

## Surfaces of Finite Total Curvature

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### Introduction

Let  $M$  be a connected, oriented, noncompact, complete surface in the 3-dimensional Euclidean space  $E^3$ . We denote the second fundamental form of  $M$  by  $B$ , the mean curvature by  $H$  and Gaussian curvature by  $K$ .

In this paper, we study the geometry of a surface  $M$  which satisfies  $\int_M |B| dM < \infty$ , where  $dM$  is the area element of  $M$ . For surfaces which satisfy  $\int_M |B|^2 dM < \infty$ , rather than  $\int_M |B| dM < \infty$ , extensive studies have been made. Since  $|B|^2 = 4H^2 - 2K \geq 2|K|$ , the condition  $\int_M |B|^2 dM < \infty$  implies  $\int_M |K| dM < \infty$ . A theorem by Huber ([H]) says that an abstract noncompact complete 2-dimensional Riemannian manifold satisfying  $\int_M |K| dM < \infty$  is conformally equivalent to a compact Riemann surface  $\bar{M}$  punctured at a finite number of points  $p_1, \dots, p_k$ . If  $M$  is a minimal surface with  $\int_M |B|^2 dM < \infty$  (or equivalently,  $|\int_M K dM| < \infty$ ), Osserman showed that the Gauss map  $G: M \rightarrow S^2$  is continuously extended to a map from  $\bar{M}$  to  $S^2$  ([O]). White ([W]) extended Osserman's theorem to surfaces with  $\int_M |B|^2 dM < \infty$  and  $K \leq 0$ . The continuous extendability of the Gauss map cannot be derived without the condition  $K \leq 0$ , but Müller-Šverák ([M-S]) showed that if  $M$  satisfies  $\int_M |B|^2 dM < \infty$ , then  $M$  is properly immersed.

Let  $M$  be a connected, oriented, noncompact, complete, properly immersed surface which satisfies a pointwise condition  $|B(x)| \leq C|x|^{-1-\varepsilon}$  for  $x$  in  $M$ , where  $|x|$  is the Euclidean distance from the origin and  $C$  and  $\varepsilon$  are positive constants. In [E], the author showed that for  $M$  satisfying this condition we have  $\int_M |B|^2 dM < \infty$  and the Gauss map is continuously extended to a map from  $\bar{M}$ . Basic tools to study those surfaces are given by Kasue and Sugahara in [K] and [K-S]. In this paper we will study a surface  $M$  which satisfies a stronger condition  $|B(x)| \leq C|x|^{-2-\varepsilon}$  and show that we have  $\int_M |B| dM < \infty$  (Theorem 1). We will also show that such a surface has a property that each end has an asymptotic plane (Theorem 2).

In general, of course, the condition  $\int_M |B| dM < \infty$  does not give any pointwise estimate for  $|B|$ . Moreover, since  $\int_M |B| dM < \infty$  does not necessarily imply  $\int_M |K| dM < \infty$ , Huber's theorem cannot be applied for these surfaces and the geometry or the topology of  $M$  near infinity can be very complicated in general. But if  $M$  is a surface of revolution, we can say something about the geometry of the ends. In Theorem 3, we will show that if  $M$  is a surface of revolution and satisfies  $\int_M |B| dM < \infty$ , then each end of  $M$  has an asymptotic plane. We note that even when  $M$  is a surface of revolution,  $\int_M |B| dM < \infty$  does not imply that  $|B|$  is uniformly bounded.

### 1. Surfaces whose second fundamental forms decay uniformly

Let  $M$  be a connected, oriented, noncompact, complete, properly immersed surface in  $E^3$ . For  $x$  in  $M$ ,  $|x|$  will denote the Euclidean distance to  $x$  from the origin.

**Theorem 1.** If there exist positive constants  $C$  and  $\varepsilon$  such that  $|B(x)| \leq C|x|^{-2-\varepsilon}$  for all  $x$  in  $M$ , then we have  $\int_M |B| dM < \infty$ .

**Proof.** Since  $M$  is properly immersed and the second fundamental form satisfies  $|B(x)| \leq C|x|^{-2-\varepsilon}$ ,  $M(t) = \{x \in M : |x| \geq t\}$  is a union of a finite number of surfaces  $M_1(t), \dots, M_q(t)$ , and for  $\lambda = 1, \dots, q$  and sufficiently large  $t$ ,  $M_\lambda(t)$  is diffeomorphic to  $\partial M_\lambda(t) \times [t, \infty)$  ([K] Lemma 2).  $\partial M_\lambda(t)$  is a closed curve whose length is less than  $C_1 t$ , where  $C_1$  is a constant which does not depend on  $t$  ([K-S] Lemma 6). Let  $r : M \rightarrow \mathbf{R}$  be a function on  $M$  which is defined by  $r(x) = |x|$  for  $x$  in  $M$ . Then there exists a positive constant  $C_2$  such that  $|\nabla r| \geq C_2$  if  $r$  is sufficiently large ([K] Lemma 2). Hence, denoting the line element of  $\partial M_\lambda(t)$  by  $ds$ , we have  $dM \leq C_3 dr ds$  for some positive constant  $C_3$ . Now, if  $R$  is sufficiently large, we have

$$\begin{aligned} \int_{M_\lambda(R)} |B| dM &\leq C_3 \int_R^\infty \left( \int_{\partial M_\lambda(r)} |B| ds \right) dr \\ &\leq C_3 \int_R^\infty (C r^{-2-\varepsilon} \text{Length}(\partial M_\lambda(r))) dr \\ &\leq C_3 \int_R^\infty C C_1 r^{-1-\varepsilon} dr \\ &< \infty. \end{aligned}$$

This proves the theorem.

**Theorem 2** If there exist positive constants  $C$  and  $\varepsilon$  such that  $|B(x)| \leq C|x|^{-2-\varepsilon}$  for all  $x$  in  $M$ , then each end of  $M$  has an asymptotic plane, i.e., for each  $\lambda = 1, \dots, q$  there exists a plane  $P_\lambda$  such that  $\sup_{x \in M_\lambda(t)} \inf_{y \in P_\lambda} |x - y|$  tends to 0 as  $t \rightarrow \infty$ .

**Proof.** By Lemma 1.4 in [E], our assumption implies that the normal component of the position vector on each end of  $M$  tends to a constant vector. To make it more precise, let  $x$  be a point in  $M_\lambda(t)$  and  $N = N(x)$  be a unit normal vector of  $M$  at  $x$ . Then, as  $|x| \rightarrow \infty$ ,  $N(x)$  converges to a constant unit vector  $E_\lambda$  and  $\langle x, N(x) \rangle$  converges to a constant  $a_\lambda$ . Let  $P_\lambda$  be the plane defined by  $\{y \in E^3 : \langle y, E_\lambda \rangle = a_\lambda\}$ . First we note that  $\inf_{y \in P_\lambda} |x - y| = \langle x, E_\lambda \rangle - a_\lambda$  for  $x$  in  $M_\lambda$ . By moving the origin of  $E^3$  if necessary, we may assume that  $a_\lambda \neq 0$ . Since

$$\begin{aligned} \lim_{|x| \rightarrow \infty} \langle x, a_\lambda E_\lambda \rangle &= \lim_{|x| \rightarrow \infty} \langle x, \langle x, N \rangle N \rangle \\ &= \lim_{|x| \rightarrow \infty} \langle x, N \rangle^2 \\ &= a_\lambda^2, \end{aligned}$$

we see that  $\langle x, E_\lambda \rangle - a_\lambda$  tends to 0 as  $|x| \rightarrow \infty$ . This shows that each end of  $M$  has an asymptotic plane.

## 2. Surfaces of revolution

In this section,  $M$  will be a surface of revolution in  $E^3$  which is defined by  $M = \{(x, y, z) = (r \cos \theta, r \sin \theta, f(r)) : r_0 \leq r < \infty, 0 \leq \theta < 2\pi\}$ . The norm of the second fundamental form and the area element of  $M$  are given as

$$\begin{aligned} |B| &= r^{-1}(1 + f'^2)^{-3/2}(r^2 f''^2 + f'^2(1 + f'^2)^2)^{1/2} \\ dM &= r(1 + f'^2)^{1/2} dr d\theta. \end{aligned}$$

**Theorem 3.** Let  $M$  be a surface of revolution given above. If  $\int_M |B| dM < \infty$ , then  $\lim_{r \rightarrow \infty} f(r) = C$  for some constant  $C$ , i.e., each end of  $M$  has an asymptotic plane.

**Proof.** Since

$$\int_M |B| dM = \int_{r_0}^{\infty} dr \int_0^{2\pi} \frac{(r^2 f''^2 + f'^2(1 + f'^2)^2)^{1/2}}{1 + f'^2} d\theta$$

$$\begin{aligned}
&= 2\pi \int_{r_0}^{\infty} \frac{(r^2 f'^2 + f'^2(1 + f'^2)^2)^{1/2}}{1 + f'^2} dr \\
&\geq 2\pi \int_{r_0}^{\infty} |f'| dr,
\end{aligned}$$

the condition  $\int_M |B| dM < \infty$  implies that  $\lim_{r \rightarrow \infty} f(r) = C$  for some constant  $C$ .

The following example shows that  $|B|$  may not be bounded on a surface of revolution with  $\int_M |B| dM < \infty$ .

**Example.** Let  $f(r)$  be a function which has the following property;

$$f''(r) = \begin{cases} n & \text{for } n + \frac{1}{n^4} \leq r \leq n + \frac{2}{n^4} \\ 0 & \text{for } n + \frac{3}{n^4} \leq r \leq n + 1 \\ < n & \text{for } n + \frac{2}{n^4} < r < n + \frac{3}{n^4} \\ \geq 0 & \text{for all } r \end{cases}$$

Then for  $n \leq r < n + 1$ , we have

$$\sum_{k=1}^{n-1} \frac{1}{k^3} < f'(r) - f'(0) < \sum_{k=1}^n \frac{2}{k^3}.$$

Hence, by a suitable choice for  $f'(0)$ , we can have  $f'(r) = O(r^{-2})$ . If  $M$  is a surface of revolution obtained from  $f(r)$ , we have

$$\begin{aligned}
\int_M |B| dM &= 2\pi \int_0^{\infty} \frac{(r^2 f'^2 + f'^2(1 + f'^2)^2)^{1/2}}{1 + f'^2} dr \\
&\leq 2\pi \int_0^{\infty} r |f''| dr + 2\pi \int_0^{\infty} |f'| dr \\
&< \infty.
\end{aligned}$$

But  $|B|$  is not bounded, since  $|B| \approx n$  if  $n + \frac{1}{n^4} \leq r \leq n + \frac{2}{n^4}$ .

## References

- [E] Compactification of submanifolds in Euclidean space by the inversion, Advanced Studies in Pure Mathematics, Vol.22 (K.Shiohama, eds.). Kinokuniya Company, 1993, 1–11.
- [H] On subharmonic functions and differential geometry in the large, Comment.

Math. Helv. **32** (1957) 13–72.

[K] A. Kasue, Gap theorems for minimal submanifolds of Euclidean space, J. Math. Soc. Japan **38** (1986) 473–492.

[K–S] A. Kasue and K. Sugahara, Gap theorems for certain submanifolds of Euclidean spaces and hyperbolic space forms, Osaka J. Math. **24** (1987) 679–704.

[M–S] S. Müller and V. Šverák, On surfaces of finite total curvature, J. Differential Geom. **42** (1995) 229–258.

[O] R. Osserman, Global properties of minimal surfaces in  $E^3$  and  $E^n$ , Ann. of Math. (2) **80** (1964) 340–364.

[W] B. White, Complete surfaces of finite total curvature, J. Differential Geom. **26** (1987) 315–326.

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