

ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. Let $p(z)$ be analytic in the open unit disk $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. S.S.Miller and P.T.Mocanu (J. Math. Anal. Appl. 276(2002)) have shown some interesting subordination theorems for such functions $p(z)$. The object of the present paper is to discuss some sufficient conditions for arguments of $p(z)$ to be $|\arg p(z)| < \frac{\pi}{2}\rho$ for $z \in \mathbb{U}$.

1. INTRODUCTION

Let $p(z)$ be analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $p(0) = 1$ and $p'(0) = 0$. For such functions $p(z)$, Miller and Mocanu [3] have shown some interesting subordination theorems.

Theorem A. ([3]) For $\frac{1}{2} < \rho \leq 1$ define the function $q(z)$ by

$$q(z) = q_\rho(z) = \left(\frac{1+z}{1-z} \right)^\rho,$$

and let $t_0 \in (0, 1)$ be the unique solution of

$$t^\rho \left\{ (1-\rho)t^2 \cos\left(\frac{\pi}{2}\rho\right) + t \sin\left(\frac{\pi}{2}\rho\right) - (1-\rho) \cos\left(\frac{\pi}{2}\rho\right) \right\} + t^2 - 1 = 0.$$

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1, p'(0) = 0$ and

$$|\arg(zp'(z) + p(z)^2 + p(z))| < \frac{\pi}{2}(\rho+1) - \tan^{-1}\left(\frac{t_0}{1+\rho-(1-\rho)t_0^2}\right),$$

then $p(z) \prec q_\rho(z)$, where the symbol " $\prec$ " means the subordinations.

To discuss our problems for functions $p(z)$, we need the following lemma due to Hal-
lenbeck and Ruscheweyh [2] which is the same as one by Fukui and Sakaguchi [1].

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Lemma 1.1. *Let $p(z)$ be analytic in $|z| < R$ and $p^{(k)}(0) = 0$ ($0 \leq k \leq n$). Then if $|p(z)|$ attains its maximum value on the circle $|z| = r < R$ at a point z_0 , we have*

$$(1.1) \quad \frac{z_0 p'(z_0)}{p(z_0)} \geq n + 1.$$

Applying the above lemma, we derive

Lemma 1.2. *Let $p(z)$ be analytic in $\mathbb{U}$, $p(0) = 1$, $p'(0) = 0$, and let $p(z) \neq 0$ ($z \in \mathbb{U}$). If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2} \alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2} \alpha$$

for some $\alpha > 0$, then we have

$$(1.2) \quad \frac{z_0 p'(z_0)}{p(z_0)} = i \alpha k,$$

where

$$k \geq \left(a + \frac{1}{a}\right) \geq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \alpha$$

and

$$k \leq -\left(a + \frac{1}{a}\right) \leq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \alpha,$$

where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

Proof. We use the same manner which was used by Nunokawa [4] for the proof of the lemma. Let us put

$$(1.3) \quad q(z) = p(z)^{1/\alpha}.$$

Then we see that $\operatorname{Re} q(z) > 0$ ($|z| < |z_0|$), $\operatorname{Re} q(z_0) = 0$, $q(0) = 1$ and $q'(0) = 0$. Defining the function $\phi(z)$ by

$$(1.4) \quad \phi(z) = \frac{1 - q(z)}{1 + q(z)},$$

we have that $\phi(0) = 0$, $|\phi(z)| < 1$ ($|z| < |z_0|$), and $|\phi(z_0)| = 1$. In view of Lemma 1.1, we know that

$$(1.5) \quad \frac{z_0 \phi'(z_0)}{\phi(z_0)} = \frac{-2z_0 q'(z_0)}{1 - q(z_0)^2}$$

$$= \frac{-2z_0q'(z_0)}{1 + |q(z_0)|^2} \geq 2.$$

It follows from (1.5) that

$$(1.6) \quad -z_0q'(z_0) \geq (1 + |q(z_0)|^2)$$

and $z_0q'(z_0)$ is a negative real number. Since $q(z_0)$ is a non-vanishing pure imaginary number, we can put $q(z_0) = ia$, where a is a non-vanishing real number.

We have, for $a > 0$,

$$(1.7) \quad \operatorname{Im} \left(\frac{z_0q'(z_0)}{q(z_0)} \right) = \operatorname{Im} \left(-\frac{iz_0q'(z_0)}{|q(z_0)|} \right) \geq \left(\frac{1+a^2}{a} \right) \geq 2$$

and, for $a < 0$,

$$(1.8) \quad \operatorname{Im} \left(\frac{z_0q'(z_0)}{q(z_0)} \right) = \operatorname{Im} \left(\frac{iz_0q'(z_0)}{|q(z_0)|} \right) \leq -\left(\frac{1+a^2}{a} \right) \leq -2$$

On the other hand, it follows that

$$(1.9) \quad \frac{z_0q'(z_0)}{q(z_0)} = \frac{1}{\alpha} \left(\frac{z_0p'(z_0)}{p(z_0)} \right).$$

This completes the proof of Lemma 1.2. □

2. ARGUMENT ESTIMATES

Our first property for argument estimates of analytic function $p(z)$ is contained in

Theorem 2.1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$(2.1) \quad |\arg(zp'(z) + p(z)^2 + \alpha p(z))| < \pi\rho \quad (z \in \mathbb{U})$$

for some α ($\alpha > 0$), ρ ($0 < \rho \leq \rho_0$), where ρ_0 ($0 < \rho_0 < 1$) is given by

$$\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho_0,$$

then

$$(2.2) \quad |\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}).$$

Proof. Let a function $p(z)$ satisfy the conditions of the theorem. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\rho,$$

then applying Lemma 1.2, we have that

$$(2.3) \quad \frac{z_0 p'(z_0)}{p(z_0)} = i\rho k,$$

where

$$k \geq a + \frac{1}{a} \geq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\rho$$

and

$$k \leq -\left(a + \frac{1}{a}\right) \leq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\rho$$

with $p(z_0)^{1/\rho} = \pm ia$ ($a > 0$). It follows that, for $\arg p(z_0) = \frac{\pi}{2}\rho$ and $k \geq a + \frac{1}{a} \geq 2$,

$$(2.4) \quad \begin{aligned} \arg(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) &= \arg p(z_0) \left(\frac{z_0 p'(z_0)}{p(z_0)} + p(z_0) + \alpha \right) \\ &= \frac{\pi}{2}\rho + \arg(i\rho k + a^\rho e^{i\frac{\pi}{2}\rho} + \alpha) = \frac{\pi}{2}\rho + \tan^{-1} \left(\frac{\rho k + a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)} \right) \end{aligned}$$

Since, by $0 < \rho \leq \rho_0 < 1$ and $k \geq 2$,

$$(2.5) \quad \tan^{-1} \left(\frac{\rho k + a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)} \right) \geq \tan^{-1} \left(\frac{2\rho + a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)} \right) > 0,$$

we define $g(a)$ by

$$(2.6) \quad g(a) = \frac{2\rho + a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)} \quad (a > 0).$$

Noting that

$$(2.7) \quad g'(a) = \frac{\alpha \rho a^{\rho-1} \cos(\frac{\pi}{2}\rho) (\tan(\frac{\pi}{2}\rho) - \frac{2\rho}{\alpha})}{(\alpha + a^\rho \cos(\frac{\pi}{2}\rho))^2},$$

we define $h(\rho)$ by

$$(2.8) \quad h(\rho) = \tan\left(\frac{\pi}{2}\rho\right) - \frac{2\rho}{\alpha} \quad (0 < \rho \leq \rho_0 < 1).$$

Then $h(0) = 0$, $h(\rho_0) = 0$, and

$$(2.9) \quad h''(\rho) = \frac{\pi^2}{2} \sec^2\left(\frac{\pi}{2}\rho\right) \tan\left(\frac{\pi}{2}\rho\right) > 0.$$

This shows that $g'(a) \leq 0$ for $a > 0$, that is, that

$$(2.10) \quad \tan^{-1} \left(\frac{\rho k + a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)} \right) \geq \tan^{-1} \left(\tan\left(\frac{\pi}{2}\rho\right) \right) = \frac{\pi}{2}\rho.$$

Therefore, we conclude that

$$(2.11) \quad \arg(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) \geq \pi \rho$$

when $\arg p(z_0) = \frac{\pi}{2}\rho$.

Similarly, for $\arg p(z_0) = -\frac{\pi}{2}\rho$ and $k \leq -\left(a + \frac{1}{a}\right) \leq -2$, we have that

$$\begin{aligned} (2.12) \quad \arg(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) &= -\frac{\pi}{2}\rho + \arg(i\rho k + a^\rho e^{-i\frac{\pi}{2}\rho} + \alpha) \\ &= -\frac{\pi}{2}\rho + \tan^{-1}\left(\frac{\rho k - a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)}\right) \\ &\leq -\frac{\pi}{2}\rho + \tan^{-1}\left(\frac{-2\rho - a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)}\right) \\ &= -\frac{\pi}{2}\rho - \tan^{-1}\left(\frac{2\rho + a^\rho \sin(\frac{\pi}{2}\rho)}{\alpha + a^\rho \cos(\frac{\pi}{2}\rho)}\right) \\ &\leq -\frac{\pi}{2}\rho - \frac{\pi}{2}\rho = -\pi\rho. \end{aligned}$$

Thus, for such a point $z_0 \in \mathbb{U}$, we see that

$$(2.13) \quad |\arg(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0))| \geq \pi\rho,$$

which contradicts our condition for $p(z)$.

Consequently, we conclude that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}).$$

□

Example 2.1. Let us consider the function $p(z)$ defined by

$$p(z) = 1 + \frac{1}{5}z^2.$$

Then we see that

$$zp'(z) + p(z)^2 + \frac{1}{2}p(z) = \frac{3}{2} + \frac{9}{10}z^2 + \frac{1}{25}z^4.$$

Letting $\alpha = \frac{1}{2}$ and

$$\rho = \frac{1}{\pi} \sin^{-1}\left(\frac{19}{30}\right)$$

in Theorem 2.1, we have that

$$\left| \arg\left(zp'(z) + p(z)^2 + \frac{1}{2}p(z)\right) \right| < \pi\rho = \sin^{-1}\left(\frac{19}{30}\right)$$

and

$$|\arg p(z)| < \sin^{-1} \left(\frac{1}{5} \right) < \frac{\pi}{2} \rho.$$

If we take $\alpha = 1$ in Theorem 2.1, then

Corollary 2.1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$(2.14) \quad |\arg (zp'(z) + p(z)^2 + p(z))| < \pi \rho \quad (z \in \mathbb{U})$$

for some ρ $\left(0 < \rho \leq \frac{1}{2}\right)$, then

$$(2.15) \quad |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}).$$

Remark 2.1. (1) If $\alpha = \frac{4}{5}$, then $0 < \rho \leq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.

(2) If $\alpha = \frac{1}{2}$, then $0 < \rho \leq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.

(3) If $\alpha = \frac{1}{3}$, then $0 < \rho \leq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.

(4) If $\alpha = \frac{1}{4}$, then $0 < \rho \leq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.

(5) If $\alpha = 1.1$, then $0 < \rho \leq \rho_0$ and $0.401247 < \rho_0 < 0.491248$.

(6) If $\alpha = 1.2$, then $0 < \rho \leq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.

(7) If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan \left(\frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha} \rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.1.

Next, we derive

Theorem 2.2. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$(2.16) \quad |\arg (zp'(z) + p(z)^2 + \alpha p(z))| < \frac{\pi}{2} \rho + \tan^{-1} \left(\frac{2\rho}{\alpha} \right) \quad (z \in \mathbb{U})$$

for some α ($\alpha > 0$), ρ ($\rho_0 \leq \rho < 1$), where ρ_0 ($0 < \rho_0 < 1$) is given by $\tan \left(\frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha} \rho_0$, then

$$(1.7) \quad |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}).$$

Proof. Using the same technique as in the proof of Theorem 2.1, we know that

$$\tan^{-1} \left(\frac{2\rho + a^\rho \sin \left(\frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left(\frac{\pi}{2} \rho \right)} \right)$$

is increasing for $a > 0$. Thus, we obtain

$$(2.18) \quad |\arg(z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0))| \geq \frac{\pi}{2}\rho + \tan^{-1}\left(\frac{2\rho}{\alpha}\right)$$

for $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\rho.$$

This contradicts our condition of the theorem. Therefore,

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}).$$

□

Letting $\alpha = 1$ in Theorem 2.2, we obtain

Corollary 2.2. Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies

$$(2.19) \quad |\arg(zp'(z) + p(z)^2 + p(z))| < \frac{\pi}{2}\rho + \tan^{-1}(2\rho) \quad (z \in \mathbb{U})$$

for some ρ $\left(\frac{1}{2} \leq \rho < 1\right)$, then

$$(2.20) \quad |\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}).$$

Finally, we note that

Remark 2.2. (1) If $\alpha = \frac{4}{5}$, then $0 < \rho \leq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.

(2) If $\alpha = \frac{1}{2}$, then $0 < \rho \leq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.

(3) If $\alpha = \frac{1}{3}$, then $0 < \rho \leq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.

(4) If $\alpha = \frac{1}{4}$, then $0 < \rho \leq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.

(5) If $\alpha = 1.1$, then $0 < \rho \leq \rho_0$ and $0.401247 < \rho_0 < 0.491248$.

(6) If $\alpha = 1.2$, then $0 < \rho \leq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.

(7) If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.2.

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