

On the Craw–Ishii Conjecture

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1 The slogan

For a finite group $G \subset \mathrm{GL}_n(\mathbb{C})$,

$$\begin{array}{c} \mathbb{C}^n \\ \downarrow \\ Y \longrightarrow \mathbb{C}^n/G, \end{array}$$

let Y be an *interesting* resolution of singularities of $X := \mathbb{C}^n/G$, eg. minimal resolutions, crepant resolutions.

Our goal is to construct Y as a moduli space of G -equivariant sheaves on $\mathbb{C}^n$.

2 G -constellations

Definition 2.1. For a finite group $G \subset \mathrm{GL}_n(\mathbb{C})$, we say that:

- (i) a G -invariant subscheme Z of $\mathbb{C}^n$ is called a **G -cluster** if $O_Z \cong \mathbb{C}[G]$, the regular representation of G ;
- (ii) the **G -Hilbert Scheme** $G\text{-Hilb } \mathbb{C}^n$ is the fine moduli space of G -clusters.

Craw and Ishii generalised this notion.

Definition 2.2. A **G -constellation** is a G -equivariant sheaf $\mathcal{F}$ on $\mathbb{C}^n$ with $H^0(\mathcal{F}) \cong \mathbb{C}[G]$.

The representation ring $R(G)$ of G is $\bigoplus_{\rho \in \mathrm{Irr } G} \mathbb{Z}\rho$. Define the GIT stability parameter space

$$\Theta = \{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \}.$$

Definition 2.3. For $\theta \in \Theta$, we say that:

- (i) a G -constellation $\mathcal{F}$ is **θ -semistable** if $\theta(\mathcal{G}) \geq 0, \forall \mathcal{G} \subset \mathcal{F}$.
- (ii) a G -constellation $\mathcal{F}$ is **θ -stable** if $\theta(\mathcal{G}) > 0, \forall \mathcal{G} \subset \mathcal{F}$.
- (iii) θ is **generic** if every θ -semistable object is θ -stable.

Theorem 2.4 (King). *The moduli space can be constructed by GIT.*

(i) $\exists \mathcal{M}_\theta$ a quasiprojective scheme which is a coarse moduli space of θ -semistable G -constellations up to S -equivalence.

(ii) For generic θ , the scheme $\mathcal{M}_\theta$ is a fine moduli space of θ -stable G -constellations.

(iii) The scheme $\mathcal{M}_\theta$ is projective over $\mathcal{M}_0$.

Remark 2.5. The scheme $\mathcal{M}_\theta$ may not be irreducible.

Observation 2.6 (Ito and Nakajima). For $\theta \in \Theta_+$, the G -Hilbert Scheme $G\text{-Hilb } \mathbb{C}^n$ is canonically $\mathcal{M}_\theta$ where

$$\Theta_+ := \{ \theta \in \Theta \mid \theta(\rho) > 0 \quad \forall \rho \neq \rho_0 \}.$$

Simply the moduli space $\mathcal{M}_\theta$ coincides with $G\text{-Hilb } \mathbb{C}^n$ for a particular choice of GIT parameter θ .

3 Related works

3.1 Surface quotient singularities

For the minimal resolution Y of $\mathbb{C}^2/G$, there are interesting results.

Theorem 3.1 (Ito and Nakamura). For $G \subset \mathrm{SL}_2(\mathbb{C})$, the minimal resolution of $\mathbb{C}^2/G$ is $G\text{-Hilb } \mathbb{C}^2$.

Theorem 3.2 (Ishii, Kido, ...). For small $G \subset \mathrm{GL}_2(\mathbb{C})$, the minimal resolution of $\mathbb{C}^2/G$ is $G\text{-Hilb } \mathbb{C}^2$.

3.2 Gorenstein canonical quotient singularities

For $G \subset \mathrm{SL}_3(\mathbb{C})$,

- (i) the quotient $X = \mathbb{C}^3/G$ is a canonical Gorenstein singularity.
- (ii) X has a crepant resolution (which may not be unique).

Question 3.3. Does a crepant resolution of X have a moduli description?

Theorem 3.4 (Nakamura, Bridgeland, King and Reid). For $G \subset \mathrm{SL}_3(\mathbb{C})$, the G -Hilbert scheme $G\text{-Hilb } \mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. In particular, $G\text{-Hilb } \mathbb{C}^3$ is smooth and irreducible.

Question 3.5. For the other crepant resolutions?!

Theorem 3.6 (Bridgeland, King and Reid, Craw and Ishii). Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite group. For generic $\theta \in \Theta$, $\mathcal{M}_\theta$ is a crepant resolution of $\mathbb{C}^3/G$. In particular, $\mathcal{M}_\theta$ is smooth and irreducible.

Theorem 3.7 (Craw and Ishii). Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite **abelian** group. Then *any projective crepant resolution* of $\mathbb{C}^3/G$ is isomorphic to $\mathcal{M}_\theta$ for some $\theta \in \Theta$, i.e. a moduli space of G -constellations.

Conjecture 3.8 (Craw and Ishii). *Theorem 3.7 is also true for non-abelian cases.*

3.3 The (Generalised) Craw–Ishii Conjecture

For a finite group G in $\mathrm{GL}_d(\mathbb{C})$, $X = \mathbb{C}^d/G$.

Let $\phi: Y \rightarrow X$ be a *relative minimal model* of X , i.e.

- (i) K_Y is ϕ -nef.
- (ii) Y has only $\mathbb{Q}$ -factorial terminal singularities.
- (iii) Y is projective over X .

Conjecture 3.9 (The Craw–Ishii Conjecture). With the notation as above, if Y is smooth, then Y is isomorphic to (an irreducible component of) $\mathcal{M}_\theta$ for some $\theta \in \Theta$.

Note 3.10. This conjecture is a generalised version of the Conjecture 3.8.

4 Main results

The *quotient of type* $\frac{1}{r}(\alpha_1, \alpha_2, \dots, \alpha_n)$ is the cyclic quotient $\mathbb{C}^n/\mu_r$ where μ_r is the group of r th roots of unity and the action is:

$$\mu_r \ni \epsilon: (x_1, x_2, \dots, x_n) \mapsto (\epsilon^{\alpha_1} x_1, \epsilon^{\alpha_2} x_2, \dots, \epsilon^{\alpha_n} x_n).$$

We prove the Craw–Ishii conjecture for some abelian group cases.

Craw–MacLagan–Thomas Theorem

Theorem 4.1 (Craw, MacLagan and Thomas). Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite abelian group and $\theta \in \Theta$ generic. Then $\mathcal{M}_\theta$ has a unique irreducible com-

ponent Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover Y_θ satisfies:

- (i) Y_θ is a not-necessarily-normal toric variety which is birational to $\mathbb{C}^n/G$.
- (ii) Y_θ is projective over $\mathbb{C}^n/G$, which is given by a variation of GIT.

$$\begin{array}{c} Y_\theta \hookrightarrow \mathcal{M}_\theta \\ \downarrow \\ \mathbb{C}^n/G \hookrightarrow \mathcal{M}_0 \end{array}$$

Remark 4.2. We call the unique irreducible component Y_θ of $\mathcal{M}_\theta$ the **birational component**. ♦

Main theorem

Theorem 4.3. For a, b, c with $(a, b) = 1$, let $r = abc + a + b + 1$.

Let G be the group of type $\frac{1}{r}(1, a, b)$. Let X denote the quotient $\mathbb{C}^3/G$. Let $\phi: Y \rightarrow X$ be a relative minimal model of X . Then Y is smooth. Moreover Y is isomorphic to the birational component Y_θ of $\mathcal{M}_\theta$ for some $\theta \in \Theta$.

Conjecture 4.4. The moduli space $\mathcal{M}_\theta$ is irreducible.

The proof uses the following two new notions developed recently:

- (i) G -bricks which gives a local description of $\mathcal{M}_\theta$.
- (ii) round down functions which is compatible with toric star subdivisions.

5 G -bricks

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \dots, \alpha_n)$. Nakamura gave a local description of the birational component of $G\text{-Hilb } \mathbb{C}^n$ with introducing a G -graph which is a monomial basis of G -clusters.

(i) A G -brick is a generalised version of the G -graph.

(ii) Simply a G -brick is a Laurent monomial basis of G -constellations on Y_θ .

For each G -brick Γ , we can associate it with:

- (i) a torus invariant G -constellation $C(\Gamma)$.
- (ii) a semigroup $S(\Gamma)$ in the G -invariant Laurent monomial lattice.
- (iii) an affine toric open set $U(\Gamma) = \mathrm{Spec } \mathbb{C}[S(\Gamma)]$.

Proposition 5.1. Let Γ be a G -brick. Let Y_θ be the birational component in $\mathcal{M}_\theta$. For generic θ , assume that $C(\Gamma)$ is θ -stable. Then there exists an open immersion

$$U(\Gamma) = \mathrm{Spec } \mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta \subset \mathcal{M}_\theta.$$

Theorem 5.2. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite diagonal group and θ a generic GIT parameter for G -constellations. Assume that $\mathfrak{S}$ is the set of all θ -stable G -bricks. The birational component Y_θ of $\mathcal{M}_\theta$ is isomorphic to the set of not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$.

6 The end

In general, $\mathcal{M}_\theta$ can be very singular, eg. it doesn't need to be normal.