

Mirror symmetry of nonabelian Landau-Ginzburg orbifolds with loop type potentials

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ABSTRACT. We study the bigraded vector space structure of Landau-Ginzburg orbifolds with the permutation symmetries of variables. We calculate the formula of the Poincaré polynomial for a certain loop polynomial orbifolded by a semidirect product of a diagonal symmetry group and cyclic permutation group and check that the generalization of Berglund-Hübsch transpose gives mirror symmetric pairs for these cases.

1. Introduction

The Landau-Ginzburg mirror symmetry conjecture predicts that for a potential polynomial W with its symmetry group G , there exist the dual potential $\tilde{W}$ and the dual symmetry group $\tilde{G}$ such that the Landau-Ginzburg A-model of (W, G) is isomorphic to the Landau-Ginzburg B-model of $(\tilde{W}, \tilde{G})$. Historically, this conjecture was studied by physicists to verify mirror symmetry of Calabi-Yau manifolds using Calabi-Yau/Landau-Ginzburg correspondence conjecture. The Landau-Ginzburg mirror symmetry is more general because not every Landau-Ginzburg theory has a corresponding geometric object.

A potential polynomial is called invertible if the number of monomials equals the number of variables. For an invertible polynomial W and its diagonal symmetry group G , Krawitz [6] suggested the method to construct the dual pair (W^T, G^T) based on the Berglund-Hübsch transposed potential W^T and proved Landau-Ginzburg mirror symmetry conjecture at the level of bigraded vector spaces.

Compared to the diagonal case such as above, there are not too many efforts on the case where a symmetry group is nonabelian. In [8], the Poincaré polynomial of the Landau-Ginzburg orbifolds for a quintic potential and its certain symmetry group which is possibly nonabelian is computed. There, in addition to mirror pairs predicted by the result of Krawitz, one can find several nonabelian mirror symmetric pairs. In [1], Ebeling and Gusein-Zade suggested the method which generalize Krawitz's construction of mirror pair to the case having the symmetry group of the form $G \rtimes S$ where G is a diagonal symmetry group and S is a permutation group of variables preserving W . They conjectured, and proved for certain cases [2, 3], that $(W, G \rtimes S)$ and $(W^T, G^T \rtimes S)$ have the same orbifold Euler characteristics of the Milnor fiber up to sign if S satisfies a condition called the parity condition.

We try to verify that there is the mirror symmetric relation between Landau-Ginzburg models of $(W, G \rtimes S)$ and $(W^T, G^T \rtimes S)$. Priddis, Ward and Williams [9] studied mirror maps of such pairs for Fermat type potentials and constructed mirror isomorphism between some kind of subspaces of A- and B-model. In

this article, we consider the potential $W = x_1^{p-1}x_2 + \cdots + x_p^{p-1}x_1$ where p is an odd prime. W is invariant under cyclic permutations and the group S generated by such permutations satisfies the parity condition. We prove a formula of the Poincaré polynomial for $(W, G \rtimes S)$ from which we can observe mirror symmetry at the level of bigraded vector spaces.

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2. Landau-Ginzburg Orbifolds

Let us recall some basic facts about Landau-Ginzburg orbifolds. See [8] and references therein for more details.

Definition 2.1. Let W be a polynomial in $\mathbb{C}[x_1, \dots, x_n]$ with a critical point at the origin. The *Jacobi ring* $\mathcal{Q}_W$ of W is defined by the quotient

$$\mathcal{Q}_W = \mathbb{C}[x_1, \dots, x_n]/I_{\nabla W}$$

where $I_{\nabla W}$ is the gradient ideal

$$I_{\nabla W} = \left(\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_n} \right).$$

The critical point at the origin is said to be *isolated* if $\mathcal{Q}_W$ is finite dimensional as a vector space over $\mathbb{C}$.

Definition 2.2. Let W be a polynomial in $\mathbb{C}[x_1, \dots, x_n]$. W is said to be *quasihomogeneous* if there exist $q_1, \dots, q_n \in \mathbb{Q}_{>0}$ which satisfy

$$W(\lambda^{q_1}x_1, \dots, \lambda^{q_n}x_n) = \lambda W(x_1, \dots, x_n)$$

for all $\lambda \in \mathbb{C}$. q_i is called the *charge* of x_i and

$$\hat{c} = \sum_{i=1}^n (1 - 2q_i)$$

is called the *central charge*. A quasihomogeneous polynomial with an isolated critical point at the origin is called *nondegenerate*.

In what follows, let W be a quasihomogeneous nondegenerate polynomial.

Let V be an n -dimensional vector space over $\mathbb{C}$ with a basis $x_1, \dots, x_n$. We define the action of $GL_n(\mathbb{C})$ on V by

$$g \cdot x_i = \sum_{j=1}^n g_{ij}x_j$$

for $g = (g_{ij}) \in GL_n(\mathbb{C})$ and extending linearly. This action induces the action on $\mathbb{C}[x_1, \dots, x_n]$ by

$$(g \cdot f)(x_1, \dots, x_n) = f(g \cdot x_1, \dots, g \cdot x_n).$$

Definition 2.3. The *maximal symmetry group* G_W and *maximal diagonal symmetry group* G_W^{diag} of W are defined by

$$G_W = \{ g \in GL_n(\mathbb{C}) \mid g \cdot W = W, g_{ij} = 0 \text{ if } q_i \neq q_j \},$$

and

$$G_W^{\text{diag}} = \{ g \in G_W \mid g \text{ is a diagonal matrix} \}.$$

The *age* of $g \in G_W$ is defined by

$$\text{age}(g) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n \log \lambda_i$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of g and the branch of the logarithmic function is chosen to satisfy $0 \leq \frac{1}{2\pi\sqrt{-1}} \log z < 1$ for $|z| = 1$.

Note that $\text{age}(g) + \text{age}(g^{-1}) = n - n_g$ where $g \in G_W$ and n_g is the multiplicity of eigenvalue 1 of g . In fact, if λ is an eigenvalue of g , λ^{-1} is an eigenvalue of g^{-1} . One has $\frac{1}{2\pi\sqrt{-1}}(\log \lambda + \log \lambda^{-1})$ equals 0 if $\lambda = 1$ and 1 otherwise, so $\text{age}(g) + \text{age}(g^{-1})$ is equal to the number of eigenvalues other than 1.

In what follows, let G be a subgroup of G_W .

G_W is not trivial since it contains the *exponential grading operator*

$$J := \text{diag} \left(e^{2\pi\sqrt{-1}q_1}, \dots, e^{2\pi\sqrt{-1}q_n} \right).$$

J is an element of G_W^{diag} and, in particular, the center of G_W . J is contained in $SL_n(\mathbb{C})$ if and only if the (generalized) Calabi-Yau condition

$$\sum_{i=1}^n q_i \in \mathbb{Z}$$

is satisfied.

Let $y_1, \dots, y_n$ be eigenvectors of g which correspond to the eigenvalues $\lambda_1^g, \dots, \lambda_n^g$ of g . Since $g_{ij} = 0$ if $q_i \neq q_j$, we can assume that y_i is a linear combination of x_j 's which have the charge q_i . Let W' be a polynomial satisfying $W'(y_1, \dots, y_n) = W(x_1, \dots, x_n)$. Put $I^g = \{ 1 \leq i \leq n \mid \lambda_i^g = 1 \}$ so $n_g = \#I^g$. We define a polynomial W^g in $R^g := \mathbb{C}[y_i \mid i \in I^g]$ by $W^g(y_{i_1}, \dots, y_{i_{n_g}}) = W'(y_1, \dots, y_n)|_{R^g}$, that is, setting $y_i = 0$ if $i \notin I^g$.

Let Ω_g be a one dimensional vector space generated by a symbol $dy_{i_1} \wedge \dots \wedge dy_{i_{n_g}}$ and define the action of $h \in C_G(g)$ on Ω_g by

$$h \cdot (dy_{i_1} \wedge \dots \wedge dy_{i_{n_g}}) = \det(h|_{\text{End}(\text{Ker}(E_n - g))})(dy_{i_1} \wedge \dots \wedge dy_{i_{n_g}}).$$

Definition 2.4. Let C be a set of representatives of conjugacy classes of G . The (state space of) Landau-Ginzburg orbifold $\mathcal{H}_{W,G}$ for a pair (W, G) is defined by

$$\mathcal{H}_{W,G} = \bigoplus_{g \in C} (R^g / I_{\nabla W^g} \otimes \Omega_g)^{C_G(g)}.$$

The left and right charge of $f \otimes v \in R^g / I_{\nabla W^g} \otimes \Omega_g$ are defined by $\text{charge}(f) + \text{age}(g) - \sum_{i \notin I^g} q_i$ and $\text{charge}(f) + \text{age}(g^{-1}) - \sum_{i \notin I^g} q_i$ respectively where $\text{charge}(f) = \sum_{i \in I^g} \alpha_i q_i$ for $f = \sum_{i \in I^g} y_i^{\alpha_i}$. The dimension of the homogeneous subspace with the left and right charge (p, q) is called the Hodge number $h^{p,q}(W, G)$. The generating function

$$P_{W,G}(t, \bar{t}) = \sum_{p,q \in \mathbb{Q}} h^{p,q}(W, G) t^p \bar{t}^q$$

of the Hodge numbers is called the *Poincaré polynomial*.

Proposition 2.5 ([8]). *The Poincaré polynomial of the pair (W, G) is given by*

$$P_{W,G}(t, \bar{t}) = \sum_{g \in C} P_{W,G,g}(t, \bar{t})$$

where

$$P_{W,G,g}(t, \bar{t}) = \frac{1}{\#C_G(g)} t^{\text{age}(g)} \bar{t}^{\text{age}(g^{-1})} (t\bar{t})^{\sum_{i \in I^g} q_i} \sum_{h \in C_G(g)} \prod_{i \in I^g} \frac{\lambda_i^{h,g} - (t\bar{t})^{1-q_i}}{1 - \lambda_i^{h,g} (t\bar{t})^{q_i}}.$$

Here, $\lambda_1^{h,g}, \dots, \lambda_n^{h,g}$ are the eigenvalues of h which are chosen so that the eigenvector corresponding to $\lambda_i^{h,g}$ belongs to the eigenspace of λ_i^g .

3. Mirror Symmetry of Invertible Polynomials

In this section, we review mirror symmetry of invertible polynomials by Krawitz. See [6] and references therein for more details.

A quasihomogeneous nondegenerate polynomial $W = \sum_{i=1}^m c_i \prod_{j=1}^n x_j^{a_{ij}}$ is *invertible* if the number of monomials equals the number of variables, that is, $m = n$ and $A := (a_{ij})$ is invertible. In what follows, we assume $c_i = 1$ without loss of generality rescaling the variables if necessary.

Theorem 3.1 (Kreuzer-Skarke [7]). *Any invertible polynomial is a decoupled sum of one of the following atomic type polynomials:*

Fermat type : x^a ,

Chain type : $x_1^{a_1} x_2 + \dots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$ ($n \geq 2$),

Loop type : $x_1^{a_1} x_2 + \dots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$ ($n \geq 2$).

Let ρ_{ij} be the (i, j) -component of A^{-1} . Then G_W^{diag} is generated by

$$g_i = \text{diag}(e^{2\pi\sqrt{-1}\rho_{i1}}, \dots, e^{2\pi\sqrt{-1}\rho_{in}}).$$

In fact, if $h = \text{diag}(e^{2\pi\sqrt{-1}h_1}, \dots, e^{2\pi\sqrt{-1}h_n}) \in G_W^{\text{diag}}$ then we see

$$A \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathbb{Z}^n.$$

So $(h_1, \dots, h_n)^T$ is a $\mathbb{Z}$ -linear combination of columns of A^{-1} .

Definition 3.2. Let W be an invertible polynomial and G be its diagonal symmetry group. The *Berglund-Hübsch transpose* (W^T, G^T) of (W, G) is defined by

$$W^T(y_1, \dots, y_n) = \sum_{i=1}^n \prod_{j=1}^n y_j^{a_{ji}}$$

$$G^T = \left\{ \prod_{i=1}^n \bar{g}_i^{r_i} \in G_{W^T}^{\text{diag}} \mid (r_1, \dots, r_n) A^{-1} (\alpha_1, \dots, \alpha_n)^T \in \mathbb{Z} \text{ for all } \prod_{i=1}^n g_i^{\alpha_i} \in G \right\}$$

where $\bar{g}_i = \text{diag}(e^{2\pi\sqrt{-1}\rho_{1i}}, \dots, e^{2\pi\sqrt{-1}\rho_{ni}})$.

Note that $(W^T)^T = W$ and $(G^T)^T = G$. Besides, $J \in G$ if and only if $G^T \subset SL_n(\mathbb{C})$.

Theorem 3.3 (Krawitz [6]). *Let W be an invertible polynomial and G be a admissible group of diagonal symmetries of W . There is a bigraded vector space isomorphism between the FJRW ring of (W, G) and the Landau-Ginzburg orbifold of (W^T, G^T) .*

See [4] for details of FJRW rings. The FJRW ring of (W, G) is isomorphic to the Landau-Ginzburg orbifold of (W, G) as a vector space, but the definition of the right charge is different. The right charge is defined as $n_g - \text{charge}(g) + \text{age}(g) - 2 \sum_{i \in I^s} q_i - \sum_{i \notin I^s} q_i$ for $f \otimes v \in (R^s/I_{\nabla W^s} \otimes \Omega_g)$. Using the fact that G is admissible if $J \in G$, we see the following corollary.

Corollary 3.4. *Let W be an invertible polynomial and G be a group of diagonal symmetries of W containing J . Then we have*

$$P_{W^T, G^T}(t, \bar{t}) = \bar{t}^c P_{W, G}(t, \bar{t}^{-1}).$$

In [1], the conjectural method to find mirror pairs based on the Berglund-Hübsch transpose is suggested. This method is applicable if G is a semidirect product $G^{\text{diag}} \rtimes S$ of a diagonal symmetry group $G^{\text{diag}} \subset G_W^{\text{diag}}$ and a subgroup S of the group of permutations. Then the mirror pair is given by $(W^T, (G^{\text{diag}})^T \rtimes S)$. The data in [8] shows that this method finds mirror pairs in some cases, but fails in some other cases. Ebeling and Gusein-Zade conjectured that this method gives mirror symmetric pairs if S satisfies a certain condition, called the *parity condition*, and presented its evidence by proving duality of the orbifold Euler characteristics of Milnor fibers associated to $(W, G \rtimes S)$ [2, 3].

In Section 5, we focus on the case of $W = x_1^{p-1}x_2 + \cdots + x_p^{p-1}x_1$ where p is an odd prime. W has symmetry group S of cyclic permutation of variables. S satisfies parity condition, so it is natural to expect $(W, G \rtimes S)$ and $(W^T, G^T \rtimes S)$ to be a mirror pair.

$G \rtimes S$ is some kind of a semidirect product of two cyclic groups also known as *split metacyclic groups*. In the next section, we calculate conjugacy classes of split metacyclic groups isomorphic to $G \rtimes S$.

4. Split Metacyclic Groups

A finite group G is called *metacyclic* if there exists a cyclic normal subgroup $N \triangleleft G$ such that G/N is also cyclic [5]. Then G can be written $G = SN$ where $S \subset G$ and $N \triangleleft G$. Such a factorization is called a *metacyclic factorization*. If $S \cap N = 1$, this metacyclic factorization is called *split* and G is called *split metacyclic*.

It is known that a group is metacyclic if and only if it has a presentation of the form

$$G_{q,p,\phi,r} := \langle a, b \mid a^q = 1, b^p = a^r, bab^{-1} = a^\phi \rangle$$

where $q, p \in \mathbb{Z}_{>0}$, $\phi, r \in \mathbb{Z}$, $(q, \phi) = 1$, $q \mid \phi^p - 1$ and $q \mid r(\phi - 1)$. Note that $G_{q,p,\phi} := G_{q,p,\phi,0}$ is split metacyclic.

Consider the split metacyclic groups $G_{p^e q, p, \phi}$ where $(p, q) = (\phi - 1, q) = 1$ and $0 \leq e \leq 2$. Then q satisfies the following lemma.

Lemma 4.1. *Let p be a prime and let q, ϕ satisfy $\phi^p \equiv 1 \pmod{q}$ and $(p, q) = (\phi - 1, q) = 1$. Then every prime factor q' of q satisfies $p \mid q' - 1$.*

Proof. Let $q = \prod_i q_i^{e_i}$ be the prime factorization of q . ϕ generates the subgroup of order p in $\mathbb{Z}_q^\times$. Since $(\phi - 1, q_i) = 1$ for all q_i , images of this group of the

isomorphism

$$\mathbb{Z}_q^\times \xrightarrow{\sim} \prod_i \mathbb{Z}_{q_i}^{\times e_i}$$

followed by projections should be a group with order p . Since $q_i \neq p$ for all q_i and $\#\mathbb{Z}_{q_i}^{\times e_i} = q_i^{e_i-1}(q_i - 1)$ must be a multiple of p , we have $p \mid q_i - 1$. $\square$

Let us calculate conjugacy classes and centralizers of $G_{p^e q, p, \phi}$. If $e = 1$, one has $\phi \equiv 1 \pmod{p}$ by Fermat's little theorem, so $ba^q b^{-1} = a^{q\phi} = a^q$, i.e, a^q is a central element. Hence $G_{pq, q, \phi} = \langle a^q, a^p, b \rangle = \langle a^q \rangle \times \langle a^p, b \rangle \cong \mathbb{Z}_p \times G_{q, p, \phi}$. Similarly, if $e = 2$ and $\phi \equiv 1 \pmod{p^2}$, one has $G_{p^2 q, p, \phi} = \langle a^q, a^{p^2}, b \rangle \cong \mathbb{Z}_{p^2} \times G_{q, p, \phi}$. The conjugacy classes of these two cases could be calculated from those of $G_{q, p, \phi}$.

Lemma 4.2. *Let p be a prime and q, ϕ be positive integers satisfying $(p, q) = (\phi - 1, q) = 1$, $\phi^p \equiv 1 \pmod{q}$. Then $\phi^i - 1$ is coprime to q and q divides*

$$\sum_{j=0}^{p-1} \phi^{ij}$$

for $i = 1, \dots, p-1$.

Proof. When $q = 1$ the statement is trivial. Assume $q \neq 1$. Let l be a minimum positive integer satisfying $(\phi^l - 1, q) \neq 1$. By the assumptions, l satisfies $2 \leq l \leq p$. Since $\phi - 1$ is coprime to q , we have $(\sum_{i=0}^{l-1} \phi^i, q) \neq 1$ and $\sum_{i=0}^{p-1} \phi^i \equiv 0 \pmod{q}$. Put $p = ml + n$, $n = 0, 1, \dots, l-1$.

$$\sum_{i=0}^{p-1} \phi^i = \left(\sum_{i=0}^{m-1} \phi^{il} \right) \left(\sum_{j=0}^{l-1} \phi^j \right) + \phi^{ml} \sum_{i=0}^{n-1} \phi^i.$$

The both sides must be divided by $(\phi^l - 1, q)$, so does $\sum_{i=0}^{n-1} \phi^i$. By the minimality of l we have $n = 0$ and by the primality of p we have $l = p$.

By the assumption we have

$$\phi^{ip} - 1 = (\phi^i - 1) \sum_{j=0}^{p-1} \phi^{ij} \equiv 0 \pmod{q}.$$

From the discussion above, $\phi^i - 1$ is coprime to q . This completes the proof. $\square$

Lemma 4.3. *Let $G = G_{q, p, \phi}$ be a split metacyclic group where p is a prime and $(p, q) = (\phi - 1, q) = 1$. The conjugacy classes and centralizers of G are as follows:*

- A class $[b^i]$ of length q for $i = 1, \dots, p-1$ with the centralizer $\langle b \rangle \cong \mathbb{Z}_p$,
- A class $[a^i]$ of length p for every coset representative $i \in (\mathbb{Z}_q \setminus \{0\}) / \langle \phi \rangle$ with the centralizer $\langle a \rangle \cong \mathbb{Z}_q$, and
- The class of the identity element.

Proof. The conjugation of a general element $a^u b^v$ is

$$a^x b^y a^u b^v b^{-y} a^{-x} = a^{u\phi^y - x(\phi^v - 1)} b^v.$$

First take $v \neq 0$. Since by Lemma 4.2 $\phi^v - 1$ is invertible modulo q , we can choose x so that $u\phi^y - x(\phi^v - 1) \equiv 0 \pmod{q}$. Then the element $a^u b^v$ is conjugate

to b^v . This conjugacy class has q elements. The centralizer should have the order p , so this is generated by b .

Next take $v = 0$ and $u \neq 0$. Then a^u is conjugate to $a^{u\phi^j}$, so a conjugacy class is $\{a^u, a^{u\phi}, \dots, a^{u\phi^{p-1}}\}$. If $a^{u\phi^i} = a^{u\phi^j}$ where $0 \leq i \leq j \leq p-1$, then $u\phi^i(\phi^{j-i} - 1) \equiv 0 \pmod{q}$. Again by Lemma 4.2, we have $i = j$. So, the conjugacy class containing a^u has p elements. The centralizer should have the order q , so this is generated by a .

Finally, $v = 0$ and $u = 0$ gives the identity element of G . This completes the proof. $\square$

Lemma 4.4. *Let p be an odd positive integer. If $\phi \in \mathbb{Z}$ satisfies $\phi \equiv 1 \pmod{p}$, then*

$$\sum_{i=0}^{p-1} \phi^{ji}$$

is divisible by p but not by p^2 for all $j \geq 0$.

Proof. By the assumption, we can put $\phi = np + 1$, so

$$\begin{aligned} \sum_{i=0}^{p-1} \phi^{ji} &= \sum_{i=0}^{p-1} (np + 1)^{ij} = \sum_{i=0}^{p-1} \sum_{k=0}^{ij} \binom{ij}{k} (np)^k \\ &\equiv \sum_{i=0}^{p-1} (ijn + 1) = jnp^2 \frac{p-1}{2} + p \\ &\equiv p \pmod{p^2}. \end{aligned}$$

$\square$

Lemma 4.5. *Let $G = G_{p^2q, p, \phi}$ be a split metacyclic group where p is an odd prime, $(p, q) = (\phi - 1, q) = 1$, $\phi \not\equiv 1 \pmod{p^2}$. The conjugacy classes and centralizer of G are as follows:*

- A class $[b^j]$ of length pq for $j = 1, \dots, p-1$ with the centralizer $\langle a^{pq}, b \rangle \cong \mathbb{Z}_p^2$,
- A class $[a^i b^j]$ of length pq for $i = 1, \dots, p-1$ and $j = 1, \dots, p-1$ with the centralizer $\langle a^i b^j \rangle \cong \mathbb{Z}_{p^2}$,
- A class $[a^i]$ of length p for every coset representative $i \in (\mathbb{Z}_{p^2q} \setminus pq\mathbb{Z}_{p^2q}) / \langle \phi \rangle$ with the centralizer $\langle a \rangle \cong \mathbb{Z}_{p^2q}$, and
- A central class $[a^{pqi}]$ for $i = 0, \dots, p-1$.

Proof. The conjugation of a general element $a^u b^v$ is

$$a^x b^y a^u b^v b^{-y} a^{-x} = a^{u\phi^y - x(\phi^v - 1)} b^v.$$

First take $v \neq 0$. $\phi^v - 1$ is divisible by p but not by p^2 . Hence $(\phi^v - 1)/p$ is coprime to p . Besides, $\phi^v - 1$ is coprime to q by Lemma 4.2, so $(\phi^v - 1)/p$ is coprime to pq and invertible modulo pq . Choose x so that it satisfies $x(\phi^v - 1)/p \equiv 1 \pmod{pq}$. Then setting $y = 0$ we have

$$u\phi^y - x(\phi^v - 1) \equiv u - p \pmod{p^2q}.$$

This shows that the element $a^u b^v$ is conjugate to $a^i b^v$ where $i = u \pmod{p}$ and the corresponding conjugacy class has pq elements. The centralizer has order p^2 and

contains a^{pq} and $a^i b^j$. For $i = 0$, the centralizer is clearly $\langle a^{pq}, b \rangle \cong \mathbb{Z}_p^2$. For $i \neq 0$, we have

$$(a^i b^j)^p = a^{i \sum_{k=0}^{p-1} \phi^{jk}} b^{pj} = a^{i \sum_{k=0}^{p-1} \phi^{jk}}.$$

$\sum_{k=0}^{p-1} \phi^{jk}$ is divisible by pq but not by p^2 from Lemma 4.2 and Lemma 4.4. Hence $(a^i b^j)^p$ is not the identity and contained in the group generated by a^{pq} . This shows that the centralizer is the cyclic group generated by $a^i b^j$ with order p^2 .

Next take $v = 0$ and $pq \nmid u$. Then a^u is conjugate to $a^{u\phi^y}$, so a conjugacy class is $\{a^u, a^{u\phi}, \dots, a^{u\phi^{p-1}}\}$. Assume $a^{u\phi^i} = a^{u\phi^j}$ where $0 \leq i \leq j \leq p-1$, then $u\phi^i(\phi^{j-i} - 1) \equiv 0 \pmod{p^2q}$. If $p \mid u$, then $q \nmid u$, so we need $\phi^{j-i} \equiv 1 \pmod{q}$. This is satisfied if and only if $i = j$ by Lemma 4.2. On the other hand, if $p \nmid u$, we need $\phi^{j-i} - 1 \equiv 0 \pmod{p^2}$ which implies $i = j$ from $\phi \not\equiv 1 \pmod{p^2}$. So, the conjugacy class containing a^u has p elements. The centralizer should have the order p^2q , so this is generated by a .

Finally, $v = 0$ and $pq \mid u$ gives the central elements of G . This completes the proof. $\square$

Remark 4.6. In [10], it was shown that the number of conjugacy classes of $G_{q,p,\phi,r}$ is given by

$$p \sum_{d|q} \frac{\varphi(d)}{u_d^2}$$

where φ denotes Euler's totient function and u_d is the order of ϕ in $\mathbb{Z}_d^\times$. For the situation of Lemma 4.3 for $d|p^e q$, ϕ has order 1 if $d = 1$ and order p otherwise. Hence

$$\begin{aligned} p \sum_{d|q} \frac{\varphi(d)}{u_d^2} &= p \left(\sum_{d|q} \frac{\varphi(d)}{p^2} + \left(1 - \frac{1}{p^2}\right) \varphi(1) \right) \\ &= \frac{q-1}{p} + p. \end{aligned}$$

Similarly, for the situation of Lemma 4.5, ϕ has order 1 if $d = 1, p$ and order p otherwise. Hence we have

$$p \sum_{d|p^2q} \frac{\varphi(d)}{u_d^2} = pq + p^2 - 1.$$

These values agree with what we expected from Lemma 4.3 and Lemma 4.5.

5. Main Theorem

Let p be an odd prime. In this section, we assume $W = x_1^{p-1}x_2 + \dots + x_p^{p-1}x_1$, $G \subset G_W^{\text{diag}}$ and S is the cyclic group generated by the cyclic permutation of variables $\sigma : x_i \mapsto x_{i+1}$ where the index is considered modulo p . Every variable x_i has charge $\frac{1}{p}$, so $\hat{c} = p - 2$. G_W^{diag} is generated by

$$\text{diag} \left(\zeta_\alpha, \dots, \zeta_\alpha^{(-p+1)^{p-1}} \right)$$

where $\alpha := \#G_W^{\text{diag}} = (p-1)^p + 1$ and $\zeta_\alpha = e^{\frac{2\pi\sqrt{-1}}{\alpha}}$, whose determinant is ζ_p . Note that α is divisible by p^2 but not by p^3 . G is a subgroup of $SL_p(\mathbb{C})$ if and only if

$\#G$ is not a multiple of p^2 . Since the exponential grading operator J has order p , $J \in G$ if and only if $p \mid \#G$. In particular $J \notin G$ implies $G \subset SL_p(\mathbb{C})$.

Let q be the greatest divisor of $\#G$ which is coprime to p , i.e., $\#G = p^e q$ for $e = 0, 1, 2$. W is invariant under the action of the group generated by G and S . This group is semidirect product $G \rtimes S$, and has a presentation

$$\langle g, \sigma \mid g^{p^e q} = 1, \sigma^p = 1, \sigma g \sigma^{-1} = g^{-p+1} \rangle,$$

which shows that $G \rtimes S \cong G_{p^e q, q, -p+1}$. In what follows, we fix g by

$$g = \text{diag} \left(\zeta_{p^e q}, \dots, \zeta_{p^e q}^{(-p+1)^{p-1}} \right).$$

Since $W^T = y_1^{p-1} y_p + y_2^{p-1} y_1 + \dots + y_p^{p-1} y_{p-1}$, W and W^T coincide up to change of indexes. Note that at least one of G and G^T contains the exponential grading operator. In fact, if $J \notin G$, then $G^T \not\subset SL_p(\mathbb{C})$, so G^T contains the exponential grading operator by the discussion above.

Let us define $P_{W, G \rtimes S}^J(t, \bar{t})$ and $P_{W, G \rtimes S}^{SL}(t, \bar{t})$ as follows:

$$P_{W, G \rtimes S}^J(t, \bar{t}) = \begin{cases} \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1} & (J \in G) \\ (p-1) \sum_{i=1}^{p-1} (t\bar{t})^{\frac{i}{p} + \frac{p-1}{2} - 1} & (\text{otherwise}) \end{cases}$$

$$P_{W, G \rtimes S}^{SL}(t, \bar{t}) = \begin{cases} \frac{p-1}{p} \sum_{i=1}^{p-1} (t\bar{t})^{i-1} & (G \subset SL_p(\mathbb{C})) \\ (p-1) \sum_{i=1}^{p-1} t^{\frac{i}{p} + \frac{p-1}{2} - 1} \bar{t}^{-\frac{i}{p} + \frac{p+1}{2} - 1} & (\text{otherwise}) \end{cases}$$

By some calculation, we can show the following lemma.

Lemma 5.1. *The following relation holds:*

$$P_{W^T, G^T \rtimes S}^J(t, \bar{t}) = \bar{t}^{\hat{c}} P_{W, G \rtimes S}^{SL}(t, \bar{t}^{-1}).$$

Lemma 5.2. *Let $h = \text{diag}(h_1, \dots, h_p) \in GL_p(\mathbb{C})$ and $j = 1, \dots, p-1$. Every root of $X^p - \det h = 0$ is an eigenvalue of $h\sigma^j$.*

Proof. Let r be a root of $X^p - \det h = 0$. Define α_i^r recursively for $i = 1, \dots, p$ by

$$\begin{cases} \alpha_1^r = 1, \\ \alpha_i^r = \frac{1}{r} \alpha_{i-j}^r h_i. \end{cases}$$

where the index is considered modulo p . α_i^r is well-defined since j is coprime to q hence there exists k which satisfies $i = 1 + jk$. The latter relation is also valid for $i = 1$ by the definition of r .

Then the vector $v = \sum_{i=1}^p \alpha_i^r x_i$ is the eigenvector of $h\sigma^i$ with the eigenvalue r :

$$\begin{aligned} h\sigma^j \cdot v &= h \cdot \left(\sum_{i=1}^p \alpha_i^r x_{i+j} \right) = \sum_{i=1}^p \alpha_i^r h_{i+j} x_{i+j} \\ &= \sum_{i=1}^p r \alpha_{i+j}^r x_{i+j} = rv. \end{aligned}$$

□

Lemma 5.3. *The following relation holds:*

$$P_{W,G \rtimes S,1}(t, \bar{t}) = \begin{cases} \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} (t\bar{t})^i & (e = 0, 1), \\ \frac{1}{p} P_{W,G,1}(t, \bar{t}) & (e = 2). \end{cases}$$

Proof. By Lemma 5.2, for $j \neq 0$ the eigenvalues of $g^i \sigma^j$ are roots of $X^p - \det g^i = 0$, that is, $\zeta_p^k (\det g^i)^{\frac{1}{p}}$ where $k = 0, \dots, p-1$. One can see

$$\begin{aligned} P_{W,G \rtimes S,1}(t, \bar{t}) &= \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{1}{p^{e+1}q} \sum_{\substack{g^i \sigma^j \\ j \neq 0}}^{p-1} \prod_{k=0}^{p-1} \frac{\zeta_p^k (\det g^i)^{\frac{1}{p}} - (t\bar{t})^{\frac{p-1}{p}}}{1 - \zeta_p^k (\det g^i)^{\frac{1}{p}} (t\bar{t})^{\frac{1}{p}}} \\ &= \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{1}{p^{e+1}q} \sum_{\substack{g^i \sigma^j \\ j \neq 0}} \frac{1}{\det g^i} \frac{(t\bar{t})^{p-1} - \det g^i}{t\bar{t} - \det g^i}. \end{aligned}$$

If $e = 0$ or $e = 1$, $G \subset SL_p(\mathbb{C})$. Then the summand of the second term is

$$\frac{(t\bar{t})^{p-1} - 1}{t\bar{t} - 1} = \sum_{k=1}^{p-1} (t\bar{t})^{k-1},$$

hence

$$\begin{aligned} P_{W,G \rtimes S,1}(t, \bar{t}) &= \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{1}{p^{e+1}q} \sum_{\substack{h=g^i \sigma^j \\ j \neq 0}}^{p-1} \sum_{k=1}^{p-1} (t\bar{t})^{k-1} \\ &= \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{1}{p^{e+1}q} p^e q (p-1) \sum_{k=1}^{p-1} (t\bar{t})^{k-1} \\ &= \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{p-1}{p} \sum_{k=1}^{p-1} (t\bar{t})^{k-1}. \end{aligned}$$

On the other hand, if $e = 2$, one can show $\det g^i = \zeta_p^i$ because one has

$$\det g = \prod_{j=0}^{p-1} \zeta_{p^2q}^{(-p+1)^j} = \zeta_{p^2q}^{\sum_{j=0}^{p-1} (-p+1)^j} = \zeta_{p^2q}^{\frac{(p-1)p+1}{p}} = \zeta_p^{\frac{p}{p^2q}} = \zeta_p$$

where the last equality can be shown by using Lemma 4.1. Then

$$\begin{aligned}
\sum_{\substack{g^i \sigma^j \\ j \neq 0}} \frac{1}{\det g^i} \frac{(t\bar{t})^{p-1} - \det g^i}{t\bar{t} - \det g^{-i}} &= (p-1) \sum_{i=0}^{p^2q-1} \frac{1}{\zeta_p^i} \frac{(t\bar{t})^{p-1} - \zeta_p^i}{t\bar{t} - \zeta_p^{-i}} \\
&= p(p-1)q \sum_{i=0}^{p-1} \zeta_p^{-i} \sum_{l=0}^{p-2} \zeta_p^{-i(p-l-2)} (t\bar{t})^l \\
&= p(p-1)q \sum_{l=0}^{p-2} (t\bar{t})^l \sum_{i=0}^{p-1} \zeta_p^{i(l+1)} = 0.
\end{aligned}$$

This completes the proof. $\square$

Proposition 5.4. *With the above assumptions, one has*

$$(\star) \quad P_{W, G \rtimes S}(t, \bar{t}) = \frac{1}{p} P_{W, G}(t, \bar{t}) + P_{W, G \rtimes S}^I(t, \bar{t}) + P_{W, G \rtimes S}^{SL}(t, \bar{t}).$$

Proof. If h is conjugate to h' , one can show $P_{W, G \rtimes S, h}(t, \bar{t}) = P_{W, G \rtimes S, h'}(t, \bar{t})$. Hence

$$\begin{aligned}
P_{W, G \rtimes S}(t, \bar{t}) &= \sum_{g^i \sigma^j \in G \rtimes S} \frac{1}{\#[g^i \sigma^j]} P_{W, G \rtimes S, g^i \sigma^j}(t, \bar{t}) \\
&= P_{W, G \rtimes S, 1}(t, \bar{t}) + \sum_{i \neq 0} \frac{1}{\#[g^i]} P_{W, G \rtimes S, g^i}(t, \bar{t}) + \sum_{\substack{[g^i \sigma^j] \\ j \neq 0}} P_{W, G \rtimes S, g^i \sigma^j}(t, \bar{t})
\end{aligned}$$

where the last sum runs over all conjugacy class of the form $[g^i \sigma^j]$ for $j \neq 0$.

First, assume $e = 0$. For all conjugacy class of the form $[g^i]$ one has $\#[g^i] = p$ and $C_{G \rtimes S}(g^i) = C_G(g^i) = G$ by Lemma 4.3, so

$$\sum_{i \neq 0} \frac{1}{\#[g^i]} P_{W, G \rtimes S, g^i}(t, \bar{t}) = \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}).$$

Again by Lemma 4.3, σ^j 's are representatives of conjugacy classes of the form $[g^i \sigma^j]$. The eigenspace of σ^j with the eigenvalue 1 is spanned by $x_1 + \cdots + x_p$. Since $C_{G \rtimes S}(\sigma^j) = S$ and any element of S acts $x_1 + \cdots + x_p$ by 1, we have

$$\begin{aligned}
\sum_{\substack{[g^i \sigma^j] \\ j \neq 0}} P_{W, G \rtimes S, g^i \sigma^j}(t, \bar{t}) &= \sum_{j=1}^{p-1} P_{W, G \rtimes S, \sigma^j}(t, \bar{t}) \\
&= \sum_{j=1}^{p-1} \frac{1}{p} t^{\frac{p-1}{2}} \bar{t}^{\frac{p-1}{2}} (t\bar{t})^{-\frac{p-1}{p}} \sum_{k=0}^{p-1} \frac{1 - (t\bar{t})^{\frac{p-1}{p}}}{1 - (t\bar{t})^{\frac{1}{p}}} \\
&= (p-1) t^{\frac{p-1}{2}} \bar{t}^{\frac{p-1}{2}} (t\bar{t})^{-\frac{p-1}{p}} \sum_{l=1}^{p-1} (t\bar{t})^{\frac{l-1}{p}} \\
&= (p-1) \sum_{l=1}^{p-1} (t\bar{t})^{\frac{l}{p} + \frac{p-1}{2} - 1}.
\end{aligned}$$

Summarizing the above calculations, one has

$$\begin{aligned}
P_{W, G \rtimes S}(t, \bar{t}) &= \frac{1}{p} P_{W, G, 1}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} (t\bar{t})^i + \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}) + (p-1) \sum_{i=1}^{p-1} (t\bar{t})^{\frac{i}{p} + \frac{p-1}{2} - 1} \\
&= \frac{1}{p} P_{W, G}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} (t\bar{t})^i + (p-1) \sum_{i=1}^{p-1} (t\bar{t})^{\frac{i}{p} + \frac{p-1}{2} - 1}.
\end{aligned}$$

Recall that $e = 0$ if and only if $J \notin G \subset SL_p(\mathbb{C})$. This shows the asserted relation $(\star)$ holds for $e = 0$.

Second, assume $e = 1$. For all conjugacy class of the form $[g^i]$, one has $\#[g^i] = p$ and $C_{G \rtimes S}(g^i) = C_G(g^i) = G$ if $q \nmid i$. Otherwise g^i is a central element, so $\#[g^i] = 1$ and $C_{G \rtimes S}(g^i) = G \rtimes S$. Hence

$$\begin{aligned}
\sum_{i \neq 0} \frac{1}{\#[g^i]} P_{W, G \rtimes S, g^i}(t, \bar{t}) &= \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} P_{W, G, g^{qi}}(t, \bar{t}) \\
&= \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} t^i \bar{t}^{p-i} (t\bar{t})^{-1} \\
&= \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1}.
\end{aligned}$$

A conjugacy class of the form $[g^{i'} \sigma^j]$ where $j \neq 0$ has a representative of the form $g^{qi} \sigma^j$ and $C_{G \rtimes S}(g^{qi} \sigma^j) = \langle g^q, \sigma \rangle$ by Lemma 4.3. By Lemma 5.2, the eigenvalues of $g^{qi} \sigma^j$ are ζ_p^k for $k = 0, \dots, p-1$. Define y_k by

$$y_k = \sum_{l=1}^p \zeta_p^{(-k+1)(l-1)} x_l$$

for $k = 1, \dots, p$. y_k is an eigenvector of σ^j with eigenvalue ζ_p^{k-1} and an eigenvector of $g^i \sigma^j$ because

$$g^{qi} \sigma^j \cdot y_k = g^{qi} \zeta_p^{(k-1)j} y_k = \zeta_p^{i+(k-1)j} y_k.$$

In particular, $y_{-ij^{-1}+1}$ is an eigenvector with the eigenvalue 1 where we denote the inverse of j modulo p by j^{-1} .

Now, one can see

$$\begin{aligned}
\frac{1}{p} \sum_{k=1}^p \zeta_p^{(l-1)(k-1)} y_k &= \frac{1}{p} \sum_{k=1}^p \zeta_p^{(l-1)(k-1)} \sum_{m=1}^p \zeta_p^{(-k+1)(m-1)} x_m \\
&= \frac{1}{p} \sum_{m=1}^p \zeta_p^{-l+m} x_m \sum_{k=1}^p \zeta_p^{k(l-m)} \\
&= x_l.
\end{aligned}$$

Then

$$W = \sum_{l=1}^p x_l^{p-1} x_{l+1} = \sum_{l=1}^p \left(\frac{1}{p} \sum_{k=1}^p \zeta_p^{l(l-1)(k-1)} y_k \right)^{p-1} \left(\frac{1}{p} \sum_{k=1}^p \zeta_p^{l(k-1)} y_k \right).$$

Setting $y_k = 0$ for $k \neq -ij^{-1} + 1$ one has

$$\begin{aligned} W^{g^{i\sigma^j}} &= \frac{1}{p^p} \sum_{l=1}^p \zeta_p^{(l-1)(-ij^{-1})(p-1)+l(-ij^{-1})} y_{-ij^{-1}+1}^p \\ &= \frac{1}{p^{p-1}} \zeta_p^{-ij^{-1}} y_{-ij^{-1}+1}^p. \end{aligned}$$

So we have $R^{g^{i\sigma^j}}/I_{\nabla W^{g^{i\sigma^j}}} \otimes \Omega_{g^{i\sigma^j}} \cong \mathbb{C}[y]/(y^{p-1}) \otimes \mathbb{C}dy \cong \bigoplus_{k=0}^{p-2} y^k dy$ as a vector space, but no elements are invariant under the action of g^q . This shows

$$P_{W, G \rtimes S, g^{i\sigma^j}}(t, \bar{t}) = 0.$$

Summarizing the calculations above, we have

$$\begin{aligned} P_{W, G \rtimes S}(t, \bar{t}) &= \frac{1}{p} P_{W, G, 1}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} (t\bar{t})^i + \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1} \\ &= \frac{1}{p} P_{W, G}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} (t\bar{t})^i + \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1} \end{aligned}$$

Recall $e = 1$ if and only if $J \in G \subset SL_p(\mathbb{C})$. This shows that the asserted relation $(\star)$ holds.

At last, assume $e = 2$. By Lemma 4.5, for conjugacy classes of the form $[g^i]$, one has $\#[g^i] = p$ and $C_{G \rtimes S}(g^i) = C_G(g^i) = G$ if $pq \nmid i$. Otherwise g^i is a central element, so $\#[g^i] = 1$ and $C_{G \rtimes S}(g^i) = G \rtimes S$. So as in the case of $e = 1$, we see

$$\sum_{i \neq 0} \frac{1}{\#[g^i]} P_{W, G \rtimes S, g^i}(t, \bar{t}) = \frac{1}{p} \sum_{i \neq 0} P_{W, G, g^i}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1}.$$

Since $\det g^i = \zeta_p^i$ as shown in the proof of Lemma 5.3, g^i has eigenvalue 1 if and only if $p \mid i$. By Lemma 4.3, conjugacy class of the form $[g^{i'\sigma^j}]$ where $j \neq 0$ are represented by $g^{i\sigma^j}$ for $i = 0, \dots, p-1$. For $i = 0$, $P_{W, G \rtimes S, \sigma^j}(t, \bar{t}) = 0$ as in the case of $e = 1$. Otherwise the eigenvalues of $g^{i\sigma^j}$ are roots of $X^p - \zeta_p^i = 0$, i.e., $\zeta_{p^2}^i \zeta_p^k$ for $k = 0, \dots, p-1$. Then one has

$$\text{age}(g^{i\sigma^j}) = \sum_{k=0}^{p-1} \left(\frac{i}{p^2} + \frac{k}{p} \right) = \frac{i}{p} + \frac{p-1}{2}$$

and

$$\text{age}((g^{i\sigma^j})^{-1}) = p - \text{age}(g^{i\sigma^j}) = -\frac{i}{p} + \frac{p+1}{2}.$$

Hence we have

$$\begin{aligned} \sum_{\substack{[g^{i\sigma^j}] \\ i,j \neq 1}} P_{W,G \rtimes S, g^{i\sigma^j}}(t, \bar{t}) &= \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} t^{\text{age}(g^{i\sigma^j})} \bar{t}^{\text{age}((g^{i\sigma^j})^{-1})} (t\bar{t})^{-1} \\ &= (p-1) \sum_{i=1}^{p-1} t^{\frac{i}{p} + \frac{p-1}{2} - 1} \bar{t}^{-\frac{i}{p} + \frac{p+1}{2} - 1}. \end{aligned}$$

The calculation above and Lemma 5.3 show

$$\begin{aligned} P_{W,G \rtimes S}(t, \bar{t}) &= \frac{1}{p} P_{W,G,1}(t, \bar{t}) + \frac{1}{p} \sum_{i \neq 0} P_{W,G, g^i}(t, \bar{t}) \\ &\quad + \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1} + (p-1) \sum_{i=1}^{p-1} t^{\frac{i}{p} + \frac{p-1}{2} - 1} \bar{t}^{-\frac{i}{p} + \frac{p+1}{2} - 1} \\ &= \frac{1}{p} P_{W,G}(t, \bar{t}) + \frac{p-1}{p} \sum_{i=1}^{p-1} t^{i-1} \bar{t}^{p-i-1} + (p-1) \sum_{i=1}^{p-1} t^{\frac{i}{p} + \frac{p-1}{2} - 1} \bar{t}^{-\frac{i}{p} + \frac{p+1}{2} - 1} \end{aligned}$$

Recall that $e = 2$ if and only if $J \in G \nsubseteq SL_p(\mathbb{C})$. This completes the proof. $\square$

Theorem 5.5. *The following relations holds:*

$$P_{W^T, G^T \rtimes S}(t, \bar{t}) = \bar{t}^c P_{W, G \rtimes S}(t, \bar{t}^{-1}).$$

Proof. Recall that at least one of G and G^T contains the exponential grading operator. This theorem follows from Corollary 3.4, Proposition 5.4 and Lemma 5.1. $\square$

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