

FINITE SEMIGROUPS AND DECIDABILITY OF AMALGAMATION BASES FOR SEMIGROUPS*

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In this paper, we prove that the decision problem of whether or not a finite semigroup S is an amalgamation base for all semigroups is decidable.

1 Introduction and preliminaries

In [3], M. Spair investigated problems of amalgams in the class of finite semigroups and showed that it is undecidable whether or not an amalgam of finite semigroups is embedded in a semigroup or a finite semigroup. However it is not known whether or not the problem of decidability of amalgamation bases. In this paper we prove decidability of whether or not a finite semigroup is an amalgamation base for all semigroups.

Let S be a semigroup. Let M be a nonempty set with a unitary and associative operation of $S : S^1 \times M \longrightarrow M ((s, w) \longmapsto sw)$, where S^1 is the monoid obtained from S by adjoining a new identity 1. Then M is called a *left S -set*. Dually, *right S -set* is defined. If M is a left S -set and a right S -set satisfying that $(sm)t = s(mt)$ for all $s, t \in S$ and $m \in M$, then M is called an *S -biset*.

A relation ρ of a left S -set M [resp. right S -set] is called *S -congruence* if $(m, m') \in \rho$ and $s \in S$ implies $(sm, sm') \in \rho$ [resp. $(ms, sm's) \in \rho$]. Let M, N be a right S -sets [left S -set]. Then a map $\phi : M \longrightarrow N$ is called an *S -map* if $\phi(sm) = s\phi(m)$ for any $m \in M$ and $s \in S$ [resp. $\phi(sm) = s\phi(m)$] for any $m \in M$ and $s \in S$.

Result 1 ([5, Proposition 1.5]). *Let S be a semigroup and $A, B, C, D \in S\text{-Ens} [\text{Ens-}S, S\text{-Ens-}S]$ such that $A \supset C, B \supset D$. Let α be a bijective S -map [(S, S) -map] : $C \longrightarrow D$. Then there exist $W \in S\text{-Ens} [\text{Ens-}S, S\text{-Ens-}S]$ and injective S -maps [(S, S) -maps] $\beta : A \longrightarrow W, \lambda : B \longrightarrow W$ such that $\alpha\lambda = \beta$ on $C, W = A\beta \cup B\lambda, A\beta \cap B\lambda = C\beta$.*

*This is an abstract and the paper will appear elsewhere.

In this case, we say that the left S -set $[-\text{biset}]W$ is the left S -set gluing C and D by ξ and write $W = C \#_{\xi} D$, where $\xi = \beta^{-1}\alpha : A \longrightarrow B$.

If A, B are generated by x and y respectively and $\xi(x) = y$, then we write $C \#_{x=y} D$ instead of $C \#_{\xi} D$.

Let Y a left S -set and $s_i, t_i \in S, y_i \in Y$ with $s_i y_i = t_i y_{i+1}$ for all $1 \leq i \leq n-1$. Then for any $1 \leq i \leq n-1$, we define left congruences $\rho(s_i), \rho(t_i)$ on S^1 as follows :

$s\rho(s_i)$ [resp. $\rho(t_i)$] if and only if $ss_i y_i = tt_i y_{i+1}$ in Y for all $s, t \in S$. Let $\bar{Y}_i = S/1_S \cup \rho(s_i) \cup \rho(t_i)$ and $\bar{y}_i = (1_S \cup \rho(s_i) \cup \rho(t_i))1$. $S^1/1 \cup \rho(s) \#_{s\rho(s)=t\rho(t)} S^1/1 \cup \rho(t)$. Then we can obtain a left S -set $\bar{Y}$

$$\bar{Y} = \bar{Y}_1 \#_{s_1 \bar{y}_1 = t_1 \bar{y}_2} \bar{Y}_2 \cdots \#_{\bar{Y}_{i-1} \#_{s_i \bar{y}_i = t_i \bar{y}_{i+1}}} \bar{Y}_{i+1} \# \cdots \#_{s_{n-1} \bar{y}_{n-1} = t_{n-1} \bar{y}_n} \bar{Y}_n.$$

Then we have the set A' of equations $s_i \bar{y}_i = t_{i+1} \bar{y}_{i+1}$ in $\bar{X}$ ($1 \leq i \leq n-1$).

We call $\bar{Y}$ *relatively free* the relatively free left S -set associated to Y with respect to A .

2 The decision problem of amalgamation bases for all semigroups

A semigroup S is called an *amalgamation base* in the class of all semigroups (simply called a *semigroup amalgamation base*) if for any semigroups T_i ($i \in I$) containing S as a subsemigroup the semigroup amalgam $[T_i (i \in I) ; S]$ is embedded into a semigroup.

This is a characterization of semigroup amalgamation bases.

Result 2 [4, Theorem 2.2]. *A semigroup S is a semigroup amalgamation base if and only if for each $X \in \text{Ens-}S, Y \in S\text{-Ens}$ and $N \in S\text{-Ens-}S$ with $N \supset S^1$, the map $: X \otimes Y \longrightarrow X \otimes N \otimes Y$ ($x \otimes y \longrightarrow x \otimes 1 \otimes y$) is injective.*

We recall Bulman-Flemming and MxDowell's characterization of equality in tensor product.

Result 3 [1, Lemma 1.2]. *Let $X \in \text{Ens-}S, Y \in S\text{-Ens}$. Then $x \otimes y = x' \otimes y'$ in $X \otimes Y$ if and only if there exist $s_1, \dots, s_n, t_1, \dots, t_n \in S^1, x_1, \dots, x_n \in X$ and $y_2, \dots, y_n \in Y$*

such that

$$\begin{array}{rcl}
 x & = & x_1 s_1, \quad s_1 y = t_1 y_2 \\
 x_1 t_1 & = & x_2 s_2, \quad s_2 y_2 = t_2 y_3 \\
 & \vdots & \vdots \\
 x_{n-1} t_{n-1} & = & x_n s_n, \quad s_n y_n = t_n y' \\
 x_n t_n & = & x'
 \end{array} \tag{1}$$

Then we call the system of equations (1) a scheme of length n over X and Y joining (x, y) to (x', y') .

The main theorem . *The decision problem whether or not a finite semigroup is an amalgamation base for all semigroups is decidable.*

2.1 Schemes and Automata

We know that a semigroup S is an amalgamation base for all semigroups if and only if so is the semigroup S^1 obtained from S by adjoining an identity element. So we assume that S is a monoid.

In this section we construct an automata associated to an equation on tensor product of a certain right S -set and certain a left S -set in order to complete the proof of Theorem 1.

Let S be a finite monoid.

I. Let $RC(S)$ be the set of all right congruences of S .

Let $\{(\xi, a) \mid \xi \in RC(S), a \in S\}$ be the sets of initial vertices and terminal vertices and $\{(\xi, a, b) \mid \xi \in RC(S), a, b \in S\}$ be the sets of vertices.

Edges are of the form $(\xi, a) \xrightarrow{\theta} (\xi', a', b')$, where θ is an S -isomorphism : $(\xi a)S \rightarrow (\xi' a')S$ with $\theta(\xi a) = \xi' a'$ or of the form $(\xi, a, b) \xrightarrow{\theta} (\xi', a', b')$, where θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$.

II. Let $LC(S)$ be the set of all left congruences of S .

Let $\{(\phi, u) \mid \phi \in LC(S), u \in S\}$ be the sets of initial vertices and terminal vertices and $\{(\phi, u, v) \mid \phi \in LC(S), u, v \in S\}$ be the sets of vertices.

Edges are of the form $(\phi, u) \xrightarrow{\theta} (\phi', u', v')$, where θ is an S -isomorphism : $(\phi u)S \rightarrow (\phi' u')S$ with $\theta(\phi u) = \phi' u'$ or of the form $(\phi, u, v) \xrightarrow{\theta} (\phi', u', v')$, where θ is an S -isomorphism : $S(\phi v) \rightarrow S(\phi' u')$ with $\theta(\phi v) = \phi' u'$.

III. $\{(\xi, a, b, m, \phi, u, v) \mid \xi \in RC(S), a, b \in S, m \in E(S S_S), \phi \in LC(S), u, v \in S\}$

To obtain all sechemes joining $(\xi_0, *, 1, 1, \rho_0, *, u)$ to $(\xi'_0, 1, *, 1, \phi'_0 u', *)$ over right S -sets, the bi- S -sets $E(S S_S)$ and left S -sets, we make a non-deterministic automaton $\mathcal{A}(\xi_0, 1, \phi_0, u,$

$E({}_SSS), \xi'_0, a', \rho'_0, u')$ as follows :

Vertices are of the form $(\xi, a, b, m, \phi, u, v)$ where $\xi \in RC(S), a, b, u, v \in S, m \in E({}_SSS), \phi \in LC(S)$,

$(\xi_0, *, 1, 1, \phi_0, *, v_0)$ is the initial vertex, $(\xi'_0, 1, *, 1, \phi'_0, u_0, *)$ is the terminal vertex, where $\xi_0, \xi'_0 \in RC(S), v \in S$ and $\phi_0, \phi'_0 \in LC(S)$.

Edges are of the form

(1) $(\xi, a, b, m, \phi, u, v) \xrightarrow{\theta} (\xi', a', b', m', \phi, u, v)$, where $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, m, m' \in E({}_SSS), \phi, \phi \in LC(S)$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b') = \xi' a'$ and there exists an element $z \in E({}_SSS)$ with $m = bz$ and $m' = a'z$.

(2) $(\xi, a, b, m, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho', u', v')$, where $\xi, \xi' \in RC(S), a, b, u, u', v, v' \in S, m, m' \in E({}_SSS), \rho, \rho' \in LC(S)$, θ is an S -isomorphism : $S(\phi v) \rightarrow (\phi' u')S$ with $\theta(\phi v) = \phi' u'$ and there exists an element $w \in E({}_SSS)$ with $m = wv$ and $m' = wu'$.

(3) $(\xi_0, *, 1, 1, \rho_0, *, v_0) \xrightarrow{\theta} (\xi, a, b, a, \rho_0, *, v_0)$, where $\xi, \in RC(S), v \in S, m \in E({}_SSS), \phi, \phi \in LC(S)$, θ is an S -isomorphism : $(\xi_0 1)S \rightarrow (\xi a)S$ with $\theta(\xi_0 1) = \xi' a$, where and there exists an element $w \in E({}_SSS)$ with $a = bw$. These edges are labelled by θ .

(4) $(\xi, a, b, m, \rho_0, *, v_0) \xrightarrow{\theta} (\xi', a', b', m', \rho_0, *, v_0)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S, m, m' \in E({}_SSS)$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $z \in E({}_SSS)$ with $m = bz, m' = a'z$. These edges are labelled by θ .

(5) Edges $(\xi, a, b, b, \rho'_0, u_0, *) \xrightarrow{\theta} (\xi', 1, *, 1, \rho'_0, u_0, *)$ are with no label, where $\xi, \xi' \in RC(S), a, b, u \in S, m \in E({}_SSS)$ and there exist elements $z \in E({}_SSS)$ with $b = aw$ and the edge is labelled by θ .

(6) $(\xi, a, b, m, \rho'_0, u'_0, *) \xrightarrow{\theta} (\xi', a', b', m', \rho'_0, *, u'_0, *)$, where $\xi, \xi' \in RC(S), a, b, a', b' \in S, m, m' \in E({}_SSS)$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$, where and there exists an element $z \in E({}_SSS)$ with $m = bz, m' = a'z$. These edges are labelled by θ .

We make an automaton $\mathcal{B}(\xi_0, 1, \phi_0, v, \xi'_0, a', \rho'_0, u')$ as follows :

Vertices are of the form (ξ, a, b, ϕ, u, v) where $\xi \in RC(S), a, b, u, v \in S, m \in E({}_SSS), \phi \in LC(S)$,

$(\xi_0, *, 1, \phi_0, *, v)$ is the initial vertex, $(\xi'_0, 1, *, \phi'_0, u, *)$ is the terminal vertex, where $\xi, \xi' \in RC(S), v \in S$ and $\phi_0, \phi'_0 \in LC(S)$.

Edges are of the form : (1) $(\xi, a, b, \phi, u, v) \xrightarrow{\theta} (\xi', a', b', \phi, u, v)$, where $\xi, \xi' \in RC(S), a, a', b, b', u, v \in S, \phi, \phi \in LC(S)$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b') = \xi' a'$.

(2) $(\xi, a, b, \rho, u, v) \xrightarrow{\theta} (\xi, a, b, m', \rho', u', v')$, where $\xi, \xi' \in RC(S), a, b, u, u', v, v' \in S, m, m' \in$

$E({}_S S_S)$, $\rho, \rho' \in LC(S)$, θ is an S -isomorphism : $S(\phi v) \rightarrow (\phi' u')S$ with $\theta(\phi v) = \phi' u'$.

(3) $(\xi_0, *, 1, \rho_0, *, v_0) \xrightarrow{\theta} (\xi, a, b, a, \rho_0, *, v_0)$, where $\xi, \in RC(S)$, $v \in S$, $\phi, \phi \in LC(S)$, θ is an S -isomorphism : $(\xi_0 1)S \rightarrow (\xi a)S$ with $\theta(\xi_0 1) = \xi' a$ and these edges are labelled by θ .

(4) $(\xi, a, b, \rho_0, *, v_0) \xrightarrow{\theta} (\xi', a', b', \rho_0, *, v_0)$, where $\xi, \xi' \in RC(S)$, $a, b, a', b' \in S$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$.

These edges are labelled by θ .

(5) Edges $(\xi, a, b, \rho'_0, u_0, *) \xrightarrow{\theta} (\xi', 1, *, 1, \rho'_0, u_0, *)$ are with no label, where $\xi, \xi' \in RC(S)$, $a, b, u \in S$ and the edge is labelled by θ .

(6) $(\xi, a, b, \rho'_0, u'_0, *) \xrightarrow{\theta} (\xi', a', b', m', \rho'_0, *, u'_0, *)$, where $\xi, \xi' \in RC(S)$, $a, b, a', b' \in S$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$. These edges are labelled by θ .

(7) $(\xi, a, b; x, \rho, u, v; y) \xrightarrow{\theta} (\xi', a', b'; x', \rho', u, v; y)$, where $\xi, \xi' \in RC(S)$, $a, a', b, b', u, v \in S$, $\rho \in LC(S)$, $x \in (\xi b)S$, $x' \in (\xi' a')S$, $y \in S(\rho u) \cup S(\rho v)$, θ is an S -isomorphism : $(\xi b)S \rightarrow (\xi' a')S$ with $\theta(\xi b) = \xi' a'$ and $\theta(x) = x'$.

(8) $(\xi, a, b; x, \rho, u, v; y) \xrightarrow{\epsilon} (\xi, a, b; x', \rho, u, v; y')$, where $\xi, \xi' \in RC(S)$, $a, b, u, v \in S$, $\rho \in LC(S)$, $x, x' \in (\xi a)S \cup (\xi b)S$, $y, y' \in S(\rho u) \cup S(\rho v)$ and there exists $s \in S$ with $x' = xs$, $y = sy'$.

Lemma 1. Let S be a finite monoid. Let $E({}_S S_S)$ be the injective hull of the S -biset ${}_S S_S$. Let X be a right S -set with an element x, x' and Y be a left S -set with an element y, y' . Let ξ [resp. ξ'] be right congruences on S such that there exists an isomorphism ϕ [resp. ϕ'] of S/ξ to xS with $\theta(\xi 1) = x$ [resp. S/ξ' to $x'S$ with $\theta'(\xi' 1) = x'$]. Let ρ [resp. ρ'] left congruences on S such that there exists an isomorphism γ [resp. γ'] of S/ρ to xS with $\gamma(\rho 1) = y$ [resp. S/ρ' to $x'S$ with $\gamma'(\rho' 1) = y'$].

Then $x \otimes 1 \otimes y' = x' \otimes 1 \otimes y'$ in $X \otimes E({}_S S_S) \otimes Y$ if and only if there exists an element $v, u \in S$ such that there exists a successful pass from the initial vertex $(\xi, 1, 1, \rho, *, v)$ to the terminal vertex $(\xi', 1, 1, \rho', u)$ on the automaton $\mathcal{A}(\xi, \xi', E({}_S S_S), \rho, \rho', v, u)$.

Lemma 2. Let S be a finite semigroup. Let X be a right S -set with an element x, x' and Y be a left S -set with an element y, y' . Let ξ [resp. ξ'] be right congruences on S such that there exists an isomorphism ϕ [resp. ϕ'] of S/ξ to xS with $\theta(\xi 1) = x$ [resp. S/ξ' to $x'S$ with $\theta'(\xi' 1) = x'$]. Let ρ [resp. ρ'] left congruences on S such that there exists an isomorphism γ [resp. γ'] of S/ρ to xS with $\gamma(\rho 1) = y$ [resp. S/ρ' to $x'S$ with $\gamma'(\rho' 1) = y'$]. Let X be the right S -set associated with $\theta_0 \theta_1 \theta_2 \cdots \theta_n$ and Y the left S -set associated with $\gamma_0 \gamma_1 \gamma_2 \cdots \gamma_m$.

Then $x \otimes y' = x' \otimes y'$ in $X \otimes Y$ if and only if there exists a successful pass with the label $(\theta_0\theta_1\theta_2\cdots\theta_n, \gamma_0\gamma_1\gamma_2\cdots\gamma_m)$ from the initial vertex $(\xi, 1, 1, \rho, v)$ to the terminal vertex $(\xi', 1, 1, \rho', u)$ on the automaton $\mathcal{B}(\xi, \xi', \rho, \rho', v, u)$.

Finally, by using Lemma 1 and Lemma 2, we can prove the main theorem.

References

- [1] S. Bulman-Fleming and K. McDowell, *Absolutely flat semigroups*, Pacific J. Math. **107**(1983), 319-333.
- [2] J. Renshaw, *Extension and amalgamation in monoids and semigroups*, Proc. London Math. Soc. (3) **52**(1986), 309-334.
- [3] M. Sapir, *Algorithmic Problem for finite semigroups*, J. Algebra, **229**(2000), 514-531.
- [4] K. Shoji, *Amalgamation bases for semigroups*, Math. Japonica, **26**(1990), 43-53.
- [5] K. Shoji, *Decidability of representation extension property*, Proc. Amer. Math. Soc., **128**(2000), 1313-1317.