

ASYMMETRIC VARIATION OF CHOI INEQUALITY FOR POSITIVE LINEAR MAP

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Introduction

Let Φ be a unital positive linear map between two matrix algebras $\mathcal{A}$ and $\mathcal{M}$.

Kadison inequality [10] states that for $A \in \mathcal{A}^{sa}$ (the self adjoint elements in $\mathcal{A}$)

$$\Phi(A)^2 \leq \Phi(A^2).$$

It is known in [e.g., [1]] that $\Phi(A)^r \leq \Phi(A^r)$ holds for $A > 0$ and $r \in [-1, 0]$ and $r \in [1, 2]$, and more genarally

$$f(\Phi(A)) \leq \Phi(f(A))$$

for operator convex function f , and $A \in \mathcal{A}^{sa}$ with spectra of A in the domain of f . We cite nice references [2] and [12] to this subject. Choi [4] shows that for $A \in \mathcal{A}^+$ (the positive cone of $\mathcal{A}$);

$$(C1) \ \Phi(A^p) \leq \Phi(A)^p \text{ for } 0 \leq p \leq 1. \quad (C2) \ \Phi(A)^p \leq \Phi(A^p) \text{ for } 1 \leq p \leq 2.$$

The study of positive linear maps is of central importance in several parts of matrix analysis and functional analysis.

J-C.Bourin and E.Ricard show very interesting asymmetric extension of Kadison inequality as follows by using quite ingenious method.

Theorem A (Bourin-Ricard [3]). *Let $A \in \mathcal{A}^+$ and $0 \leq p \leq q$. Then*

$$|\Phi(A^p)\Phi(A^q)| \leq \Phi(A^{p+q}).$$

§1. A result interpolating Theorem A and Choi inequality (C2)

Löwner-Heinz inequality asserts that *If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.* As an extension of Löwner-Heinz inequality, we state the following result to give proofs of our results.

Theorem B.

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

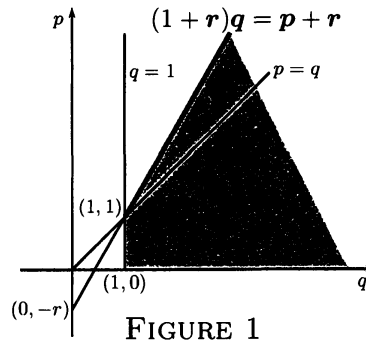


FIGURE 1

The original proof of Theorem B is shown in [6], an elementary one-page proof is in [7] and alternative ones are in [5],[9] and [8]. It is shown in [11] that the conditions p , q and r in FIGURE 1 are best possible.

Theorem 1.1. *Let $A \in \mathcal{A}^+$, (i) $0 \leq p \leq q$ and (ii) $\frac{q}{q+p} \leq r \leq \frac{2q}{q+p}$. Then*

$$(1.0) \quad |\Phi(A^p)^r \Phi(A^q)^r| \leq \Phi(A^{(p+q)r}).$$

Proof.

Put $X = \Phi(A^q)^{\frac{p}{q}}$ and $Y = \Phi(A^p)$. Then $X \geq Y \geq 0$ by Choi (C1). Put $\alpha = 2r \geq 0$ and $\beta = \frac{2qr}{p} \geq 0$. Then $(1+\beta)2 \geq \alpha + \beta$ holds by (i) and (ii), so that (ii) of Theorem B ensures

$$(1.1) \quad \Phi(A^q)^{\frac{p}{q}(\frac{\alpha+\beta}{2})} \geq \left(\Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}} \Phi(A^p)^\alpha \Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}} \right)^{\frac{1}{2}}$$

and (1.1) yields

$$(1.2) \quad \Phi(A^q)^{\frac{(p+q)r}{q}} \geq \left(\Phi(A^q)^r \Phi(A^p)^{2r} \Phi(A^q)^r \right)^{\frac{1}{2}}$$

and

$$(1.3) \quad \Phi(A^{(p+q)r}) \geq \Phi(A^q)^{\frac{(p+q)r}{q}} \quad \text{by Choi (C2) and (ii)}$$

so that we have the desired result (1.0) by (1.2) and (1.3)

$$(1.0) \quad \Phi(A^{(p+q)r}) \geq \left(\Phi(A^q)^r \Phi(A^p)^{2r} \Phi(A^q)^r \right)^{\frac{1}{2}} = |\Phi(A^p)^r \Phi(A^q)^r|. \quad \square$$

Remark 1. Theorem 1.1 implies Theorem A by putting $r = 1$ and also Theorem 1.1 implies Choi inequality (C2) by putting $p = 0$.

Theorem 1.1

$$\begin{array}{ccc} & r = 1 \swarrow & \searrow p = 0 \\ \text{Theorem A} & & \text{Choi inequality (C2)} \end{array}$$

Theorem 1.1 can be extended to the class of positive, sub-unital linear maps. The result also holds in the general setting of positive linear maps between unital C^* -algebra.

Corollary 1.2. Let $A \in \mathcal{A}^+$ and $0 \leq p \leq q$. Then

$$(1.4) \quad |\Phi(A^p)^{\frac{q}{q+p}} \Phi(A^q)^{\frac{q}{q+p}}| \leq \Phi(A^q)$$

and

$$(1.5) \quad |\Phi(A^p)^{\frac{2q}{q+p}} \Phi(A^q)^{\frac{2q}{q+p}}| \leq \Phi(A^{2q}).$$

Proof. Put $r = \frac{q}{q+p}$ and $r = \frac{2q}{q+p}$ in Theorem 1.1 respectively.

§2. Asymmetric variations of $\Phi(A)^{-1} \leq \Phi(A^{-1})$ paralleled to Theorem 1.1

Let $A \in \mathcal{A}^{++}$ be defined by $A \in \mathcal{A}^+$ and A is invertible and let Φ be strictly positive and unital. By the almost similar way to Theorem 1.1, we show the following result.

Theorem 2.1. Let $A \in \mathcal{A}^{++}$, (i) $0 \leq p \leq q$ and (ii) $\frac{q}{q+p} \leq r \leq \frac{2q}{q+p}$. Then

$$(2.0) \quad |\Phi(A^{-p})^{-r} \Phi(A^q)^r| \leq \Phi(A^{(p+q)r}).$$

Proof. Since $f(t) = t^s$ is operator convex for $s \in [-1, 0]$, $\Phi(A)^s \leq \Phi(A^s)$ holds for $A > 0$ and $s \in [-1, 0]$ as stated in Introduction. Put $X = \Phi(A^q)^{\frac{p}{q}}$ and $Y = \Phi(A^{-p})^{-1}$. Then $X \geq Y > 0$. Put $\alpha = 2r \geq 0$ and $\beta = \frac{2qr}{p} \geq 0$. Then $(1 + \beta)2 \geq \alpha + \beta$ by (i) and (ii). so that (ii) of Theorem B ensures

$$(2.1) \quad \Phi(A^q)^{\frac{p}{q}(\frac{\alpha+\beta}{2})} \geq \left(\Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}} \Phi(A^{-p})^{-\alpha} \Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}} \right)^{\frac{1}{2}}$$

and (2.1) yields

$$(2.2) \quad \Phi(A^q)^{\frac{(p+q)r}{q}} \geq \left(\Phi(A^q)^r \Phi(A^{-p})^{-2r} \Phi(A^q)^r \right)^{\frac{1}{2}}$$

and

$$(2.3) \quad \Phi(A^{(p+q)r}) \geq \Phi(A^q)^{\frac{(p+q)r}{q}} \quad \text{by Choi (C2) and (ii)}$$

so that we have the desired (2.0) by (2.2) and (2.3)

$$(2.0) \quad \Phi(A^{(p+q)r}) \geq \left(\Phi(A^q)^r \Phi(A^{-p})^{-2r} \Phi(A^q)^r \right)^{\frac{1}{2}} = |\Phi(A^{-p})^{-r} \Phi(A^q)^r| \quad \square$$

Corollary 2.2. *Let $A \in \mathcal{A}^{++}$ and $0 \leq p \leq q$. Then*

$$(2.4) \quad |\Phi(A^{-p})^{\frac{-q}{q+p}} \Phi(A^q)^{\frac{q}{q+p}}| \leq \Phi(A^q).$$

$$(2.5) \quad |\Phi(A^{-p})^{\frac{-2q}{q+p}} \Phi(A^q)^{\frac{q}{q+p}}| \leq \Phi(A^{2q}).$$

Proof. Put $r = \frac{q}{q+p}$ and $r = \frac{2q}{q+p}$ in Theorem 2.1 respectively. $\square$

Remark 2. Theorem 2.1 interpolating Choi inequality (C2) by putting $p = 0$ and $|\Phi(A^{-p})^{-1} \Phi(A^q)| \leq \Phi(A^{p+q})$ for $0 \leq p \leq q$ by putting $r = 1$.

Theorem 2.1

$$r = 1 \quad \swarrow \quad \searrow \quad p = 0$$

$$|\Phi(A^{-p})^{-1} \Phi(A^q)| \leq \Phi(A^{p+q}). \quad \text{Choi inequality (C2)}$$

The complete form of this talk has been published in the following paper:

T.Furuta, Around Choi inequalities for positive linear maps, Linear Algebra Appl., **434**(2011), 14-17.

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