

A Remark on Nowhere Dense Closed P-Sets

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Abstract. Using the methods from continua theory of R^* , we prove that NCF implies that ω^* can be covered by an increasing sequence of nowhere dense closed P-sets.

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Kunen, van Mill and Mills proved in [4] that no compact space of weight 2^ω can be covered by nowhere dense closed P-sets under CH. It was proved in [1] that, in the model obtained by adding ω_1 Cohen reals to a model of $\text{MA} + \neg \text{CH}$, ω^* can be covered by nowhere dense closed P-sets. It is not difficult to show that the axiom of near coherence of filters, abbreviated as NCF (See [2]), implies that ω^* can be covered by nowhere dense closed P-sets. Our purpose in this note is to strengthen the conclusion as follows:

Theorem 1. *NCF implies that ω^* can be covered by an increasing sequence of nowhere dense closed P-sets.*

Our way is to use the methods from continua theory of $\mathbb{R}^*$ to guarantee an induction construction going smoothly through the limit steps. Actually, we shall prove, (See also Corollary 5.7 in [5]).

Theorem 2. *NCF is equivalent to that $\beta[0, \infty) - [0, \infty)$ can be covered by a strictly increasing sequence of subcontinua which are nowhere dense P-sets.*

It is not difficult to show that if ω^* can be covered by nowhere dense closed P-sets then so can R^* (See Corollary 4). But the author don't know whether or not the converse is true.

we refer to [2] for the background on NCF and [5] for continua theory of R^* .

Let Ω be the collection of all families of infinite discrete non-degenerate closed interval of the half real line $[0, \infty)$. For $\mathcal{I} \in \Omega$, we let $i: \omega \rightarrow \mathcal{I}$ be the bijection such that $i(n) < i(n+1)$ for $n \in \omega$, where $i(n) < i(n+1)$ means that $r < s$ for all $(r, s) \in i(n) \times i(n+1)$. Let $\dot{i}: \cup \mathcal{I} \rightarrow \omega$ be such that $\dot{i}(x) = n$ if and only if $x \in i(n)$. Let $\beta \dot{i}$ be the Stone-Ćech extension of $\dot{i}$ from $\text{cl}_{\beta R}(\cup \mathcal{I})$ to $\beta \omega$. For $B \subset \omega^*$, we define

$$M(\mathcal{I}, B) = \beta \dot{i}^{-1}(B)$$

and, if $B = \{u\}$, then $M(\mathcal{I}, \{u\})$ is denoted by $M(\mathcal{I}, u)$. It is well-known that $M(\mathcal{I}, u)$ is a continuum for any $u \in \omega^*$. Moreover, a subcontinuum C of $\beta[0, \infty) - [0, \infty)$ is called a standard continuum if $C = M(\mathcal{I}, u)$ for some $\mathcal{I} \in \Omega$ and $u \in \omega^*$. Note that every proper subcontinuum of $\beta[0, \infty) - [0, \infty)$ is nowhere dense since $\beta[0, \infty) - [0, \infty)$ is an indecomposable continuum.

Recall that a subset B of a space X is called a P-set

provided that the intersection of countably many neighbourhoods of B is again a neighbourhood of B . A point x of X is called a P-point if the singleton $\{x\}$ is a P-set.

For an open set U of a metric space X , we let $O(U) = \{x \in \beta X : \exists F \in \mathcal{X}(F \subset U)\}$. Then $\{O(U) : U \text{ is open in } X\}$ is a base for βX . $[\omega]^\omega$ is the set of all infinite subsets of ω . As usual, $O(A) \cap \omega^*$ is denoted by A^* for $A \in [\omega]^\omega$.

For $\mathcal{I}, \mathcal{I}' \in \Omega$, we say that $\mathcal{I}'$ is an expander of $\mathcal{I}$ if $i(n)$ is contained in the interior of $i'(n)$ for all $n \in \omega$.

Lemma 3. $B \subset \omega^*$ is a nowhere dense closed P-set if and only if $M(\mathcal{I}, B)$ is a nowhere dense closed P-set of $\beta[0, \omega) - [0, \omega)$ for $\mathcal{I} \in \Omega$.

Proof. Assume that $M(\mathcal{I}, B)$ is a nowhere dense closed P-set of $\beta[0, \omega) - [0, \omega)$. It is easily seen that B is nowhere dense closed in ω^* . Let $\mathcal{I}' \in \Omega$ be an expander of $\mathcal{I}$. Then $M(\mathcal{I}, B)$ is a P-set of $\text{cl}_{\beta R}(O\mathcal{I}')$. Suppose that $\{A_n^* : n \in \omega\}$ is a family of countably many neighbourhoods of B . Then $\{\beta i'^{-1}(A_n^*) : n \in \omega\}$ is a family of neighbourhoods of $M(\mathcal{I}, B)$. Therefore, there is a basic open set $O(U)$ such that $M(\mathcal{I}, B) \subset O(U) \cap R^* \subset \beta i'^{-1}(A_n^*)$ for all $n \in \omega$. Note that, for $u \in \omega^*$, $M(\mathcal{I}, u) = \bigcap \{\text{cl}_{\beta R}(O\mathcal{I}) : i^{-1}(\mathcal{I}) \in u\}$. Therefore, for each $u \in B$, there is $A_u \in u$ such that $\bigcup \{i(n) : n \in A_u\} \subset U$. Let $A = \{i^{-1}(I) : I \in \mathcal{I} \text{ and } I \subset U\}$. Then $A_u \subset A$ for $u \in B$. So A^* is a neighbourhood of B . Since $O(U) \cap R^* \subset \beta i'^{-1}(A_n^*)$,

we have that $A^* \subset A_n^*$ for all $n \in \omega$.

Assume that B is a nowhere dense closed P -set of ω^* . Let $O(U)$ be a basic open set of βR and $O(U) \cap M(\mathcal{G}, B) \neq \emptyset$. Let $A = \{n \in \omega : i(n) \cap U \neq \emptyset\}$. Then $A \in [\omega]^\omega$. Since B is nowhere dense, there is $A_1 \in [\omega]^\omega$ such that $A_1 \subset A$ and $A_1^* \cap B = \emptyset$. Therefore, $M(\mathcal{G}, A_1^*) \cap M(\mathcal{G}, B) = \emptyset$. But $O(U) \cap M(\mathcal{G}, A_1^*) \neq \emptyset$. So $O(U) \cap R^* \setminus M(\mathcal{G}, B) \neq \emptyset$. It follows that $M(\mathcal{G}, B)$ is nowhere dense. Suppose that $\{O(U_n) : n \in \omega\}$ is a family of neighbourhoods of $M(\mathcal{G}, B)$. Let $A_n = \{i(I) : I \in \mathcal{G} \text{ and } I \subset U_n\}$ for $n \in \omega$. As we showed in the last paragraph, A_n^* is a neighbourhood of B for all $n \in \omega$. Since B is a P -set, there is $A \in [\omega]^\omega$ such that $B \subset A^*$ and $A^* \subset A_n^*$ for all $n \in \omega$. We choose a strictly increasing sequence $\{m_n : n \in \omega\}$ of integers so that for each $n \in \omega$ $A \setminus m_n \subset A_n$ and $[m_n, m_{n+1}) \cap A \neq \emptyset$, where $[m_n, m_{n+1}) = \{i \in \omega : m_n \leq i < m_{n+1}\}$. For each $i \in [m_n, m_{n+1}) \cap A$, let J_i be an open interval of R such that $i(i) \subset J_i \subset U_n$. Let $V = \bigcup \{J_n : n \in A \setminus m_0\}$. Then $M(\mathcal{G}, B) \subset M(\mathcal{G}, A^*) \subset O(V)$ and $O(V) \subset O(U_n)$ for $n \in \omega$. This completes the proof of Lemma 3.

Since we can easily choose $\mathcal{G}, \mathcal{G}' \in \Omega$ such that $\bigcup(\mathcal{G} \cup \mathcal{G}') = [0, \infty)$ and R^* is the topological sum of $\beta(-\infty, 0] - (-\infty, 0]$ and $\beta[0, \infty) - [0, \infty)$, we have

Corollary 4. *If ω^* can be covered by nowhere dense closed P -sets, then so can R^* .*

Blass proved in [2] that, under NCF, for any $u \in \omega^*$ there is a finite-to-one non-decreasing function $f: \omega \rightarrow \omega$ such that $v = \beta f(u)$ is a P-point. It is easily seen that $\beta f^{-1}(v)$ is a nowhere dense closed P-set of ω^* and $u \in \beta f^{-1}(v)$. Therefore, NCF implies that ω^* can be covered by nowhere dense closed P-sets. Our purpose is to sharpen the conclusion so that ω^* can be covered by an increasing sequence of nowhere dense closed P-sets under NCF.

We regard ω^* as a subspace of $\beta[0, \omega) - [0, \omega)$. The following lemma is an easy observation.

Lemma 5. *If $u \in \omega^*$ is a P-point, then $M(\mathcal{I}, u) \cap \omega^*$ is a nowhere dense closed P-set of ω^* for $\mathcal{I} \in \Omega$.*

Proof. Let $X = \omega \cap (\cup \mathcal{I})$ and $Y = \{i^{-1}(I) : I \cap \omega \neq \emptyset\}$. If $Y \notin u$, then, $M(\mathcal{I}, u) \cap \omega^* = \emptyset$. So we assume that $Y \in u$. We define a finite to one function $f: X \rightarrow Y$ from X onto Y by $f(n) = m$ if and only if $n \in i(m)$. Then $M(\mathcal{I}, u) \cap \omega^* = \beta f^{-1}(u)$. Since $\beta f^{-1}(u)$ is a nowhere dense closed P-set in X^* , $M(\mathcal{I}, u) \cap \omega^*$ is a nowhere dense closed P-set in ω^* .

By Lemma 3 and 5, our Theorem 1 and 2 follows easily from the following theorem.

Theorem 2'. NCF is equivalent to that there is a family $\{(\mathcal{I}_\alpha, u_\alpha) : \alpha < \lambda\}$ such that

- (1) $\mathcal{I}_\alpha \in \Omega$ and $u_\alpha \in \omega^*$ is a P-point for all $\alpha < \lambda$;
- (2) $M(\mathcal{I}_\alpha, u_\alpha) \subset M(\mathcal{I}_\beta, u_\beta)$ for all $\alpha < \beta < \lambda$;
- (3) $\beta[0, \omega) - [0, \omega) = \bigcup \{M(\mathcal{I}_\alpha, u_\alpha) : \alpha < \lambda\}$.

Theorem 2' will be proved along the line of the proof of Corollary 5.7 in [5]. We first recall some properties of NCF and standard continua. We refer to [2] and [5] for details.

A subset C of a continuum K is a composant if, for some point $p \in C$, C is the set of all points x such that there is a proper subcontinuum of K containing both p and x . It is well-known that NCF is equivalent to that $\beta[0, \omega) - [0, \omega)$ is a composant of itself (See [3]). Therefore, our conditions in Theorem 2' implies NCF.

Recall that there is a natural partial order $<_{\mathcal{I}, u}^{\mathcal{I}}$ on $M(\mathcal{I}, u)$ for $\mathcal{I} \in \Omega$ and $u \in \omega^*$, defined as follows: For any $x, y \in M(\mathcal{I}, u)$,

$x <_{\mathcal{I}, u}^{\mathcal{I}} y$ if there are $F \in x$ and $H \in y$ such that $\{i^{-1}(I) : I \in \mathcal{I} \text{ and } F \cap I < H \cap I\} \in u$,

For $x \in M(\mathcal{I}, u)$, we let

$$[x]_u^{\mathcal{G}} = \{y \in M(\mathcal{G}, u) : y \text{ is } <_u^{\mathcal{G}}\text{-incomparable with } x \text{ or } y=x\}.$$

$[x]_u^{\mathcal{G}}$ is called a layer of $M(\mathcal{G}, u)$. It is well-known that layers are indecomposable subcontinua of $M(\mathcal{G}, u)$ and every indecomposable subcontinuum of $M(\mathcal{G}, u)$ is contained in a layer.

Lemma 6 (Corollary 2.11 in [5]). *Let C and D be subcontinua of R^* . If one of them is indecomposable, then $C \subset D$, $D \subset C$ or $C \cap D = \emptyset$.*

A point $u \in \omega^*$ is a Q-point if every finite-to-one function from ω to ω is one-to-one on a set in u . By Proposition 5.1 in [5], it is equivalent to require the functions in the definition of Q-points to be non-decreasing. Blass proved in [2] that NCF implies that there is no Q-points.

Lemma 7. *Under NCF, for every proper subcontinuum C of $\beta[0, \infty) - [0, \infty)$, there is a standard continuum $M(\mathcal{G}, u)$ and a layer T of $M(\mathcal{G}, u)$ such that $C \subset T$ and $M(\mathcal{G}, u)$ is a nowhere dense P-set of $\beta[0, \infty) - [0, \infty)$.*

Proof. Since every proper subcontinuum of $\beta[0, \infty) - [0, \infty)$ is contained in a standard subcontinuum, we assume that $C \subset M(\mathcal{G}_1, u_1)$ for some $\mathcal{G}_1 \in \Omega$ and $u \in \omega^*$. Since NCF implies that there is no

Q-points, there is a finite-to-one non-decreasing function $f:\omega\rightarrow\omega$ which witnesses that u_1 is not a Q-point. We define $\mathcal{I}_2=\{I_n:n\in\omega\}$ as follows: I_n is the convex hull of the set $\bigcup\{i_1(m):m\in f^{-1}(n)\}$. Let $u_2=f(u_1)$. Then, $M(\mathcal{I}_1,u_1)\subset M(\mathcal{I}_2,u_2)$. Moreover, for any $x,y\in M(\mathcal{I}_1,u_1)$, x and y are $<_{u_2}^{\mathcal{I}_2}$ -incomparable or $x=y$. Therefore, $M(\mathcal{I}_1,u_1)$ is contained in a layer T' of $M(\mathcal{I}_2,u_2)$. By NCF, there is a finite-to-one non-decreasing function $g:\omega\rightarrow\omega$ such that $u=g(u_2)$ is a P-point. By the same method as above, we can find $\mathcal{I}\in\Omega$ such that $M(\mathcal{I}_2,u_2)\subset M(\mathcal{I},u)$. Since T' is an indecomposable subcontinuum of $M(\mathcal{I},u)$, there is a layer T of $M(\mathcal{I},u)$ such that $C\subset T'\subset T$. By Lemma 3, $M(\mathcal{I},u)$ is a nowhere dense P-set of $\beta[0,\omega)-[0,\omega)$.

Now we are in a position to complete the proof of Theorem 2'. We assume NCF. We define, inductively, $\mathcal{I}_\alpha\in\Omega$, $u_\alpha\in\omega^*$ and a layer T_α of $M(\mathcal{I}_\alpha,u_\alpha)$ for $\alpha\geq 0$ satisfying that

- (a) u_α is a P-point for all $\alpha\geq 0$;
- (b) $M(\mathcal{I}_\alpha,u_\alpha)\subset T_\beta$ for $\alpha<\beta$.

Our induction process will stop at some λ if $\beta[0,\omega)-[0,\omega)=\bigcup\{M(\mathcal{I}_\alpha,u_\alpha):\alpha<\lambda\}$. Suppose that we have defined $\mathcal{I}_\beta$, u_β and T_β for all $\beta<\alpha$ satisfying (a) and (b). If $\alpha=0$ or $\gamma+1$, then, by Lemma 7, we can easily define $\mathcal{I}_\alpha$, u_α and T_α satisfying (a) and (b). Assume that $\alpha\neq 0$ is a limit and $\beta[0,\omega)-[0,\omega)$ is not covered by $\{M(\mathcal{I}_\beta,u_\beta):\beta<\alpha\}$. Note that by (b) $\bigcup\{M(\mathcal{I}_\beta,u_\beta):\beta<\alpha\}=$

$\cup \{T_\beta : \beta < \alpha\}$ since α is a limit. Take $x, y \in \beta[0, \omega) - [0, \omega)$ such that $x \in T_0$ and $y \in T_\beta$ for all $\beta < \alpha$. By NCF, there is a proper subcontinuum C of $\beta[0, \omega) - [0, \omega)$ containing both x and y . By Lemma 6, $T_\beta \subset C$ for all $\beta < \alpha$. By Lemma 7, there is $\mathcal{F}_\alpha \in \Omega$, $u_\alpha \in \omega^*$ and a layer T_α of $M(\mathcal{F}_\alpha, u_\alpha)$ such that $C \subset T_\alpha$ and u_α is a P-point. This completes our inductive construction. Since $\{M(\mathcal{F}_\alpha, u_\alpha) : \alpha \geq 0\}$ is a strictly increasing sequence, our induction process can not go over $|R^*|$ steps. This completes the proof of Theorem 2'.

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