

STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACE

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ABSTRACT. We prove for a nonexpansive mapping T that under certain conditions the strong $\lim_{t \rightarrow 1-} G_t(x)$ exists and is a fixed point of T , where $G_t(x) = (1-t)x + tTG_t(x)$, $0 \leq t < 1$.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E . A mapping $T : C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all x, y in C .

Let E^* be the dual space of E . Then the value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. With each $x \in E$, we associate the set

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \emptyset$ for each $x \in E$. The multivalued operator $J : E \rightarrow E^*$ is called the duality mapping of E . Let $B = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . Then the norm of E is said to be Gateaux differentiable (and E is said to be smooth) if

$$\lim_{r \rightarrow 0} \frac{\|x + ry\| - \|x\|}{r}$$

exists for each x and y in B . It is said to be Frechet differentiable if for each x in B , this limit is attained uniformly for y in B . Finally, it is said to be uniformly Frechet differentiable (and E is said to be uniformly smooth) if the limit is attained uniformly for (x, y) in $B \times B$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E has a Frechet differentiable norm, then J is norm to norm continuous.

The purpose of this note is to continue the discussion concerning the strong convergence of the path $t \rightarrow G_t(x)$, $0 \leq t < 1$ defined by (1) below for each x in C . We prove for a nonexpansive mapping T that under certain conditions the strong $\lim_{t \rightarrow 1-} G_t(x)$ exists and is a fixed point of T . The first results of this nature were established by Brower([2]) and Browder and Petryshyn([4]).

Key words and phrases. Nonexpansive mapping, Banach limit, Fixed points.

2. Lemmas

Let E be a Banach space. Then the modulus of convexity of E is defined as $\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in B_E \text{ and } \|x - y\| \geq \varepsilon\}$, where $B_E = \{x \in E : \|x\| \leq 1\}$ is the closed unit ball of E . We recall that E is said to have the modulus of convexity of power type $p \geq 2$ (and E is said to be p -uniformly convex) if there exists a constant $c > 0$ such that

$$\delta_E(\varepsilon) \geq c\varepsilon^p$$

for $0 < \varepsilon \leq 2$.

We now define the mapping $G_t : C \rightarrow C$ by

$$G_t(x) = (1 - t)x + tTG_t(x) \quad (1)$$

for all x in C and $0 \leq t < 1$. It is clear that for each $0 \leq t < 1$, the fixed point set of G_t coincides with that of T .

We also recall that a Banach limit LIM is a bounded linear functional on ℓ^∞ of norm 1 such that

$$\liminf_{n \rightarrow \infty} x_n \leq \text{LIM}\{x_n\} \leq \limsup_{n \rightarrow \infty} x_n$$

and

$$\text{LIM}\{x_n\} = \text{LIM}\{x_{n+1}\}$$

for all $\{x_n\}$ in ℓ^∞ .

LEMMA 1. (Prus and Smarzewski [6]) *Let E be a p -uniformly convex Banach space ($p > 1$). Then there exists a constant $c > 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - cW_p(\lambda)\|x - y\|^p \quad (2)$$

for all $x, y \in E$ and $\lambda \in [0, 1]$, where $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

LEMMA 2. *Let C be a nonempty closed convex and bounded subset of a p -uniformly convex Banach space E , and let $\{x_n\}$ be a bounded sequence in E . We define the functional $r : C \rightarrow R$ by the formula*

$$r(x) = \text{LIM}\{\|x_n - x\|^p\}.$$

Then $r(\cdot)$ is continuous and convex.

Proof. For $x, y \in C$, we have

$$|\|x_n - x\|^p - \|x_n - y\|^p| \leq p(\text{diam } C)^{p-1}|\|x_n - x\| - \|x_n - y\||$$

and

$$\begin{aligned} |r(x) - r(y)| &= |\text{LIM}\{\|x_n - x\|^p\} - \text{LIM}\{\|x_n - y\|^p\}| \\ &\leq p(\text{diam } C)^{p-1} \text{LIM}\{|\|x_n - x\| - \|x_n - y\||\} \\ &\leq p(\text{diam } C)^{p-1} \text{LIM}\{\|x - y\|\} \\ &\leq p(\text{diam } C)^{p-1}\|x - y\|. \end{aligned}$$

For any fixed $n \in N$ and $0 < t < 1$, by the inequality (2), we get

$$\begin{aligned} \|x_n - ((1-t)x + ty)\|^p &= \|(1-t)(x_n - x) + t(x_n - y)\|^p \\ &\leq (1-t)\|x_n - x\|^p + t\|x_n - y\|^p - cW_p(t)\|x - y\|^p \\ &\leq (1-t)\|x_n - x\|^p + t\|x_n - y\|^p. \end{aligned}$$

Taking the Banach limit LIM on each side, we obtain

$$\text{LIM}\{\|x_n - ((1-t)x + ty)\|^p\} \leq (1-t)\text{LIM}\{\|x_n - x\|^p\} + t\text{LIM}\{\|x_n - y\|^p\}.$$

Therefore we get

$$r((1-t)x + ty) \leq (1-t)r(x) + tr(y).$$

LEMMA 3. Let C be a nonempty closed convex subset of a p -uniformly convex and uniformly smooth Banach space E . Let $\{x_n\}$ be a bounded sequence in E . Then for $z_0 \in C$,

$$\text{LIM}\{\|x_n - z_0\|^p\} = \min_{y \in C} \text{LIM}\{\|x_n - y\|^p\}$$

if and only if

$$\text{LIM}\{\langle z - z_0, J(x_n - z_0) \rangle\} \leq 0$$

for all $z \in C$.

Proof. We first assume that $\text{LIM}\{\|x_n - z_0\|^p\} = \min_{y \in C} \text{LIM}\{\|x_n - y\|^p\}$. For $z \in C$ and $\lambda: 0 \leq \lambda \leq 1$, we have

$$\begin{aligned} \|x_n - z_0\|^p &= \|x_n - \lambda z_0 - (1-\lambda)z + (1-\lambda)(z - z_0)\|^p \\ &\geq \|x_n - \lambda z_0 - (1-\lambda)z\|^p \\ &\quad + p(1-\lambda) \langle z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) \rangle \end{aligned}$$

since $J(x)$ is subdifferential of the convex function $\frac{1}{p}\|x\|^p$ ([3,p97]). Let $\varepsilon > 0$ be given. Since E is uniformly smooth, the duality map is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^* ([3]). Therefore,

$$|\langle z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) - J(x_n - z_0) \rangle| < \varepsilon$$

if λ is close enough to 1. Consequently, we have

$$\begin{aligned} \langle z - z_0, J(x_n - z_0) \rangle &< \varepsilon + \langle z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) \rangle \\ &\leq \varepsilon + \frac{1}{p(1-\lambda)} \{\|x_n - z_0\|^p \\ &\quad - \|x_n - \lambda z_0 - (1-\lambda)z\|^p\}. \end{aligned}$$

Therefore, we have

$$\text{LIM}\{\langle z - z_0, J(x_n - z_0) \rangle\} \leq 0$$

for all $z \in C$.

To prove reverse, let $z \in C$. Then since

$$\|x_n - z\|^p - \|x_n - z_0\|^p \geq p \langle z_0 - z, J(x_n - z_0) \rangle$$

for all $n \in N$ and $\text{LIM}\{\langle z - z_0, J(x_n - z_0) \rangle\} \leq 0$ for all $z \in C$, we have

$$\text{LIM}\{\|x_n - z_0\|^p\} = \min_{z \in C} \text{LIM}\{\|x_n - z\|^p\}.$$

LEMMA 4. Let C be a closed convex and bounded subset of a p -uniformly convex and uniformly smooth Banach space E , and $\{x_n\}$ be a bounded sequence of E . Then, the set

$$M = \{u \in C : LIM\{\|x_n - u\|^p\} = \min_{z \in C} LIM\{\|x_n - z\|^p\}\}$$

consists of one point.

Proof. Let $g(z) = LIM\{\|x_n - z\|^p\}$ for every $z \in C$ and $r = \inf\{g(z) : z \in C\}$. Then, by Lemma 2, the function g on C is convex and continuous and $g(z) \rightarrow \infty$ as $\|z\| \rightarrow \infty$. From [1, p79], there exists $u \in C$ with $g(u) = r$. Therefore M is nonempty. By Lemma 3, we know that $u \in M$ if and only if $LIM\{\langle z - u, J(x_n - u) \rangle\} \leq 0$ for all $z \in C$. We show that M consists of one point. Let $u, v \in M$ and suppose $u \neq v$. Then by [7, Theorem 1], there exists a positive number ε such that

$$\langle x_n - u - (x_n - v), J(x_n - u) - J(x_n - v) \rangle \geq \varepsilon$$

for every $n \in N$. Therefore

$$LIM\{\langle v - u, J(x_n - u) - J(x_n - v) \rangle\} \geq \varepsilon > 0.$$

On the other hand, since $u, v \in M$, we have

$$LIM\{\langle u - v, J(x_n - v) \rangle\} < 0$$

and

$$LIM\{\langle v - u, J(x_n - u) \rangle\} < 0.$$

Then we have

$$LIM\{\langle v - u, J(x_n - u) - J(x_n - v) \rangle\} < 0.$$

This is a contradiction. Therefore $u = v$.

3. Main Results

THEOREM 1. Let C be a closed convex and bounded subset of a p -uniformly convex and uniformly smooth Banach space E , and $\{x_n\}$ be a bounded sequence of E such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. If $T : C \rightarrow C$ is a nonexpansive mapping, then

$$M = \{u \in C : LIM\{\|x_n - u\|^p\} = \min_{z \in C} LIM\{\|x_n - z\|^p\}\}$$

is a fixed point set of T .

Proof. We will show that the set M is invariant under T . In fact, since $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, we have, for $u \in M$,

$$\begin{aligned} LIM\{\|x_n - Tu\|^p\} &= LIM\{\|Tx_n - Tu\|^p\} \\ &\leq LIM\{\|x_n - u\|^p\} \end{aligned}$$

and hence $Tu \in M$. On the other hand, by Lemma 4, we know that M consists of one point. Therefore this point is a fixed point of T and M is a fixed point set of T .

It is well known in ([8]) that a uniform smooth space has normal structure. Since such a space is also reflexive, each bounded closed convex subset of it has the fixed point property for nonexpansive mappings ([5]).

THEOREM 2. Let C be a closed convex and bounded subset of a p -uniformly convex and uniformly smooth Banach space E , $T : C \rightarrow C$ a nonexpansive mapping, and $G_t : C \rightarrow C$, $0 < t < 1$, the family of mappings defined by (1). Then, for each x in C , the strong $\lim_{t \rightarrow 1^-} G_t(x)$ exists and is a fixed point of T .

Proof. Note that from the preceding statement T has a fixed point in C . Let w be a fixed point of T . Fix a point x in C , denote $G_t(x)$ by $y(t)$. Since $y(t) - w = (1 - t)(x - w) + t(Ty(t) - Tw)$,

$$\|y(t) - w\| \leq \|x - w\|$$

and $\{y(t)\}$ remains bounded as $t \rightarrow 1^-$. We also have

$$\begin{aligned} \lim_{t \rightarrow 1^-} \|y(t) - Ty(t)\| &= \lim_{t \rightarrow 1^-} \|(1 - t)x - (1 - t)Ty(t)\| \\ &= 0. \end{aligned}$$

Now let $t_n \rightarrow 1^-$ and $y_n = y(t_n)$. Define $f : C \rightarrow [0, \infty)$ by $f(z) = \text{LIM}\{\|y_n - z\|^p\}$. From Lemma 2 f is continuous and convex, $f(z) \rightarrow \infty$ as $\|z\| \rightarrow \infty$, which implies that f attains its infimum over C . That is, there exists a $z_0 \in C$ such that

$$\text{LIM}\{\|y_n - z_0\|^p\} = \min_{y \in C} \text{LIM}\{\|y_n - y\|^p\}.$$

Let M be the set of minimizers of T . By Theorem 1, $z_0 \in M$ is the fixed point of T . Therefore

$$\begin{aligned} \langle y_n - Ty_n, J(y_n - z_0) \rangle &= \langle y_n - Tz_0 + Tz_0 - Ty_n, J(y_n - z_0) \rangle \\ &= \|y_n - Tz_0\|^2 - \langle Ty_n - Tz_0, J(y_n - z_0) \rangle \\ &\geq \|y_n - Tz_0\|^2 - \|Ty_n - Tz_0\| \|y_n - z_0\| \\ &\geq \|y_n - Tz_0\|^2 - \|y_n - Tz_0\|^2 = 0 \end{aligned}$$

for all n . It follows that for $x \in C$,

$$\begin{aligned} 0 &\leq \langle y_n - Ty_n, J(y_n - z_0) \rangle \\ &= \langle (1 - t_n)x + t_nTy_n - Ty_n, J(y_n - z_0) \rangle \\ &= \langle (1 - t_n)x - (1 - t_n)Ty_n, J(y_n - z_0) \rangle \\ &= (1 - t_n) \langle x - Ty_n, J(y_n - z_0) \rangle \end{aligned}$$

for all n . Thus, we get for $x \in C$,

$$\langle y_n - x, J(y_n - z_0) \rangle \leq 0 \quad (3)$$

for all n . From Lemma 3

$$\text{LIM}\{\langle z - z_0, J(y_n - z_0) \rangle\} \leq 0 \quad (4)$$

for all $z \in C$. Choosing $z = y_n$ in (4), we conclude that

$$\text{LIM}\{\|y_n - z_0\|\} \leq 0.$$

Thus there is a subsequence of $\{y_n\}$ which converges strongly to z_0 . To complete the proof, suppose that $y_{n_k} \rightarrow z_1$ and $y_{m_k} \rightarrow z_2$. Then by (3),

$$\langle z_1 - x, J(z_1 - z_2) \rangle \leq 0$$

and

$$\langle z_1 - z_2, J(z_1 - z_2) \rangle \leq 0.$$

Hence $z_1 = z_2$ and the strong $\lim_{t \rightarrow 1^-} y(t)$ exists, which completes the proof.

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