquad\text{but}\qquad \lambda_{[1:1]}([2^n:1],\,v_2)=0.$$

Vojta's conjecture is an inequality of global and local heights, and so this can be interpreted as a way geometric information contained in global heights controls the arithmetic information contained in local heights. Before stating the conjecture, let us recall that a normal crossings divisor D on a smooth variety is a divisor such that at each point P, D is defined by $f_1\cdots f_l$ where $(f_1,\ldots,f_l)$ form a regular sequence at P. In particular, a normal crossings divisor is by definition effective and has no multiple components.

Vojta's Conjecture. *Let X be a smooth projective variety over a number field k, K a canonical divisor of X, A an ample divisor and D a normal crossings divisor defined over k. Fix height functions $\lambda_D(-,v)$, h_K, and h_A. Let S be a finite subset of M_k. Then given $\epsilon>0$, there exists a Zariski-closed $Z_\epsilon\subsetneq X$ and a bounded function $O(1)$ such that*

$$\sum_{v\in S}\lambda_D(P,v)+h_K(P) < \epsilon h_A(P)+O(1) \tag{2.2}$$

for $P \in X(k)$ not on Z_ϵ.

Because of ϵ, the inequality (2.2) essentially says that the left hand side cannot be too big. As mentioned earlier, a local height $\lambda_D(P, v)$ is big when the point P is v-adically close to D. Since the canonical divisor and its global height h_K is geometric (it does not depend on the field k or S), the conjecture says that how close a point P can be to D in the v-adic sense is controlled by the geometry of X. In this sense, Vojta's conjecture is a Diophantine approximation with inputs from geometry, as shown in the fundamental example below.

Example 2.1. When $X = \mathbb{P}^1$, let α be a real algebraic number not in $\mathbb{Q}$, and let f be its minimal polynomial over $\mathbb{Q}$. We can take $K = -2H$ and $A = H$ for a hyperplane H, so for $k = \mathbb{Q}$ and $S = \{\infty\}$, the conjecture says that

$$-\log|f(p/q)| < (2+\epsilon)\log\max(|p|,|q|) + C.$$

If p/q is close to α, then it is at least a fixed distance apart from all other roots of f, so adjusting constants, this is equivalent to

$$\left|\frac{p}{q} - \alpha\right| > \frac{C}{|q|^{2+\epsilon}}.$$

This is exactly Roth's theorem, and Vojta's conjecture gives the justification of the exponent 2: it comes from the degree of the canonical divisor of $\mathbb{P}^1$. In this example, the exception Z_ϵ to the inequality (2.2) as specified in the conjecture is a set of finite points, so we can remove it by adjusting the constant.

The restriction to normal crossings divisor is necessary. Since this hypothesis will be important later, we will briefly discuss two examples which demonstrate how the conjecture fails for non-normal crossings divisors.

Example 2.2. When we include multiplicities in the divisor, the conjecture does not hold. For example, let $X = \mathbb{P}^1$, $D = 2(0) + (\infty)$, $k = \mathbb{Q}$, $S = \{v_\infty, v_2\}$. Since $\lambda_{(0)}([x:1], v) = \max(0, -\log|x|_v)$ and $\lambda_{(\infty)}([x:1], v) = \max(0, -\log|1/x|_v)$, we have

$$\sum_{v\in S} \lambda_{(0)}([2^n:1], v) = \sum_{v\in S} \lambda_{(\infty)}([2^n:1], v) = n\log 2.$$

Thus the left hand side of (2.2) is $3n\log 2 - 2n\log 2$, so for $\epsilon < 1$, the conjecture will not hold no matter what Z_ϵ we choose.

Example 2.3. We also cannot allow more than $\dim X$ components of D to meet at any point. Let $X = \mathbb{P}^2$, $D = (X) + (Y) + (Y - X) + (Z)$, $k = \mathbb{Q}$, and $S = \{v_\infty, v_2\}$.

Using (2.1), we will calculate the local heights at $[2^n : 2^m : 1]$, $m > n$. For $v = v_\infty$, the coordinate with minimum v is 2^m, so

$$\lambda_D([2^n : 2^m : 1], v_\infty) = (-\log 2^n + \log 2^m) + (-\log(2^m - 2^n) + \log 2^m) + \log 2^m.$$

On the other hand, for $v = v_2$, the coordinate with minimum v is 1, so

$$\lambda_D([2^n : 2^m : 1], v_2) = \log 2^n + \log 2^m + \log 2^n.$$

Combining, we get $3 \log 2^m + \log 2^n + (\log 2^m - \log(2^m - 2^n)) \geq 3 \log 2^m + \log 2^n$. The exception Z_ϵ can only cover finitely many lines of the form $Y = 2^l X$, so if we look at points with m large and $n > m\epsilon$, this will contradict the conjecture, as the degree of the canonical divisor is -3.

We now discuss some preliminaries from arithmetic dynamics. For a more complete overview of this field, see for example [11]. Here, we will introduce basic definitions and concepts necessary for the main theorem. The field of dynamics studies behavior of self-maps $\phi : X \longrightarrow X$ as we iterate. We will use $\phi^{(m)}$ for the m-fold composition $\phi \circ \cdots \circ \phi$. When $P \in X$, the (forward) orbit $\mathcal{O}_\phi(P)$ of P under ϕ is the set $\{P, \phi(P), \phi^{(2)}(P), \ldots\}$ of points traversed via applications of ϕ when we start from P. When $\phi^{(m)}(P) = P$ for some m, then we say that P is a *periodic* point of ϕ. If P is a preimage of a periodic point under some $\phi^{(n)}$, we say that P is *preperiodic*. It is easy to check that P is preperiodic if and only if $\mathcal{O}_\phi(P)$ is finite. It is often helpful to analyze the family $\{\phi^{(m)}\}$ around (pre)periodic points in dynamics. In fact, using the information around the periodic points, McMullen [6] constructed a finite-to-one cover of the moduli space of all rational maps up to a change of coordinates on $\mathbb{P}^1$.

If $P \in X(k)$ and ϕ is defined over k, then $\phi(P) \in X(k)$. So for non-preperiodic P, $\mathcal{O}_\phi(P)$ contains infinitely many rational points. On the other hand, if we look at a subset of the set of rational points, we will not know how frequently the orbit intersects with this subset. One natural arithmetic candidate for such a subset is a set of integral points, i.e. points with coordinates in $\mathcal{O}_k$ on X minus an ample divisor (in some affine coordinates). Even if we start from an integral point P, $\phi(P)$ is not necessarily integral, and it is natural to ask how frequently integral points appear in an orbit. The following theorem describes the situation in the case of $X = \mathbb{P}^1$, where integral points are $\{[x : 1] : x \in \mathbb{Z}\}$.

Theorem 2.4 (Silverman [10]). *Let $\phi \in \mathbb{Q}(z)$ be a rational map of degree ≥ 2. Then if $\phi^{(2)}$ is not a polynomial (i.e. not in $\mathbb{Q}[z]$), then the number of integers in an orbit $\mathcal{O}_\phi(P)$ is finite for any $P \in \mathbb{P}^1(\mathbb{Q})$. Moreover, if P is not preperiodic and if we let $\phi^{(m)}(P) = a_m/b_m$ in a reduced form, then unless $\phi^{(2)}$ or $\frac{1}{\phi^{(2)}(1/z)}$ is not a polynomial, we must have*

$$\lim_{m \longrightarrow \infty} \frac{\log |a_m|}{\log |b_m|} = 1.$$

We will make several remarks about this theorem, as a preparation for proving Theorem 1.1. First, the finiteness of integers in an orbit also holds for a general number field. Secondly, the hypothesis that $\phi^{(2)}$ is not a polynomial is necessary, as we obviously get infinitely many integers if $\phi^{(2)} \in \mathbb{Z}[z]$ and $P \in \mathbb{Z}$. Thirdly, one of the most elementary but powerful tools that we have for maps on $\mathbb{P}^1$ is the Riemann–Hurwitz formula, and this is at play in this theorem as well. For one, we can only have two totally ramified fixed point, so one can show that if $\phi^{(m)}$ is a polynomial for some m, then $\phi^{(2)}$ is already a polynomial. This is the reason we only need to assume information about $\phi^{(2)}$ in the above theorem. For two, the Riemann–Hurwitz formula together with purely combinatorial arguments show that $\phi^{(4)}$ has to have at least 3 preimage points of ∞ unless $\phi^{(2)}$ is a polynomial. This allows us to use Siegel's theorem and prove the first part of the theorem. The second part is much more subtle. If the numerator grows much more rapidly than the denominator, then $\phi^{(m)}(P)$ approximates ∞ very well. By employing Riemann–Hurwitz and some case-by-case ramification analyses, one can show that $\phi^{(m-n)}(P)$ must well-approximate one of the preimages of ∞ under $\phi^{(n)}$, contradicting Roth's theorem if one chooses m and n appropriately.

§ 3. Theorem 1.1

We first start by noting some challenges for generalizing Silverman's theorem to higher dimensions. These explain the hypotheses in the statement of Theorem 1.1. In $\mathbb{P}^N$, we do not have results like Riemann–Hurwitz which greatly controls ramification. So rather than dealing with all maps, we restrict to a (relatively large) subset of maps where the ramification behavior is under control. This is the normal crossings conditions on $\left(\phi^{(n)}\right)^*(H)$ in Theorem 1.1. Moreover, Roth's theorem on $\mathbb{P}^1$ describes everything one needs to know about approximating a divisor by a rational point, since divisors on $\mathbb{P}^1$ are sum of points. On $\mathbb{P}^N$, Roth's theorem has a natural generalization in the form of Schmidt's subspace theorem, but this still only covers the linear divisors. For Diophantine approximation to general non-linear divisors, one naturally comes to Vojta's conjecture. These explain why normal crossings condition and Vojta's conjecture were used in the statement of Theorem 1.1. We are now ready to prove the theorem.

Proof of Theorem 1.1. We will begin by proving (b) first. Assume on the contrary, then there exists $\epsilon > 0$ such that there exists an infinite set I with

$$\frac{\log |a_0^{(m)}|}{\log \max_i |a_i^{(m)}|} < 1 - \epsilon \quad \text{for } m \in I. \tag{3.1}$$

Rearranging terms and using (2.1), we have

$$\epsilon h(\phi^{(m)}(P)) = \epsilon \log \max_i |a_i^{(m)}| \ < \ \left(\log \max_i |a_i^{(m)}|\right) - \log |a_0^{(m)}| = \lambda_H(\phi^{(m)}(P), v_\infty). \tag{3.2}$$

Fix n large enough so that $\epsilon d^n > N+1$. By assumption, $\left(\phi^{(n)}\right)^*(H)$ is a normal crossings divisor. Then by Vojta's conjecture, given $\epsilon' > 0$, there exists $Z_{\epsilon'}$ such that

$$\lambda_{\left(\phi^{(n)}\right)^*(H)}(\phi^{(m-n)}(P), v_\infty) + h_{(-N-1)H}(\phi^{(m-n)}(P)) < \epsilon' h_H(\phi^{(m-n)}(P)) + C$$

for $\phi^{(m-n)}(P)$ outside of $Z_{\epsilon'}$. Combining (3.2) and using functoriality of local heights, we have

$$\epsilon h(\phi^{(m)}(P)) < (N+1+\epsilon')h(\phi^{(m-n)}(P)) + C'.$$

Since ϕ is a morphism, there exists a constant C'' such that $h(\phi^n(Q)) \geq d^n h(Q) + C''$ for all Q, so

$$\epsilon\left[d^n h(\phi^{(m-n)}(P)) + C''\right] < (N+1+\epsilon')h(\phi^{(m-n)}(P)) + C'.$$

We now choose ϵ' small enough so that $\epsilon d^n > N+1+\epsilon'$. For each $m \in I$ with $m > n$, either $\phi^{(m-n)}(P)$ is in $Z_{\epsilon'}$ or it has a bounded height. Since any infinite subset of the orbit was assumed to be Zariski-dense, this is a contradiction.

For (a), if on the contrary there are infinitely many integers in an orbit, then $a_0^{(m)}$ is equal to 1 for infinitely many m's, so (3.1) has to hold for ϵ arbitrarily close to 1. Then as long as there is one n with $d^n > N+1$ such that $\left(\phi^{(n)}\right)^*(H)$ is normal crossings, we choose $\epsilon < 1$ with $\epsilon d^n > N+1$ and then choose $\epsilon' = \frac{\epsilon d^n - (N+1)}{2}$, we get a contradiction by the same argument as above. □

Example 3.1. We discuss some examples of the theorem. Let ϕ be the morphism $[X^2+Y^2+Z^2 : Y^2 : Z^2]$ on $\mathbb{P}^2$. We prove by induction that $(\phi^{(n)})^*(X=0)$ is a smooth divisor, so in particular a normal crossings divisor. Let $\phi^{(n)} = [F_n : G_n : H_n]$, and let us assume that $P = [x : y : z]$ is a singular point on $F_n = 0$. By the Jacobian criterion,

$$F_n(P) = 0 \quad \text{that is,} \quad F_{n-1}(x^2+y^2+z^2, y^2, z^2) = 0 \tag{3.3}$$

$$\frac{\partial}{\partial x}F_n(P) = 0 \quad \text{that is,} \quad \frac{\partial}{\partial x}F_{n-1}(\phi(P)) \cdot 2x = 0 \tag{3.4}$$

$$\frac{\partial}{\partial y}F_n(P) = 0 \quad \text{that is,} \quad \frac{\partial}{\partial x}F_{n-1}(\phi(P)) \cdot 2y + \frac{\partial}{\partial y}F_{n-1}(\phi(P)) \cdot 2y = 0 \tag{3.5}$$

$$\frac{\partial}{\partial z}F_n(P) = 0 \quad \text{that is,} \quad \frac{\partial}{\partial x}F_{n-1}(\phi(P)) \cdot 2z + \frac{\partial}{\partial z}F_{n-1}(\phi(P)) \cdot 2z = 0 \tag{3.6}$$

By inductive hypothesis, $F_{n-1} = 0$ is smooth, so at $\phi(P)$, one of the partial derivatives of F_{n-1} is nonzero. So by (3.4)–(3.6), $xyz = 0$. If $x = 0$, then $F_n(P) = y^{2^n} + z^{2^n}$, so by (3.5) and (3.6), $y = z = 0$, contradiction. So assume $x \neq 0$ and assume $y = 0$. If in addition $z = 0$, then F_n becomes a pure power of x, so (3.4) gives $x = 0$, contradiction. If $z \neq 0$, then (3.4) and (3.6) show $\frac{\partial}{\partial x}F_{n-1}$ and $\frac{\partial}{\partial z}F_{n-1}$ are both zero at

$\phi(P)$, contradicting smoothness of $F_{n-1} = 0$. The case $x \neq 0$ and $z = 0$ is similar, so we prove that $F_n = 0$ is smooth.

So this ϕ satisfies the normal crossings condition appearing in part (b) of the theorem. However, this particular ϕ has the property that the x-coordinate is larger than the y or the z-coordinate after the second iterate, so we will trivially have finitely many points in the orbit in $(\mathbb{P}^2 \backslash (X = 0))(\mathbb{Z})$. If we try $\psi = [X^2 - Y^2 + Z^2 : Y^2 + Z^2 : Z^2]$, then the argument in the above paragraph carries over and shows that $(\psi^{(n)})^*(X = 0)$ is normal crossings for all n. But for an orbit of $[1 : 2 : 4]$, the first coordinate is always odd, so again it is trivial to show finiteness of integers in the orbit. We can argue similarly for other initial points. Of course, if we allow mixed terms like XY in the map, then a simple size argument or a modular arithmetic argument usually will not immediately give us finiteness of integers. On the other hand, for these maps, it becomes harder to show normal crossings condition for all n.

Regarding the orbit density, at least for $\mathbb{P}^2$, there is often an ad hoc argument to show that a curve cannot contain infinitely many points in the orbit. For example, for maps of the form ϕ or ψ above, the z-coordinate is a pure power, so every point in the orbit is an S-integer in $\mathbb{P}^2 \backslash (Z = 0)$ for an appropriately chosen finite set S of primes. Thus, a curve containing infinitely many of the points has geometric genus 0, and we can write the x and the y coordinate as some Laurent polynomials in S-units. By numerically analyzing the asymptotic ratios of the coordinates of the points in the orbit, one then often derives a contradiction.

In the examples of ϕ and ψ above, it is easier to show finiteness of integers than to show denseness. In general, in fact, it is a well-known *conjecture* by Zhang [14, Conjecture 4.1.6] that a polarized endomorphism on a projective variety has a rational point over a number field whose orbit is Zariski-dense.

We make several additional remarks about Theorem 1.1. As in Silverman's theorem, (a) generalizes to arbitrary number fields directly. (b) actually holds for any embedding of the number field into $\mathbb{C}$, because all we are using is (3.2).

The normal crossings condition in the theorem is certainly geometric, and independent of P. In this sense, the situation is analogous to Silverman's theorem: on $\mathbb{P}^1$, being a polynomial is a geometric property about existence of a totally ramified fixed point, and once we remove this possibility, the orbit is finite for all points P. Moreover, d^2 is already greater than $1 + 1$, so Theorem 1.1 (a) on $\mathbb{P}^1$ is also a condition on the second iterate. On the other hand, the hypothesis of Theorem 1.1 in the case of $\mathbb{P}^1$ is stronger than what Silverman required. Indeed, normal crossings divisor on $\mathbb{P}^1$ means no multiplicity, while non-polynomial means no total ramification (multiplicity d). So a more appropriate weaker hypothesis is desirable for Theorem 1.1.

On the other hand, "most" ϕ do satisfy the hypothesis given in (a) of the theorem.

For example, if one looks at a map $[F_0 : \cdots F_N]$ with F_i homogeneous of degree at least $N+2$, then the hypothesis of (a) is satisfied if the hypersurface $F_0 = 0$ is smooth. Such a collection of maps form a nonempty Zariski-open set among maps of the same degree. Even when the degree of F_i is lower than $N+2$, we could look at the first n such that $d^n > N+1$. The smooothness of the 0-th coordinate of $\phi^{(n)}$ can be detected by a rank of an appropriate Jacobian matrix, which can be described by polynomials in the coefficients of F_i's. So again, for most ϕ of this degree, $\left(\phi^{(n)}\right)^*(H)$ is smooth, in particular normal crossings. Similarly, for a fixed ϕ, $\left(\phi^{(n)}\right)^*(H)$ will be normal crossings for the first n with $d^n > N+1$ for most choices of H.

§ 4. Some Unconditional Results

We now discuss some results that are related to Theorem 1.1, without assuming Vojta's conjecture. The first example involves morphisms on $\mathbb{P}^2$ of lowest nontrivial degree, i.e. $\phi = [F_0 : F_1 : F_2]$ with F_i homogeneous of degree 2.

Theorem 4.1. *Let ϕ be a morphism on $\mathbb{P}^2$ of degree 2 defined over a number field such that it fixes a line, i.e. there exists a line L with $\phi(L) = L$ (as a point set). If ϕ does not fix any other line, then for any $P \in \mathbb{P}^2(\mathbb{Q})$, $\mathcal{O}_\phi(P) \cap \mathbb{Z}(\mathbb{P}^2 \backslash L)$ is finite.*

If ϕ fixes another line, then ϕ restricted to this line could behave like an integer-coefficient polynomial on $\mathbb{P}^1$, creating many integer points in an orbit. So this hypothesis is necessary. A complete proof of this theorem is in [13], but we discuss the general ideas of the proof here. Since $\phi(L) = L$, by an appropriate change of coordinate, we can assume that L is the line $X_2 = 0$ and $F_2 = X_1X_2$. Because ϕ is assumed to be a morphism, $h(\phi^{(n)}(P))$ is roughly $2^n h(P)$. Since each F_i is a degree-2 polynomial, this means that the common divisors of the coordinates are fairly small. That is, if $\phi^{(n-1)}(P) = [a : b : c]$ with $a, b, c \in \mathbb{Z}$ with no common divisors (call this the reduced form), then common divisor of $F_0(a,b,c), F_1(a,b,c), F_2(a,b,c)$ is not too big. So in order to get many integer points (points with the last coordinate equal to 1 in the reduced form), neither the X_1 nor the X_2 coordinates of $\phi^{(n)}(P)$ can grow. This forces the height of $\phi^{(n)}(P)$ to come solely from X_0. This is a contradiction unless F_1 has X_0X_1 and X_0X_2 terms just in the right ratio to cancel each other out when we plug in P. This corresponds to having a second fixed line, ending the proof. The idea is straightforward, but as in any Diophantine approximations, we need some delicacy in handling various height inequalities.

The second example is an example more in spirit, because the maps involved are rational maps rather than morphisms. We introduce several results from the joint work [4] with Areyh Gregor. We analyze maps on $\mathbb{P}^2$ having very simple forms, namely those whose F_0, F_1, F_2 are all monomials. These are rarely defined on all of $\mathbb{P}^2$, so this is

not an example of Theorem 1.1. In fact, in the proof of Theorem 1.1, we use a height inequality ($h(\phi^{(n)}(P)) \geq d^n h(P) + C''$) that requires ϕ to be a morphism, so the same result does not have to hold. On the other hand, these monomial maps provide nice special cases where finiteness of integers in an orbit can be explored directly. Note that the only points where ϕ is undefined are $[0:0:1], [0:1:0], [1:0:0]$, and away from these three points, a forward orbit is well-defined. Moreover, the pullback of $Z = 0$ by $\phi^{(n)}$ is just the triangle XYZ albeit with high multiplicities. So monomial maps are rational maps, but points with bad behavior are very well-contained.

Dividing by F_2 and using $x = X/Z$ and $y = Y/Z$, we can write a monomial map as $(x^i y^j, x^k y^l)$ for $i, j, k, l \in \mathbb{Z}$. So we could look at these maps as algebraic group morphisms on $\mathbb{G}_m \times \mathbb{G}_m$, but since we are interested in integral points of $\mathbb{P}^2 \backslash \{Z = 0\}$, we will not pursue this. For $P \in \mathbb{P}^2(\mathbb{Q})$, we would like to know if the orbit $\mathcal{O}_\phi(P)$ contains only finitely many integers, i.e. points of the form $[a : b : 1]$ with $a, b \in \mathbb{Z}$. Just as in the $\mathbb{P}^1$ case, when we have a polynomial (i.e. $i, j, k, l \geq 0$), we trivially get infinitely many integers by starting from integral points. On $\mathbb{P}^1$, we only needed to check to see if the second iterate was a polynomial. The following describes the analog for monomial maps on $\mathbb{P}^2$.

Theorem 4.2 (AY [4]). *Let ϕ be a monomial map on $\mathbb{P}^2$. If $\phi^{(n)}$ is a polynomial for some n, then first such n is $1, 2, 3, 4, 6, 8, 12$.*

So we can check $\phi^{(24)}$ to see if we ever get a polynomial and thus trivially obtain infinitely many integers in some orbit. The proof uses linear algebra together with some Galois theory. If we write the exponents of ϕ in a matrix format $A = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$, then the exponent matrix for $\phi^{(n)}$ is A^n. Perron's theorem in linear algebra provides necessary conditions for non-negative matrices: there exists a positive real eigenvalue which is bigger than or equal to the absolute value of all other eigenvalues. After some algebraic manipulations, we show that for the first n making $\phi^{(n)}$ a polynomial, the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ has Galois group which is a subgroup of a Klein 4 group, proving the theorem.

In [4], we then completely characterize when there exists a point P whose orbit $\mathcal{O}_\phi(P)$ contains infinitely many integers. Because monomial maps are rational maps, unlike Theorem 1.1, one can often actually get infinitely many integers in an orbit. We state two representative cases here.

Theorem 4.3 (AY [4]). *Let $\phi = (x^i y^j, x^k y^l)$ and let A as above.*

(a) *If A has two real eigenvalues and the two coordinates of an eigenvector of the bigger eigenvalue have opposite signs, then all orbits contain just finitely many integers.*

(b) *If A has complex eigenvalues, then there exists $P \in \mathbb{P}^2(\mathbb{Q})$ such that $\mathcal{O}_\phi(P)$ contains infinitely many integers.*

An example for (a) is $\phi = (y/x, x)$, and in this case exponents appearing in $\phi^{(n)}$ are Fibonacci numbers. An example for (b) is $\phi = (x/y, xy^2)$. To determine finiteness of integers, we only need asymptotic analyses of A^n, so eigenvalues and eigenvectors play important roles. In the situation of (a), we also have an analog of Theorem 1.1 (b): if $\phi^{(m)}(P) = (\alpha_m/\beta_m,\ \gamma_m/\delta_m)$ in a reduced form, then the ratio of $\log\max(\alpha_m, \gamma_m)$ and $\log\max(\beta_m, \delta_m)$ goes toward some finite number as $m \longrightarrow \infty$, but not necessarily to 1.

We have now described two very special examples in the direction of finiteness of integers in an orbit, without assuming Vojta's conjecture. Generalizing these even slightly already seems to be tricky. In fact, an analog of Theorem 4.2 is not apparent from computer calculations when we analyze monomial maps on $\mathbb{P}^3$, and an analog of Theorem 4.1 for $\mathbb{P}^3$ or $d \geq 3$ cannot be proved by the same methods. So we need much more further research in this direction.

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